Random Measures, Point Processes, and Stochastic Geometry

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Preface

This book is centered on the mathematical analysis of random structures embedded in the Euclidean space or more general topological spaces, with a main focus on random measures, point processes, and stochastic geometry. Such random structures have been known to play a key role in several branches of natural sciences (cosmology, ecology, cell biology) and engineering (material sciences, networks) for several decades. Their use is currently expanding to new fields like data sciences. The book was designed to help researchers finding a direct path from the basic definitions and properties of these mathematical objects to their use in new and concrete stochastic models.

The theory part of the book is structured to be self-contained, with all proofs included, in particular on measurability questions, and at the same time comprehensive. In addition to the illustrative examples which one finds in all classical mathematical books, the document features sections on more elaborate examples which are referred to as *models* in the book. Special care is taken to express these models, which stem from the natural sciences and engineering domains listed above, in clear and self-contained mathematical terms. This continuum from a comprehensive treatise on the theory of point processes and stochastic geometry to the collection of models that illustrate its representation power is probably the main originality of this book.

The book contains two types of mathematical results: (1) structural results on stationary random measures and stochastic geometry objects, which do not rely on any parametric assumptions; (2) more computational results on the most important parametric classes of point processes, in particular Poisson or Determinantal point processes. These two types are used to structure the book.

The material is organized as follows. Random measures and point processes are presented first, whereas stochastic geometry is discussed at the end of the book. For point processes and random measures, parametric models are discussed before non-parametric ones. For the stochastic geometry part, the objects as point processes are often considered in the space of random sets of the Euclidean space. Both general processes are discussed as, e.g., particle processes, and parametric ones like, e.g., Poisson Boolean models of Poisson hyperplane processes.

We assume that the reader is acquainted with the basic results on measure and probability theories. We prove all technical auxiliary results when they are not easily available in the literature or when existing proofs appeared to us not sufficiently explicit. In all cases, the corresponding references will always be given.

Request for feedback and Acknowledgements

The present web version is a first version meant to trigger feedback. Comments from readers are most welcomed and should be sent to the authors at any of the following email addresses:
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Contents

Notation

General

The notation $x := y$ means that x is defined as y. The punctuation mark ':' means '*such that*'.

For any $x \in \mathbb{R}$, we denote $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$.

The notation $x_n \uparrow x$ means that $\lim_{n\to\infty} x_n = x$ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ is nondecreasing; that is $x_n \leq x_{n+1}$. A similar convention applies for $x_n \downarrow x$ when $\{x_n\}_{n\in\mathbb{N}}$ is nonincreasing.

Sets

The *indicator function* of a set A is denoted by $\mathbf{1}_A$. We sometimes use $\mathbf{1}\{x \in A\}$ instead of $\mathbf{1}_A(x)$.

The *complement* of a set A is denoted by A^{c}. For two sets A and B, the notation $A \subset B$ means A is a *subset* of B (A may be equal to B). We denote the union $A \cup B = \{x : x \in A \text{ or } x \in B\}$, the difference $A \backslash B = \{x \in A : x \notin B\}$, and the *symmetrical difference* $A \triangle B = (A\backslash B) \cup (B \backslash A)$. For all $A, B \subset \mathbb{R}^d$, we denote $A \oplus B = \{x + y : x \in A, y \in B\}$. For any set A, let $A^n = A \times \ldots \times A$ be the cartesian product of A with it self n times; we denote also $A^{(n)} = \{(x_1, \ldots, x_n) \in A^n : x_i \neq x_j \text{ for any } i \neq j\}$.

We denote the sets of integers by

$$\mathbb{N} = \{0, 1, 2, \ldots\} \quad \text{and} \quad \mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}.$$

Moreover $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. Similarly $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, and $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. We denote by $\boldsymbol{i}$ is the imaginary unit complex number, by $\mathbb{C}$ the set of complex numbers and by $\bar{\mathbb{C}}$ the set of complex numbers whose real and imaginary parts are in $\bar{\mathbb{R}}$. The complex-conjugate of $z \in \bar{\mathbb{C}}$ is denoted by z^*. For any $z \in \mathbb{C}, n \in \mathbb{N}^*$, we denote by

$$z^{(n)} := z(z-1)\ldots(z-n+1)$$

the n-th *factorial power* of z.

For a finite set A, the *cardinality* (i.e., the number of elements) of A is denored by $|A|$.

The closed intervals in $\bar{\mathbb{R}}$ are denoted by $[a,b]=\left\{x\in\bar{\mathbb{R}}:a\leq x\leq b\right\}$ where $a<b\in\bar{\mathbb{R}}$; the open intervals are denoted by $(a,b)=\{x\in\mathbb{R}:a<x<b\}$; similarly, $[a,b)=\left\{x\in\bar{\mathbb{R}}:a\leq x<b\right\}$ and $(a,b]=\left\{x\in\bar{\mathbb{R}}:a<x\leq b\right\}$.

Vectors and matrices

The vector are by default considered as column vectors. The *transpose* of a matrix or a vector M is denoted by M^{T} and the *transpose-conjugate* is denoted by M^*.

The Euclidean norm of $x=(x_1,\ldots,x_d)\in\mathbb{C}^d$ is denoted by $|x|=\left(\sum_{i=1}^d|x_i|^2\right)^{1/2}$.

For any $\lambda_1,\ldots,\lambda_n\in\mathbb{C}$, the notation $A=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ means that A is a *diagonal matrix*; i.e., a square matrix of size n with $A_{ij}=\lambda_i\mathbf{1}\{i=j\}$ for $i,j\in\{1,\ldots n\}$.

The determinant of a matrix A is denoted by $\det(A)$.

For two vectors $u=(u_1,\ldots,u_n)\,,v=(v_1,\ldots,v_n)$, we denote $u\cup v=(u_1,\ldots,u_n,v_1,\ldots,v_n)$.

Functions

Consider two functions $f,g:\mathbb{R}\to\mathbb{R}$. The notation $f=o(g)$ at t_0 means that $\lim_{t\to t_0}\frac{f(t)}{g(t)}=0$. The notation $f\sim g$ at t_0 means that $\lim_{t\to t_0}\frac{f(t)}{g(t)}=1$; two functions f and g are then said *equivalent* at t_0.

For any function $f:\mathbb{G}\to\mathbb{C}$ we denote $\|f\|_\infty:=\sup_{x\in\mathbb{G}}|f(x)|$. If f is n times differentiable, then its n-th derivative is denoted by $f^{(n)}(x)=\frac{\mathrm{d}^nf(x)}{\mathrm{d}x}$. By convention $f^{(0)}(x)=f(x)$. We will say that f is of class C^n when it is n times differentiable and its n-th derivative is continuous.

A *constant function* f whose value is c will be denoted $f\equiv c$. The *support of a function* $f:\mathbb{G}\to\bar{\mathbb{C}}$, denoted $\operatorname{supp}(f)$, is $\{x\in\mathbb{G}:f(x)\neq0\}$.

Topology

For a topological space $\mathbb{G}$, we will denote by $\mathcal{B}(\mathbb{G})$ the Borel σ-algebra (i.e., the σ-algebra generated by the topology) on $\mathbb{G}$ and by $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ the set of relatively compact sets in $\mathcal{B}(\mathbb{G})$. Moreover, we denote by $\mathfrak{F}_+(\mathbb{G})$ the class of all measurable functions $f:\mathbb{G}\to\mathbb{R}_+$ and by $\mathfrak{F}_c(\mathbb{G})$ the subclass of functions in $\mathfrak{F}_+(\mathbb{G})$ which are bounded and continuous with support in $\mathcal{B}_c(\mathbb{G})$.

We denote by $B(x,r)$ the open ball of center $x\in\mathbb{R}^d$ and radius r; that is

$$B(x,r)=\left\{y\in\mathbb{R}^d:|y-x|<r\right\}.$$

The closure of any set A in a topological space is denoted by $\bar{A}$; in particular $\bar{B}(x,r)$ is the closed ball of center x and radius r. The *boundary* of A is denoted by ∂A (which is the closure minus the interior of A).

A topological space is said to be *separable* if it admits a countable dense subset.

Measures

The *Dirac measure* is denoted by δ_x; that is $\delta_x(A) = \mathbf{1}\{x \in A\}$. The *Lebesgue measure* on $\mathbb{R}^d$ is denoted by ℓ^d and the Lebesgue measure of a set $A \in \mathcal{B}(\mathbb{R}^d)$ is denoted by $\ell^d(A)$ or $|A|$.

Let $(\mathbb{G}, \mathcal{G}, \mu)$ be a measure space. The (Lebesgue) integral of some measurable function $f : \mathbb{G} \to \bar{\mathbb{C}}$ with respect to μ is denoted by

$$\mu(f) = \int f(x)\,\mu(\mathrm{d}x) = \int f \mathrm{d}\mu,$$

provided it is well defined. We say that f is *μ-integrable* if $\int |f(x)|\,\mu(\mathrm{d}x) < \infty$. Let $p \in \mathbb{N}^*$ and $\mathbb{F} = \mathbb{R}$, $\bar{\mathbb{R}}$, $\mathbb{C}$ or $\bar{\mathbb{C}}$, we denote by $L^p_{\mathbb{F}}(\mu, \mathbb{G})$ the set of functions $f : \mathbb{G} \to \mathbb{F}$ such that f^p is integrable with respect to μ; that is $\int_{\mathbb{G}} |f(x)|^p\,\mu(\mathrm{d}x) < \infty$.

A measure μ on a measurable space $(\mathbb{G}, \mathcal{G})$ is called *σ-finite* if there is a countable family of measurable sets of finite measure μ, covering $\mathbb{G}$.

Let μ be a measure on a measurable space $(\mathbb{G}, \mathcal{G})$. The restriction of μ to $B \in \mathcal{G}$ is denoted by $\mu|_B$; i.e., $\mu|_B(\cdot) = \mu(\cdot \cap B)$. We shall sometimes denote $\mu|_B$ simply by μ_B.

Let $(\mathbb{G}_1, \mathcal{G}_1)$ and $(\mathbb{G}_2, \mathcal{G}_2)$ be two measurable spaces, we denote by $\mathcal{G}_1 \otimes \mathcal{G}_2$ the *product σ-algebra* of $\mathcal{G}_1$ and $\mathcal{G}_2$; cf. [44, §33]. In particular, for a measurable space $(\mathbb{G}, \mathcal{G})$ and $n \in \mathbb{N}^*$, we denote by $\mathcal{G}^{\otimes n}$ its *n-th power* in the sense of products of σ-algebra.

Given two σ-finite measures μ_1 and μ_2 on measurable spaces $(\mathbb{G}_1, \mathcal{G}_1)$ and $(\mathbb{G}_2, \mathcal{G}_2)$ respectively, their *product measure* (cf. [44, Theorem 35.B]) is denoted by $\mu_1 \times \mu_2$. In particular, for a σ-finite measure μ on a measurable space $(\mathbb{G}, \mathcal{G})$ and $n \in \mathbb{N}^*$, we denote by μ^n its *n-th power* in the sense of products of measures.

Let μ and ν be two measures on the same measurable space $(\mathbb{G}, \mathcal{G})$. We say that μ is *absolutely-continuous* with respect to ν, denoted $\mu \ll \nu$, if for each $A \in \mathcal{G}$, $\nu(A) = 0 \Rightarrow \mu(A) = 0$. Two measures are said *equivalent* if they are absolutely-continuous with respect to each other. Equivalence of measures is denoted by $\mu \sim \nu$.

Random variables

The *basic probability space* is denoted by $(\Omega, \mathcal{A}, \mathbf{P})$. A *random variable* X is a measurable mapping from Ω to an arbitrary measurable space $(\mathbb{G}, \mathcal{G})$; we say that X is $\mathbb{G}$-*valued*. When $\mathbb{G} = \mathbb{R}$ (resp. $\mathbb{C}$), we say that X is a *real* (resp.

complex) random variable. When $\mathbb{G} = \mathbb{R}^n$ (resp. $\mathbb{C}^n$), we say that X is a *real* (resp. *complex*) random vector.

A sequence of random variables $X_0, X_1, \ldots$ is denoted by $\{X_k\}_{k\in\mathbb{N}}$. A stochastic process indexed by an arbitrary index set I is denoted by $\{X(i)\}_{i\in I}$. When the $X(i)$'s are real (resp. complex) random variables, we say that $\{X(i)\}_{i\in I}$ is a *real* (resp. *complex*) stochastic process.

The *distribution* of a random variable X is denoted by $\mathbf{P}_X$; that is

$$\mathbf{P}_X(A) = \mathbf{P}(X \in A) = \mathbf{P}(\{\omega \in \Omega : X(\omega) \in A\}).$$

The notation $X \overset{\text{dist.}}{\sim} Q$ means that $\mathbf{P}_X = Q$. The *expectation* of a random variable X is denoted by $\mathbf{E}[X]$.

For two random variables X and Y, the notation $X \overset{\text{dist.}}{=} Y$ means that they have the same distribution.

The *moments* of a random variable X with values in $\mathbb{C}$ are $\mathbf{E}[X], \mathbf{E}\left[X^2\right], \ldots$

The random variables $X_0, X_1, \ldots$ are said to be *i.i.d* when they are independent and identically distributed.

Part I

Random measures and point processes

Chapter 1

Foundations

A point process may be seen as a random object taking as values locally finite configurations of points or, equivalently, counting measures. We will consider the more general notion of random measure; that is a random object taking measures as possible realizations.

1.1 Framework

Let $(\mathbb{G},\mathfrak{T})$ be a topological space which is locally compact, second countable (i.e., its topology $\mathfrak{T}$ has a countable basis), and Hausdorff (i.e., distinct points have disjoint neighborhoods). This will be abbreviated by *l.c.s.h.*. Such a space is *Polish* [56, Theorem 5.3 p.29]; i.e., there exists some metric d on $\mathbb{G}$ such that the topology induced by d is equal to $\mathfrak{T}$ and such that $(\mathbb{G}, d)$ is a complete separable metric space.

Our basic measurable space will be $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$, where $\mathcal{B}(\mathbb{G})$ is the associated *Borel* σ-algebra, namely, the σ-algebra generated by the topology $\mathfrak{T}$. A set $B \in \mathcal{B}(\mathbb{G})$ is called a *Borel set*.

A set $B \in \mathcal{B}(\mathbb{G})$ is called *relatively compact* if its closure is compact. Let $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ denote the set of relatively compact sets in $\mathcal{B}(\mathbb{G})$. Moreover, we denote by $\mathfrak{F}_+(\mathbb{G})$ the class of all measurable functions $f : \mathbb{G} \to \mathbb{R}_+$ and by $\mathfrak{F}_{\mathrm{c}}(\mathbb{G})$ the subclass of functions in $\mathfrak{F}_+(\mathbb{G})$ which are bounded and continuous with support in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$.

We will always assume that $\mathbb{R}^d$ (for some integer d), $\mathbb{R}_+$, $\mathbb{C}$, etc., are endowed with the *usual topology* induced by Euclidean norm.

Example 1.1.1. *Let $\mathbb{G} = \mathbb{R}^d$ and $\mathfrak{T}$ be the usual topology on $\mathbb{R}^d$ induced by Euclidean norm. The topological space $(\mathbb{R}^d,\mathfrak{T})$ is l.c.s.h. A Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ is compact iff it is closed and bounded. A Borel set is relatively compact iff it is bounded.*

Example 1.1.2. *Let $\mathbb{G} = \mathbb{Z}^d$ and $\mathfrak{T}$ be the topology which contains all subsets of $\mathbb{Z}^d$ as open sets called the* discrete topology*. It is the subspace topology induced* *by the usual topology of* $\mathbb{R}^d$. *The topological space* $(\mathbb{Z}^d, \mathfrak{T})$ *is l.c.s.h. A Borel set* $B \subset \mathbb{Z}^d$ *is compact (or relatively compact) iff it is finite.*

Remark 1.1.3. *Let* $\mathbb{G}$ *be a l.c.s.h. space and let* $\mathcal{F}(\mathbb{G})$ *be the space of closed subsets of* $\mathbb{G}$. *In Chapter 9, we will construct a topology on* $\mathcal{F}(\mathbb{G})$ *making it also l.c.s.h.*

Lemma 1.1.4. *A l.c.s.h. space* $\mathbb{G}$ *may be covered by a countable union of relatively compact open sets. Moreover, there is a countable partition of* $\mathbb{G}$ *into relatively compact sets.*

Proof. By local compactness, for every $x \in \mathbb{G}$ there is an open neighborhood U_x of x with compact closure. On the other hand, since the topological space $(\mathbb{G}, \mathfrak{T})$ has a countable basis $\mathcal{C}$, then, for all $x \in \mathbb{G}$, there exists some $C_x \in \mathcal{C}$ such that $C_x \subset U_x$. Clearly, C_x is relatively compact and $\{C_x\}_{x \in \mathbb{G}}$ is countable and covers $\mathbb{G}$. Then $\mathbb{G}$ may be covered by a countable union of relatively compact sets, say $C_0, C_1, \ldots$. Finally, the sets $B_0, B_1, \ldots$ constructed recursively as follows

$$B_0 = C_0, \quad B_k = C_k \backslash \bigcup_{j=0}^{k-1} B_j, \quad k = 1, 2, \ldots$$

are relatively compact and constitute a partition of $\mathbb{G}$. □

A measure μ on $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$ is said to be *locally finite* if $\mu(B) < \infty$ for all $B \in \mathcal{B}_c(\mathbb{G})$. By Lemma 1.1.4 such a measure is in particular σ-finite.

Let $\bar{\mathbb{M}}(\mathbb{G})$ be the space of locally finite measures on $\mathbb{G}$ and $\bar{\mathcal{M}}(\mathbb{G})$ be the σ-algebra on $\bar{\mathbb{M}}(\mathbb{G})$ generated by the mappings $\mu \mapsto \mu(B), B \in \mathcal{B}(\mathbb{G})$ (i.e., generated by the family of sets $\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B) \le x\right\}$ where $B \in \mathcal{B}(\mathbb{G}), x \in \mathbb{R}_+$). Given $\mu \in \bar{\mathbb{M}}(\mathbb{G})$, for all measurable functions f defined on $\mathbb{G}$, we define $\mu(f)$ by

$$\mu(f) = \int_{\mathbb{G}} f(s)\,\mu(\mathrm{d}s),$$

when the integral in the right-hand side of the above equality is well defined in the sense of Lebesgue.

Lemma 1.1.5. *Let* $\mathbb{G}$ *be a l.c.s.h. space. For every* $f \in \mathfrak{F}_+(\mathbb{G})$, *the mapping* $\mu \mapsto \mu(f)$ *defined from* $\bar{\mathbb{M}}(\mathbb{G})$ *to* $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ *(equipped with the* σ*-algebras* $\bar{\mathcal{M}}(\mathbb{G})$ *and* $\mathcal{B}\left(\bar{\mathbb{R}}_+\right)$, *respectively) is measurable.*

Proof. Cf. [52, p.12]. Recall that a measurable function $f : \mathbb{G} \to \mathbb{R}$ is called simple if it is of the form $f = \sum_{j=1}^{k} \alpha_j \mathbf{1}_{C_j}$ for some $k \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ and $C_j \in \mathcal{B}(\mathbb{G})$ $(1 \le j \le k)$, where the C_j 's can be chosen mutually disjoint without loss of generality, and where $\mathbf{1}_C$ denotes the indicator function of the set C. The statement is true for simple f by definition of $\bar{\mathcal{M}}(\mathbb{G})$. For any $f \in \mathfrak{F}_+(\mathbb{G})$, the simple approximation theorem [11, Theorem 13.5 p.185] ensures that there exists a nondecreasing sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{F}_+(\mathbb{G})$ such that $f_n(x) \uparrow f(x), \forall x \in \mathbb{G}$. For a given $n \in \mathbb{N}$, let $f_n = \sum_{j=1}^{k} \alpha_j \mathbf{1}_{C_j}$. We have

$$\mu(f_n) = \int_{\mathbb{G}} f_n \mathrm{d}\mu = \sum_{j=1}^{k} \alpha_j \int_{\mathbb{G}} \mathbf{1}_{C_j} \mathrm{d}\mu = \sum_{j=1}^{k} \alpha_j \mu(C_j).$$

Each of the mappings $\mu \mapsto \mu(C_j)$ is measurable. Then $\mu \mapsto \mu(f_n)$ is measurable. For a given $\mu \in \bar{\mathbb{M}}(\mathbb{G})$, $\mu(f_n) = \int_{\mathbb{G}} f_n \mathrm{d}\mu \to \int_{\mathbb{G}} f \mathrm{d}\mu = \mu(f)$ by the monotone convergence theorem [11, Theorem 16.2 p.208]. Since $\mu \mapsto \mu(f)$ is the limit of the sequence $\{\mu \mapsto \mu(f_n)\}_{n\in\mathbb{N}}$ of measurable functions, it is measurable. □

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space.

Definition 1.1.6. *Let $\mathbb{G}$ be a l.c.s.h. space. A* random measure *on $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$ is a measurable mapping $\Phi : \Omega \to \bar{\mathbb{M}}(\mathbb{G})$ (Ω and $\bar{\mathbb{M}}(\mathbb{G})$ being equipped with the σ-algebras $\mathcal{A}$ and $\bar{\mathcal{M}}(\mathbb{G})$, respectively). The probability distribution of Φ is the probability measure on $\left(\bar{\mathbb{M}}(\mathbb{G}), \bar{\mathcal{M}}(\mathbb{G})\right)$ induced by Φ; that is $\mathbf{P}_\Phi = \mathbf{P} \circ \Phi^{-1}$. In other words*

$$\mathbf{P}_\Phi(L) = \mathbf{P}(\Phi \in L), \quad L \in \bar{\mathcal{M}}(\mathbb{G}).$$

Proposition 1.1.7. *Let $\mathbb{G}$ be a l.c.s.h. space. For a mapping $\Phi : \Omega \to \bar{\mathbb{M}}(\mathbb{G})$ the three following statements are equivalent:*

(i) Φ is a random measure on $\mathbb{G}$.

(ii) $\Phi(f)$ is a random variable for all $f \in \mathfrak{F}_+(\mathbb{G})$.

(iii) $\Phi(B)$ is a random variable for all $B \in \mathcal{B}(\mathbb{G})$.

Proof. (i)⇒(ii). $\Phi(f)$ is the composition of Φ with the mapping $\mu \mapsto \mu(f)$ which is measurable by Lemma 1.1.5. (ii)⇒(iii). Take $f = \mathbf{1}_B$. (iii)⇒(i). The sets $L = \left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B) \leq x\right\}, x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{G})$ generate $\bar{\mathcal{M}}(\mathbb{G})$. But $\Phi^{-1}(L) = \{\omega \in \Omega : \Phi(\omega)(B) \leq x\} \in \mathcal{A}$. Hence Φ is measurable. □

The above proposition says that for any random measure Φ on $\mathbb{G}$, $\{\Phi(B)\}_{B\in\mathcal{B}(\mathbb{G})}$ is a stochastic process with values in $\mathbb{R}_+$ indexed by $\mathcal{B}(\mathbb{G})$. Conversely:

Corollary 1.1.8. *Let $\mathbb{G}$ be a l.c.s.h. space and let Φ be a mapping from Ω to the set of measures on $\mathbb{G}$ such that $\{\Phi(B)\}_{B\in\mathcal{B}(\mathbb{G})}$ is a stochastic process. Then Φ is a random measure iff $\Phi(\omega)$ is locally finite for all $\omega \in \Omega$.*

Example 1.1.9. Randomized Lebesgue measure. *Let X be a nonnegative random variable and let*

$$\Phi(B) = X|B|, \quad B \in \mathcal{B}\left(\mathbb{R}^d\right), \tag{1.1.1}$$

where $|B|$ denotes the Lebesgue measure of B. Since for all $B \in \mathcal{B}\left(\mathbb{R}^d\right)$, $\Phi(B)$ is a random variable, it follows from Proposition 1.1.7(iii) that Φ is a random measure on $\mathbb{R}^d$.

Example 1.1.10. Integral of a stochastic process. *Let* $\{\lambda(x)\}_{x\in\mathbb{R}^d}$ *be a non-negative measurable stochastic process (i.e., the mapping from* $\mathbb{R}^d \times \Omega$ *into* $\mathbb{R}_+$ *defined by* $(x,\omega) \to \lambda(x,\omega)$ *is measurable). Assume that, for almost all* $\omega \in \Omega$*, the function* $x \mapsto \lambda(x,\omega)$ *is locally integrable. Let*

$$\Phi(B) = \int_B \lambda(x)\, \mathrm{d}x, \quad B \in \mathcal{B}(\mathbb{G}).$$

Then Φ *is a random measure on* $\mathbb{R}^d$ *by Proposition 1.1.7(iii).*

Example 1.1.11. Sample and Binomial point process. *Let* $k \in \mathbb{N}$ *and let* $X_1, \ldots, X_k$ *be random variables with values in a l.c.s.h. space* $\mathbb{G}$*. We denote by* δ_x *the* Dirac measure *on* $\mathbb{G}$*; that is* $\delta_x(B) = \mathbf{1}\{x \in B\}, B \in \mathcal{B}(\mathbb{G})$*. Then* $\Phi = \sum_{j=1}^k \delta_{X_j}$ *is a random measure on* $\mathbb{G}$ *called a* sample point process*. Indeed, for any* $B \in \mathcal{B}(\mathbb{G})$*,* $\Phi(B) = \sum_{j=1}^k \mathbf{1}\{X_j \in B\}$ *is a random variable. In the particular case when* $X_1, \ldots, X_k$ *are i.i.d,* $\Phi = \sum_{j=1}^k \delta_{X_j}$ *is called a* Binomial point process*. We will see in Section 2.1 that this example is closely related to an important class called Poisson point processes (where k will be random).*

By definition, the *finite-dimensional distributions* of a random measure Φ are the probability distributions of the random vectors $(\Phi(B_1), \ldots, \Phi(B_k))$, for all $k \in \mathbb{N}$, $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G})$.

Lemma 1.1.12. *Let* $\mathbb{G}$ *be a l.c.s.h. space. The probability distribution of a random measure* Φ *on* $\mathbb{G}$ *is characterized by its finite-dimensional distributions.*

Proof. Let $\mathcal{C}$ be the class of subsets of $\bar{\mathcal{M}}(\mathbb{G})$ of the form

$$\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B_1) \in A_1, \ldots, \mu(B_k) \in A_k\right\},$$

where $k \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}, B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G}), A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R})$. Since, $\mathcal{C}$ is non-empty and stable by finite intersections (i.e., a π-system) then two probability measures which agree on $\mathcal{C}$ agree on $\sigma(\mathcal{C}) = \bar{\mathcal{M}}(\mathbb{G})$; cf. [11, Theorem 10.3 p.163]. □

Definition 1.1.13. *Let* Φ *be a random measure on a l.c.s.h. space* $\mathbb{G}$*, then* $\left(\bar{\mathbb{M}}(\mathbb{G}), \bar{\mathcal{M}}(\mathbb{G}), \mathbf{P}_\Phi\right)$ *is called the* canonical probability space *associated to* Φ*. The identity is a random measure in this space with probability distribution* $\mathbf{P}_\Phi$*.*

1.2 Mean measure, Laplace transform and void probability

Definition 1.2.1. *Let* Φ *be a random measure on a l.c.s.h. space* $\mathbb{G}$*.*

- *Its* mean measure *is a measure defined on* $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$ *by*

$$M_\Phi(B) = \mathbf{E}[\Phi(B)], \quad B \in \mathcal{B}(\mathbb{G}).$$

- *Its* Laplace transform*, denoted by $\mathcal{L}_\Phi$, is a functional defined on the set of all measurable functions $f:\mathbb{G}\to\bar{\mathbb{R}}_+$ by*

$$\mathcal{L}_\Phi(f)=\mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f\mathrm{d}\Phi\right)\right].$$

- *Its* void probability *function is a set function defined on $\mathcal{B}(\mathbb{G})$ by*

$$\nu_\Phi(B)=\mathbf{P}(\Phi(B)=0),\quad B\in\mathcal{B}(\mathbb{G}).$$

The fact that M_Φ is a measure on $(\mathbb{G},\mathcal{B}(\mathbb{G}))$ may be proved as follows. Indeed, M_Φ inherits the property of finite additivity from the underlying random measure Φ; moreover, if the sequence $\{B_n\}_{n\in\mathbb{N}}$ of Borel sets is increasing to B, then, by monotone convergence, $M_\Phi(B_n)\uparrow M_\Phi(B)$. However, M_Φ is not necessarily locally finite.

If Φ is a random measure on $\mathbb{G}=\mathbb{R}^d$ and its mean measure has the form

$$M_\Phi(B)=\int_B \lambda(x)\,\mathrm{d}x,$$

for some locally integrable measurable function $\lambda:\mathbb{R}^d\to\bar{\mathbb{R}}_+$, then this last function is called the *intensity function* of Φ. If λ is constant then it is called the *intensity* of Φ.

The Laplace transform plays an important role for random measures in particular due to the following result.

Corollary 1.2.2. *The probability distribution of a random measure Φ on a l.c.s.h. space $\mathbb{G}$ is characterized by its Laplace transform.*

Proof. Recall that the probability distribution of a random vector is characterized by its Laplace transform. Moreover, observe that for all $k\in\mathbb{N}^*$ and $t_1,\ldots,t_k\in\mathbb{R}_+, B_1,\ldots,B_k\in\mathcal{B}(\mathbb{G})$,

$$\mathcal{L}_\Phi\left(\sum_{j=1}^k t_j\mathbf{1}_{B_j}\right)=\mathbf{E}\left[\exp\left(-\sum_{j=1}^k t_j\Phi(B_j)\right)\right]=\mathcal{L}_{\Phi(B_1),\ldots,\Phi(B_k)}(t_1,\ldots,t_k),$$

which is the Laplace transform of the random vector $(\Phi(B_1),\ldots,\Phi(B_k))$. Then the Laplace transform of Φ characterizes its finite-dimensional distributions which, by Lemma 1.1.12, characterize the probability distribution of Φ. □

Example 1.2.3. Integral of a stochastic process, cont'd. *The mean measure of the random measure $\Phi(B)=\int_B\lambda(x)\,\mathrm{d}x$ of Example 1.1.10 is*

$$M_\Phi(B)=\mathbf{E}\left[\int_B\lambda(x)\,\mathrm{d}x\right]=\int_B\mathbf{E}[\lambda(x)]\,\mathrm{d}x.$$

The void probability function is

$$\nu_\Phi(B)=\mathbf{P}(\lambda(x)=0 \textit{ for Lebesgue-almost all } x\in B).$$

Example 1.2.4. Sample and Binomial point process, cont'd. *Consider a sample point process as in Example 1.1.11; that is $\Phi = \sum_{j=1}^{k} \delta_{X_j}$. Its mean measure is*

$$M_\Phi(B) = \mathbf{E}[\Phi_k(B)] = \sum_{j=1}^{k} \mathbf{P}(X_j \in B), \quad B \in \mathcal{B}(\mathbb{G}).$$

In the particular case when Φ is a Binomial point process (i.e., $X_1, \ldots, X_k$ are i.i.d), the mean measure is $M_\Phi(B) = k\mathbf{P}_{X_1}(B), B \in \mathcal{B}(\mathbb{G})$; the Laplace transform is

$$\mathcal{L}_\Phi(f) = \mathbf{E}\left[e^{-\sum_{j=1}^{k} f(X_j)}\right] = \left(\mathbf{E}\left[e^{-f(X_1)}\right]\right)^k = \left[\mathcal{L}_{f(X_1)}(1)\right]^k, \quad f \in \mathfrak{F}_+(\mathbb{G}),$$

and the void probability function is

$$\nu_\Phi(B) = [\mathbf{P}(X_1 \notin B)]^k, \quad B \in \mathcal{B}(\mathbb{G}).$$

1.2.1 Campbell's averaging formula

We denote by $\bar{\mathbb{C}}$ the set of complex numbers whose real and imaginary parts are in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For any measure μ on $\mathbb{G}$ and $p \in \mathbb{N}^*$ we denote by $L^p_{\bar{\mathbb{R}}}(\mu, \mathbb{G})$ (resp. $L^p_{\bar{\mathbb{C}}}(\mu, \mathbb{G})$) the set of measurable functions $f : \mathbb{G} \to \bar{\mathbb{R}}$ (resp. $f : \mathbb{G} \to \bar{\mathbb{C}}$) such that

$$\int_{\mathbb{G}} |f(x)|^p \mu(\mathrm{d}x) < \infty.$$

In particular, $L^1_{\bar{\mathbb{R}}}(\mu, \mathbb{G})$ is the set of measurable functions $f : \mathbb{G} \to \bar{\mathbb{R}}$ which are integrable with respect to μ.

Theorem 1.2.5. Campbell averaging formula. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with mean measure M_Φ. Then for any measurable function $f : \mathbb{G} \to \bar{\mathbb{C}}$ which is either nonnegative or in $L^1_{\bar{\mathbb{C}}}(M_\Phi, \mathbb{G})$, the integral $\int_{\mathbb{G}} f \mathrm{d}\Phi$ is a well defined random variable. Moreover,*

$$\mathbf{E}\left[\int_{\mathbb{G}} f \mathrm{d}\Phi\right] = \int_{\mathbb{G}} f \mathrm{d}M_\Phi. \tag{1.2.2}$$

The above result holds also true for all $f \in L^1_{\bar{\mathbb{C}}}(M_\Phi, \mathbb{G})$.

Proof. Consider first a simple function $f = \sum_{j=1}^{k} a_j \mathbf{1}_{B_j}$, where $a_j \geq 0$ and $B_j \in \mathcal{B}(\mathbb{G})$. Then

$$\mathbf{E}\left[\int f \mathrm{d}\Phi\right] = \mathbf{E}\left[\sum_{j=1}^{k} a_j \Phi(B_j)\right] = \sum_{j=1}^{k} a_j M_\Phi(B_j) = \int f \mathrm{d}M_\Phi.$$

For any measurable function $f : \mathbb{G} \to \bar{\mathbb{R}}_+$, there exists a nondecreasing sequence of simple functions $\{f_n\}_{n \in \mathbb{N}}$ in $\mathfrak{F}_+(\mathbb{G})$ such that $f_n \uparrow f$ as $n \to \infty$ pointwise. It follows from the monotone convergence theorem [11, Theorem 16.2 p.208] that, for all $\omega \in \Omega$, $\int f_n \mathrm{d}\Phi(\omega) \uparrow \int f \mathrm{d}\Phi(\omega)$ as $n \to \infty$. And again by monotone convergence,

$$\mathbf{E}\left[\int f \mathrm{d}\Phi\right] = \lim_{n\to\infty} \mathbf{E}\left[\int f_n \mathrm{d}\Phi\right] = \lim_{n\to\infty} \int f_n \mathrm{d}M_\Phi = \int f \mathrm{d}M_\Phi.$$

Let now $f \in L^1_{\mathbb{R}}(M_\Phi, \mathbb{G})$. We may decompose $f = f^+ - f^-$ where $f^+, f^- \in \mathfrak{F}_+(\mathbb{G})$. Note that $\mathbf{E}\left[\int f^\pm \mathrm{d}\Phi\right] = \int f^\pm \mathrm{d}M_\Phi < \infty$. Then the random variables $\int f^\pm \mathrm{d}\Phi$ are almost surely finite. Therefore $\int f \mathrm{d}\Phi = \int f^+ \mathrm{d}\Phi - \int f^- \mathrm{d}\Phi$ is a well-defined almost surely finite random variable, and

$$\mathbf{E}\left[\int f \mathrm{d}\Phi\right] = \mathbf{E}\left[\int f^+ \mathrm{d}\Phi\right] - \mathbf{E}\left[\int f^- \mathrm{d}\Phi\right] = \int f^+ \mathrm{d}M_\Phi - \int f^- \mathrm{d}M_\Phi = \int f \mathrm{d}M_\Phi.$$

Finally, for $f \in L^1_{\mathbb{C}}(M_\Phi, \mathbb{G})$, its real and imaginary parts are in $L^1_{\mathbb{R}}(M_\Phi, \mathbb{G})$. Then it is enough to apply (1.2.2) to each of these parts and then use linearity of integral and expectation to prove that (1.2.2) holds also true for f. □

We will show later in Section 3.1 how to extend (1.2.2) to the case when f is a function defined on $\Omega \times \mathbb{G}$ using Palm theory.

1.3 Distribution characterization

This section gathers a few technical results allowing us to refine some previously presented results regarding the measurability and the distribution characterization of a random measure. In particular, we will show that it is enough to check the measurability of $\Phi(B)$ in Proposition 1.1.7(iii) for some subclass of $B \in \mathcal{B}(\mathbb{G})$. To do so, we need first the following lemma.

Lemma 1.3.1. *Let $\mathbb{G}$ be a l.c.s.h. space and let $\mathcal{B}_0(\mathbb{G})$ be a subclass of $\mathcal{B}_c(\mathbb{G})$ closed under finite intersections, generating the σ-algebra $\mathcal{B}(\mathbb{G})$ and which contains a sequence increasing to $\mathbb{G}$ (or a countable partition of $\mathbb{G}$). Then $\bar{\mathcal{M}}(\mathbb{G})$ is generated by the mappings $\mu \mapsto \mu(B), B \in \mathcal{B}_0(\mathbb{G})$.*

Proof. Let $\mathcal{C}$ be the class of subsets of $\bar{\mathcal{M}}(\mathbb{G})$ of the form

$$\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(I_1) \in A_1, \ldots, \mu(I_k) \in A_k\right\},$$

where $k \in \mathbb{N}^*, I_1, \ldots, I_k \in \mathcal{B}_0(\mathbb{G}), A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R})$. We aim to show that $\sigma(\mathcal{C}) = \bar{\mathcal{M}}(\mathbb{G})$. Note first that since $\mathcal{B}_0(\mathbb{G}) \subset \mathcal{B}_c(\mathbb{G})$ then $\mu(I \cap B) < \infty$ for all $I \in \mathcal{B}_0(\mathbb{G}), B \in \mathcal{B}(\mathbb{G})$. Now, let

$$\mathcal{H} = \{B \in \mathcal{B}(\mathbb{G}) : \mu \mapsto \mu(I \cap B) \text{ is } \sigma(\mathcal{C})/\mathcal{B}(\mathbb{R}) \text{ measurable}, \forall I \in \mathcal{B}_0(\mathbb{G})\}.$$

Note that $\mathcal{H}$ contains $\mathbb{G}$ and is clearly closed under proper differences (for $C \subset B$, $\mu(I \cap (B - C)) = \mu(I \cap B) - \mu(I \cap C)$ which is well defined because the two last terms are finite) and nondecreasing limits; that is $\mathcal{H}$ is a *Dynkin system.*

Furthermore, $\mathcal{H}$ contains the class $\mathcal{B}_0(\mathbb{G})$ which is non-empty and closed under finite intersections ($\mathcal{B}_0(\mathbb{G})$ is called a π-*system*), it follows from Dynkin's theorem [11, Theorem 3.2 p.42] that $\mathcal{H}$ contains the σ-algebra $\sigma(\mathcal{B}_0(\mathbb{G}))$ which is equal to $\mathcal{B}(\mathbb{G})$, this proves that $\mathcal{H} = \mathcal{B}(\mathbb{G})$. Let $I_1, I_2, \ldots$ be a sequence in $\mathcal{B}_0(\mathbb{G})$ increasing to $\mathbb{G}$ (or a countable partition of $\mathbb{G}$), then for each $B \in \mathcal{B}(\mathbb{G})$, the mapping $\mu \mapsto \mu(B) = \lim_j \uparrow \mu(I_j \cap B)$ (or $\sum_j \mu(I_j \cap B)$) is $\sigma(\mathcal{C}) / \mathcal{B}(\bar{\mathbb{R}})$ measurable. It follows that $\bar{\mathcal{M}}(\mathbb{G}) \subset \sigma(\mathcal{C})$ and hence $\sigma(\mathcal{C}) = \bar{\mathcal{M}}(\mathbb{G})$. □

For example, for $\mathbb{G} = \mathbb{R}^d$, the class $\mathcal{B}_0(\mathbb{R}^d)$ of sets of the form $I = \prod_{k=1}^d (a_k, b_k]$, where $a_k, b_k \in \mathbb{R}$ satisfies the conditions of Lemma 1.3.1.

Corollary 1.3.2. *Let $\mathbb{G}$ be a l.c.s.h. space and let $\mathcal{B}_0(\mathbb{G})$ be as in Lemma 1.3.1. A mapping $\Phi : \Omega \to \bar{\mathbb{M}}(\mathbb{G})$ is a random measure on $\mathbb{G}$ iff $\Phi(B)$ is a random variable for all $B \in \mathcal{B}_0(\mathbb{G})$.*

Proof. This follows from Lemma 1.3.1 and [11, Theorem 13.1 p.182]. □

Example 1.3.3. Lebesgue-Stieltjes random measure. *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a real-valued, nondecreasing with right-continuous trajectories. Let*

$$\Phi(B) = \int_B X(\mathrm{d}t), \quad B \in \mathcal{B}(\mathbb{R}),$$

where the integral in the right-hand side of the above equation is the Lebesgue-Stieltjes integral. Thus Φ is the Lebesgue-Stieltjes measure *associated with X. For any $B \in \mathcal{B}_c(\mathbb{R})$, let $(a, b]$ be an interval containing B, then*

$$\Phi(B) \leq \Phi((a, b]) = X(b) - X(a) < \infty,$$

which shows that $\Phi(\omega)$ is locally finite. For any interval $(a, b]$, $\Phi((a, b]) = X(b) - X(a)$ is a random variable. It follows that Φ is a random measure by Corollary 1.3.2.

We show now that the probability distribution of a random measure is characterized by the finite-dimensional distributions where $B_1, \ldots, B_k$ are in some subclass of $\mathcal{B}(\mathbb{G})$.

Corollary 1.3.4. *The probability distribution of a random measure Φ on a l.c.s.h. space $\mathbb{G}$ is characterized by the distributions of the random vectors $(\Phi(B_1), \ldots, \Phi(B_k))$ where $k \in \mathbb{N}$, $B_1, \ldots, B_k \in \mathcal{B}_0(\mathbb{G})$; where $\mathcal{B}_0(\mathbb{G})$ is as in Lemma 1.3.1.*

Proof. Let $\mathcal{C}$ be the class of subsets of $\bar{\mathcal{M}}(\mathbb{G})$ of the form

$$\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B_1) \in A_1, \ldots, \mu(B_k) \in A_k\right\},$$

where $k \in \mathbb{N}^*, B_1, \ldots, B_k \in \mathcal{B}_0(\mathbb{G}), A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R})$. If follows from Lemma 1.3.1 that $\sigma(\mathcal{C}) = \bar{\mathcal{M}}(\mathbb{G})$. Since, $\mathcal{C}$ is non-empty and stable by finite intersections (i.e., a π-system), then two σ-finite measures which agree on $\mathcal{C}$ would agree on $\sigma(\mathcal{C}) = \bar{\mathcal{M}}(\mathbb{G})$; cf. [11, Theorem 10.3 p.163]. □

Corollary 1.3.5. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ and let $\mathcal{B}_0(\mathbb{G})$ be as in Lemma 1.3.1. Let $\sigma(\Phi)$ be the σ-algebra generated by Φ, then $\sigma(\Phi)$ is generated by $\Phi(B)$ where $B \in \mathcal{B}_0(\mathbb{G})$; that is*

$$\sigma(\Phi) = \sigma(\{\Phi(B) : B \in \mathcal{B}_0(\mathbb{G})\}).$$

Proof. This follows from Lemma 1.3.1. □

Lemma 1.3.6. *For any l.c.s.h. $\mathbb{G}$,*

(i) there exists a countable base of the topology on $\mathbb{G}$ consisting of open relatively compact sets;

(ii) there exists a countable class $\mathcal{B}_0(\mathbb{G})$ as in Lemma 1.3.1.

Proof. (i) Every point x in $\mathbb{G}$ has an open neighborhood in $\mathcal{B}_c(\mathbb{G})$, say U_x. Collect all $U_x \cap O$ for every $x \in \mathbb{G}$ and open set O containing x. This collection is a base consisting of $\mathcal{B}_c(\mathbb{G})$-sets. But since $\mathbb{G}$ is second-countable, such a base has a countable subfamily, say $\mathcal{D}$, which is still a base. (ii) Then $\mathcal{D}$ generates the Borel σ-algebra $\mathcal{B}(\mathbb{G})$. Let $\mathcal{B}_0$ be the ring (i.e., a family closed under unions and set differences) generated by $\mathcal{D}$. Then $\mathcal{B}_0$ is countable by [44, Theorem C p.23]. Since $\mathcal{B}_0$ is a ring, it is closed under finite intersections (since $A \cap B = A \backslash (A \backslash B)$). Let B_n be the union of the first n sets in $\mathcal{D}$, then $B_n \uparrow \mathbb{G}$. If we denote $A_0 = B_0, A_{n+1} = B_{n+1} \backslash B_n$, then $A_0, A_1, \ldots$ is a countable partition of $\mathbb{G}$ into $\mathcal{B}_c(\mathbb{G})$-sets. □

1.3.1 Powers and moment measures

Given a σ-finite measure μ on a measurable space and $n \in \mathbb{N}^*$, recall the notation μ^n of its n-th power in the sense of products of measures. This extends to random measures as follows.

Lemma 1.3.7. Random measure power. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$. Then, its n-th power Φ^n is itself a random measure for any $n \in \mathbb{N}^*$.*

Proof. It follows from (14.E.3) that, for any $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{G})$,

$$\Phi^n(B_1 \times \ldots \times B_n) = \Phi(B_1) \ldots \Phi(B_n)$$

is a random variable. Then for any finite disjoint union B of sets of the form $B_1 \times \ldots \times B_n$, $\Phi^n(B)$ is also a random variable. The class of such unions is closed under finite intersections, generates $\mathcal{B}(\mathbb{G})^{\otimes n}$ and contains $\mathbb{G}^n$. Then, Φ^n is a random measure by Corollary 1.3.2. □

Definition 1.3.8. Moment measures. *For a random measure Φ on a l.c.s.h. space $\mathbb{G}$, let Φ^n be the n-th power of Φ. We call M_{Φ^n} the* n-th moment measure *(the first moment measure is the mean measure) of Φ.*

Example 1.3.9. *For any random measure Φ on a l.c.s.h. space $\mathbb{G}$, and any $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{G})$,*

$$M_{\Phi^n}(B_1 \times \ldots \times B_n) = \mathbf{E}[\Phi^n(B_1 \times \ldots \times B_n)] = \mathbf{E}[\Phi(B_1)\ldots\Phi(B_n)].$$

where the second equality is due to (14.E.3). In particular, $M_{\Phi^n}(B^n) = \mathbf{E}[\Phi(B)^n]$, $B \in \mathcal{B}(\mathbb{G})$.

Example 1.3.10. *The Campbell averaging theorem 1.2.5 applied to the random measure Φ^n shows that for all measurable functions $f : \mathbb{G}^n \to \bar{\mathbb{R}}_+$,*

$$\mathbf{E}\left[\int_{\mathbb{G}^n} f \mathrm{d}\Phi^n\right] = \int_{\mathbb{G}^n} f \mathrm{d}M_{\Phi^n}.$$

1.3.2 Laplace transform characterization

Proposition 1.3.11. *The probability distribution of a random measure Φ on on a l.c.s.h. space $\mathbb{G}$ is characterized by its Laplace transform $\mathcal{L}_\Phi(f)$ for measurable functions $f : \mathbb{G} \to \mathbb{R}_+$ which are bounded with support in $\mathcal{B}_c(\mathbb{G})$.*

Proof. For any measurable $f : \mathbb{G} \to \bar{\mathbb{R}}_+$, consider an increasing sequence of measurable functions $f_n : \mathbb{G} \to \mathbb{R}_+$ which are bounded with support in $\mathcal{B}_c(\mathbb{G})$, converging pointwise to f. Invoking the monotone convergence theorem, we get $\mathcal{L}_\Phi(f_n) \to \mathcal{L}_\Phi(f)$ as $n \to \infty$. Corollary 1.2.2 allows one to conclude. □

Corollary 1.3.12. *Let $\varepsilon \in \mathbb{R}_+^*$. Let Φ be a random measure on on a l.c.s.h. space $\mathbb{G}$ such that: (i) its moment measures are locally finite and (ii) for any $D \in \mathcal{B}_c(\mathbb{G})$, the radius of convergence $R_{\mathcal{L}_{\Phi(D)}}$ (cf. Definition 13.B.3) is positive. Assume that $\tilde{\Phi}$ is a random measure on $\mathbb{G}$ such that $\mathcal{L}_\Phi(f) = \mathcal{L}_{\tilde{\Phi}}(f)$ for any measurable function $f : \mathbb{G} \to \mathbb{R}_+$ with support in $\mathcal{B}_c(\mathbb{G})$ such that $\|f\|_\infty < \varepsilon$. Then Φ and $\tilde{\Phi}$ are equal in distribution.*

Proof. Consider a bounded measurable function $g : \mathbb{G} \to \mathbb{R}_+$ with support in $\mathcal{B}_c(\mathbb{G})$. For any $t < \varepsilon / \|g\|_\infty$, $\mathcal{L}_\Phi(tg) = \mathcal{L}_{\tilde{\Phi}}(tg)$; that is

$$\mathbf{E}\left[e^{-t\Phi(g)}\right] = \mathbf{E}\left[e^{-t\tilde{\Phi}(g)}\right].$$

Applying Corollary 13.B.5 to the random variables $X = \Phi(g)$ and $Y = \tilde{\Phi}(g)$ shows that $\Phi(g) \stackrel{\text{dist.}}{=} \tilde{\Phi}(g)$. Then the above equality holds for any $t \in \mathbb{R}_+$; in particular for $t = 1$. Proposition 1.3.11 allows one to conclude. □

1.3.3 Independence

The independence of random measures is defined as usually through their distributions.

Definition 1.3.13. *Two random measures Φ_1 and Φ_2 on a l.c.s.h. space $\mathbb{G}$ are called* independent *if*

$$\mathbf{P}(\Phi_1 \in L_1, \Phi_2 \in L_2) = \mathbf{P}(\Phi_1 \in L_1)\mathbf{P}(\Phi_2 \in L_2), \quad L_1, L_2 \in \bar{\mathcal{M}}(\mathbb{G}).$$

More generally, a family $\{\Phi_j\}_{j\in I}$ of random measures on $\mathbb{G}$ indexed by an arbitrary set I is called independent *if for all finite $J \subset I$,*

$$\mathbf{P}\left(\bigcap\nolimits_{j\in J}\{\Phi_j \in L_j\}\right) = \prod\nolimits_{j\in J}\mathbf{P}(\Phi_j \in L_j).$$

The following result characterizes the independence of random measures by the independence of their values on measurable sets.

Corollary 1.3.14. *A family of random measures $\{\Phi_j\}_{j\in I}$ on a l.c.s.h. space $\mathbb{G}$ is independent iff the family of random variables $\{\Phi_j(B_j)\}_{j\in I}$ is independent for all $B_j \in \mathcal{B}_0(\mathbb{G})$ $(j \in I)$; where $\mathcal{B}_0(\mathbb{G})$ is as in Lemma 1.3.1.*

Proof. The direct sense is obvious. The converse follows from Corollary 1.3.5. □

We now characterize the independence of random measures through their Laplace transforms.

Proposition 1.3.15. *A family $\{\Phi_j\}_{j\in I}$ of random measures (where I is an arbitrary index set) on a l.c.s.h. space $\mathbb{G}$ is independent iff for all finite $J \subset I$, and all $f_j \in \mathfrak{F}_+(\mathbb{G})$ $(j \in J)$,*

$$\mathbf{E}\left[e^{-\sum_{j\in J}\Phi_j(f_j)}\right] = \prod\nolimits_{j\in J}\mathbf{E}\left[e^{-\Phi_j(f_j)}\right], \tag{1.3.1}$$

where $\Phi_j(f_j) = \int f_j \mathrm{d}\Phi_j$.

Proof. The direct sense is obvious. For the converse, let $B_j \in \mathcal{B}(\mathbb{G})$ $(j \in J)$. For any $\alpha_j \in \mathbb{R}_+$ $(j \in J)$, applying (1.3.1) to $f_j = \alpha_j \mathbf{1}_{B_j}$ $(j \in J)$ we get

$$\mathbf{E}\left[e^{-\sum_{j\in J}\alpha_j\Phi_j(B_j)}\right] = \prod\nolimits_{j\in J}\mathbf{E}\left[e^{-\alpha_j\Phi_j(B_j)}\right].$$

Then the random variables $\{\Phi_j(B_j)\}_{j\in J}$ are independent. Then $\{\Phi_j\}_{j\in J}$ are independent by Corollary 1.3.14 which completes the proof. □

1.4 Stochastic integral

Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ and let $\{Z(t)\}_{t\in\mathbb{G}}$ be a measurable stochastic process with values in $\mathbb{R}_+$; i.e., the mapping from $\mathbb{G}\times\Omega$ (endowed with the product σ-algebra $\mathcal{B}(\mathbb{G})\otimes\mathcal{A}$) into $\mathbb{R}_+$ defined by $(t,\omega) \to Z(t,\omega)$ is measurable. Then one can define the Lebesgue integrals

$$\tilde{\Phi}(\omega, B) = \int_B Z(t,\omega)\Phi(\omega, \mathrm{d}t), \quad \omega \in \Omega, B \in \mathcal{B}(\mathbb{G}) \tag{1.4.1}$$

called a *stochastic integral*. For each $\omega \in \Omega$, $\tilde{\Phi}(\omega, \cdot)$ is a clearly a measure.

Recall from Definition 14.D.1 that a *measure kernel* from Ω to $\mathbb{G}$ is a mapping Φ from $\Omega \times \mathcal{B}(\mathbb{G})$ to $\mathbb{R}_+$ such that: for all $B \in \mathcal{B}(\mathbb{G})$ the map $\omega \mapsto \Phi(\omega, B)$ is measurable, and for all $\omega \in \Omega$, $\Phi(\omega, \cdot)$ is a measure on $\mathbb{G}$. Then it follows from Proposition 1.1.7(iii) that a random measure is indeed a measure kernel from Ω to $\mathbb{G}$ such that for all $\omega \in \Omega$, $\Phi(\omega, \cdot)$ is locally finite. In particular, any probability kernel from $\Omega \times \mathcal{B}(\mathbb{G})$ to $\mathbb{R}_+$ is a Random measure.

Proposition 1.4.1. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$, let $\{Z(t)\}_{t \in \mathbb{G}}$ be a measurable stochastic process with values in $\mathbb{R}_+$, and assume that, for each $\omega \in \Omega$, the measure $\tilde{\Phi}(\omega, \cdot)$ defined by (1.4.1) is locally finite. Then $\tilde{\Phi}$ is a random measure.*

Proof. From Corollary 1.3.2, it is enough to show that for each $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, the mapping $\omega \mapsto \tilde{\Phi}(\omega, B)$ is measurable. Note first that, by definition of a random measure, $\Phi(\omega, B) < \infty$ for all $\omega \in \Omega$. Then by appropriate normalization, it is enough to show that $\omega \mapsto \tilde{\Phi}(\omega, B)$ is measurable when Φ is a probability kernel from Ω to B. Under this assumption, it follows from the measure mixture theorem 14.D.4, that there exists a unique probability measure λ on $\Omega \times B$ such that

$$\lambda(A \times C) = \int_A \Phi(\omega, C) \mathbf{P}(\mathrm{d}\omega), \quad A \in \mathcal{A}, C \in \mathcal{B}(\mathbb{G}), C \subset B$$

(λ is called a *mixture* of $\Phi(\omega, \mathrm{d}t)$ with respect to $\mathbf{P}(\mathrm{d}\omega)$). Moreover, this theorem shows that for all nonnegative measurable functions Z on $\mathbb{G} \times \Omega$, the mapping

$$\omega \mapsto \int_B Z(t, \omega) \Phi(\omega, \mathrm{d}t)$$

is measurable. $\square$

Proposition 1.4.2. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure and let $\{Z(t)\}_{t \in \mathbb{G}}$ be a measurable stochastic process with values in $\bar{\mathbb{R}}_+$* independent *of Φ. Then*

$$\mathbf{E}\left[\int_B Z(t, \omega) \Phi(\omega, \mathrm{d}t)\right] = \int_B \mathbf{E}[Z(t, \omega)] M_\Phi(\mathrm{d}t) \quad B \in \mathcal{B}(\mathbb{G}). \tag{1.4.2}$$

Proof. Let $Z := \{Z(t)\}_{t \in \mathbb{G}}$. For any $B \in \mathcal{B}(\mathbb{G})$, we have

$$\begin{aligned}
\mathbf{E}\left[\int_B Z(t, \omega) \Phi(\omega, \mathrm{d}t)\right] &= \mathbf{E}\left[\mathbf{E}\left[\int_B Z(t, \omega) \Phi(\omega, \mathrm{d}t) \middle| Z\right]\right] \\
&= \mathbf{E}\left[\int_B Z(t, \omega) M_\Phi(\mathrm{d}t)\right] \\
&= \int_B \mathbf{E}[Z(t, \omega)] M_\Phi(\mathrm{d}t),
\end{aligned}$$

where the second equality is due to the Campbell averaging formula (1.2.2) and the independence of Φ from Z; the last equality is due to Fubini-Tonelli theorem. $\square$

When Φ and Z are independent, the reader will find more properties of the stochastic integral (1.4.1) in [55]. We will see later in Section 3.1 how to extend Equation (1.4.2) to the case when Z and Φ are not necessarily independent using Palm theory.

1.5 Vague topology on $\bar{\mathbb{M}}(\mathbb{G})$

Lemma 1.5.1. *Let $\mathbb{G}$ be a l.c.s.h. space. The σ-algebra $\bar{\mathcal{M}}(\mathbb{G})$ is generated by the mappings $\mu \mapsto \mu(f)$, $f \in \mathfrak{F}_c(\mathbb{G})$.*

Proof. Cf. [52, Lemma 1.4]. Let $\bar{\mathcal{M}}'$ be the σ-algebra on $\bar{\mathbb{M}}(\mathbb{G})$ generated by the mappings $\mu \mapsto \mu(f)$, $f \in \mathfrak{F}_c(\mathbb{G})$. It follows from Lemma 1.1.5 that $\bar{\mathcal{M}}' \subset \bar{\mathcal{M}}(\mathbb{G})$. To prove the converse, let B be a compact in $\mathbb{G}$ and let $f_1, f_2, \ldots \in \mathfrak{F}_c(\mathbb{G})$ be such that $f_n \downarrow 1_B$ (cf. [52, §15.6.1]). Then $f_1 - f_n \uparrow f_1 - 1_B$ and by the monotone convergence theorem $\mu(f_1) - \mu(f_n) \uparrow \mu(f_1) - \mu(B)$. Since $\mu(f_1) < \infty$, we get $\mu(f_n) \downarrow \mu(B)$. Then the mapping $\mu \mapsto \mu(B)$ is $\bar{\mathcal{M}}'$-measurable for all compacts B. Since, by Lemma 1.3.1, $\bar{\mathcal{M}}(\mathbb{G})$ is generated by $\mu \mapsto \mu(B)$, B compact, it follows that $\bar{\mathcal{M}}(\mathbb{G}) \subset \bar{\mathcal{M}}'$. □

Definition 1.5.2. *Let $\mathbb{G}$ be a l.c.s.h. space and $\mu, \mu_1, \mu_2, \ldots$ be in $\bar{\mathbb{M}}(\mathbb{G})$. We say that the sequence μ_n* converges vaguely *to μ, and write $\mu_n \overset{v}{\to} \mu$, if, for $f \in \mathfrak{F}_c(\mathbb{G})$, $\mu_n(f) \to \mu(f)$. Let $\mathcal{T}$ be the* vague topology *on $\bar{\mathbb{M}}(\mathbb{G})$; that is the topology associated with the vague convergence.*

Proposition 1.5.3. *Let $\mathbb{G}$ be a l.c.s.h. space. The σ-algebra $\bar{\mathcal{M}}(\mathbb{G})$ coincides with the σ-algebra generated by the vague topology $\mathcal{T}$.*

Proof. Cf. [52, Lemma 4.1]. By definition of the vague topology, for all $f \in \mathfrak{F}_c(\mathbb{G})$, the mapping $\mu \mapsto \mu(f)$ is $\mathcal{T}$ continuous and hence $\sigma(\mathcal{T})$ measurable. By Lemma 1.5.1, $\bar{\mathcal{M}}(\mathbb{G}) \subset \sigma(\mathcal{T})$. To prove the converse, note first that all finite intersections of sets of the form $\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : s < \mu(f) < t\right\}$ where $f \in \mathfrak{F}_c(\mathbb{G}), s, t \in \mathbb{R}$ is a base of $\mathcal{T}$ (cf. [52, §15.7]). Moreover, since $\left(\bar{\mathbb{M}}(\mathbb{G}), \bar{\mathcal{M}}(\mathbb{G})\right)$ is Polish in the vague topology (cf. [52, §15.7.7]) and since every Polish space is second countable, any set in $\mathcal{T}$ may be formed by countable set operations from sets of the form $\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : s < \mu(f) < t\right\}$ with $f \in \mathfrak{F}_c(\mathbb{G}), s, t \in \mathbb{R}$. Then $\mathcal{T} \subset \bar{\mathcal{M}}(\mathbb{G})$, hence $\sigma(\mathcal{T}) \subset \bar{\mathcal{M}}(\mathbb{G})$. □

1.6 Point processes

A *counting measure* is a locally finite measure μ on $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$ such that $\mu(B) \in \mathbb{N}$ for all $B \in \mathcal{B}_c(\mathbb{G})$. Let $\mathbb{M}(\mathbb{G})$ be the set of counting measures on $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$. Let $\mathcal{M}(\mathbb{G})$ be the σ-algebra on $\mathbb{M}(\mathbb{G})$ generated by the mappings $\mu \mapsto \mu(B), B \in \mathcal{B}(\mathbb{G})$, i.e., the smallest σ-algebra making these mappings measurable.

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space.

Definition 1.6.1. *Let $\mathbb{G}$ be a l.c.s.h. space. A* point process *is a measurable mapping $\Phi : \Omega \to \mathbb{M}(\mathbb{G})$ (Ω and $\mathbb{M}(\mathbb{G})$ being equipped with the σ-algebras $\mathcal{A}$ and $\mathcal{M}(\mathbb{G})$, respectively). The probability distribution of Φ is $\mathbf{P}_\Phi = \mathbf{P} \circ \Phi^{-1}$.*

1.6.1 Simple point processes

Definition 1.6.2. *Let $\mathbb{G}$ be a l.c.s.h. space. A counting measure $\mu \in \mathbb{M}(\mathbb{G})$ is said to be simple if*

$$\mu(\{x\}) \leq 1, \quad \forall x \in \mathbb{G}.$$

A point process Φ is called simple *if $\Phi(\omega)$ is simple $\mathbf{P}$-almost surely; that is*

$$\mathbf{P}(\forall x \in \mathbb{G}, \Phi(\{x\}) \leq 1) = 1. \tag{1.6.1}$$

In order to prove that the event considered in the above definition is measurable, we need the following technical lemma which will be used later many times.

Lemma 1.6.3. *[52, p.19] Let $\mathbb{G}$ be a l.c.s.h. space. There exists a* sequence of nested partitions *$\mathcal{K}_n = \{K_{n,j}\}_{j\in\mathbb{N}}$ of $\mathbb{G}$, i.e., for each $n \in \mathbb{N}$, $\mathcal{K}_n$ is a partition of $\mathbb{G}$, and $\mathcal{K}_n$ is nested in $\mathcal{K}_{n-1}$, such that $K_{n,j} \in \mathcal{B}_c(\mathbb{G})$ and*

$$\lim_{n\to\infty} \left(\sup_{j\in\mathbb{N}} |K_{n,j}| \right) = 0, \tag{1.6.2}$$

where $|\cdot|$ denotes the diameter in any fixed metric. Moreover, for all $B \in \mathcal{B}_c(\mathbb{G})$, $n \in \mathbb{N}$, only finitely many sets in $\mathcal{K}_n$ intersect B. Denoting by k_n the number of these sets, we say that $\{K_{n,j} \cap B\}_{j=1}^{k_n}$ is a sequence of nested partitions *of B.*

Proof. Let $B(x, r)$ denotes the open ball of center $x \in \mathbb{G}$ and radius r w.r.t. a metric d on $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$ making it separable and complete. Let $U_0, U_1, \ldots$ be locally compact sets constituting a countable base of the topology of $\mathbb{G}$. For all $i = 0, 1, \ldots$, let a_i be a point of U_i. For all nonnegative integers i and j, let

$$V_{i,j}^0 = U_i \cap B(a_j, 1).$$

We now show that this countable collection of open sets covers $\mathbb{G}$. Since

$$\bigcup_{i,j} V_{i,j}^0 = \bigcup_j \left(B(a_j, 1) \bigcap \left(\bigcup_i U_i \right) \right) = \bigcup_j B(a_j, 1),$$

it is enough to prove that the balls $B(a_j, 1)$, $j = 0, 1, \ldots$ cover $\mathbb{G}$. For all x and all neighborhoods V_x of x, it follows from the fact that $\{U_i\}$ is a basis that the set $V_x \cap B(x, 1)$ contains one of the sets $\{U_i\}$, say U_{i^*}. Hence $x \in B(a_{i^*}, 1)$, which completes the proof of coverage. Let $\mathcal{W}_0 = \{W_{0,k}\}_{k\in\mathbb{N}}$ be some enumeration of these sets. Then define the partition $\mathcal{K}_0 = \{K_{0,k}\}_{k\in\mathbb{N}}$ of order 0 by

$$K_{0,0} = W_{0,0}, \quad K_{0,k} = W_{0,k} \setminus \bigcup_{i=1}^{k-1} W_{0,l}, \quad k = 1, 2, \ldots$$

The construction of the partitions of order $n > 0$ proceeds by induction on n. Once $\mathcal{K}_{n-1}$ is constructed, the next partition $\mathcal{K}_n$ is constructed as follows. Let

$$V_{i,j}^n = U_i \cap B\left(a_j, 2^{-n}\right), \quad i, j \in \mathbb{N}$$

and let $\mathcal{W}_n = \{W_{n,k}\}_{k\in\mathbb{N}}$ be some enumeration of these sets. Then define a partition $\mathcal{C}_n = \{C_{n,k}\}_{k\in\mathbb{N}}$ by

$$C_{n,0} = W_{n,0}, \quad C_{n,k} = W_{n,k} \backslash \bigcup_{i=1}^{k-1} W_{n,l}, \quad k = 1, 2, \ldots$$

and let

$$\mathcal{K}_n = \{C_{n,k} \cap K_{n-1,j} : j, k \in \mathbb{N}\}.$$

Clearly $\mathcal{K}_n$ is a partition nested in $\mathcal{K}_{n-1}$ and $\sup_{j\in\mathbb{N}} |K_{n,j}| \leq 2^{-n} \to 0$ as $n \to \infty$. Let $B \in \mathcal{B}_c(\mathbb{G})$. We show now by induction that for all $n \in \mathbb{N}$ only finitely many sets in $\mathcal{K}_n$ intersect B. **Initialization**. Since B is relatively compact, it may be covered by a finite number of $W_{0,j}$ and thus by finitely many $K_{0,j}$. **Heredity**. Assume that the announced property holds for $n-1$. B may be covered by a finite number of $W_{n,k}$ and thus by finitely many $C_{n,j}$. Since, by induction assumption, B is covered by finitely many $K_{n-1,j}$, then B is covered by finitely many $C_{n,i} \cap K_{n-1,j}$. □

Arguing as in [52, Lemma 1.1], it may be shown that the condition (1.6.2) is independent of the choice of the metric. Moreover, this condition ensures that $\mathcal{K}_n$ separates eventually the points of $\mathbb{G}$; i.e., for all $x \neq y \in \mathbb{G}$, there exists n such that x and y belong to different sets in $\mathcal{K}_n$.

We will show now that a point process is indeed a random measure.

Corollary 1.6.4. Point process versus random measure. *Let $\mathbb{M}(\mathbb{G})$ the set of counting measures on a l.c.s.h. space $\mathbb{G}$ and $\bar{\mathbb{M}}(\mathbb{G})$ be the set of locally finite measures on $\mathbb{G}$. Let $\mathcal{M}(\mathbb{G})$ and $\bar{\mathcal{M}}(\mathbb{G})$ be the respective σ-algebras generated by the mappings $\mu \mapsto \mu(B), B \in \mathcal{B}(\mathbb{G})$. Then:*

(i)
$$\mathcal{M}(\mathbb{G}) \subset \bar{\mathcal{M}}(\mathbb{G}).$$

(ii) A point process is a random measure. Conversely, a random measure with values in $\mathbb{M}(\mathbb{G})$ is a point process.

Proof. (i) Observe first that

$$\mathcal{E} := \left\{\mathbb{M}(\mathbb{G}) \cap B : B \in \bar{\mathcal{M}}(\mathbb{G})\right\}$$

is a σ-algebra of subsets of $\mathbb{M}(\mathbb{G})$. Moreover, for all $B \in \mathcal{B}(\mathbb{G})$ and all $a \in \mathbb{R}_+$,

$$\{\mu \in \mathbb{M}(\mathbb{G}) : \mu(B) \leq a\} = \mathbb{M}(\mathbb{G}) \cap \left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B) \leq a\right\}$$

belongs to $\mathcal{E}$. Then $\mathcal{M}(\mathbb{G}) \subset \mathcal{E}$. It is therefore enough to prove that $\mathbb{M}(\mathbb{G}) \in \bar{\mathcal{M}}(\mathbb{G})$ to get the announced inclusion. Let $\mathcal{K}_n = \{K_{n,j}\}_{j\in\mathbb{N}}$ be a sequence of nested partitions of $\mathbb{G}$ as in Lemma 1.6.3. Let $\mathcal{K} = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n = \{K_{n,j}\}_{n,j\in\mathbb{N}}$ and define

$$N = \{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B) \in \mathbb{N} \text{ for all } B \in \mathcal{K}\}.$$

Clearly $\mathbb{M}(\mathbb{G}) \subset N$. We prove that $N \subset \mathbb{M}(\mathbb{G})$ by showing that for any $\mu \in N$ and K compact subset of $\mathbb{G}$, $\mu(K) \in \mathbb{N}$. Since any $\mu \in \bar{\mathbb{M}}(\mathbb{G})$ is inner regular by [26, Proposition 7.2.3], it will follow that $\mu \in \mathbb{M}(\mathbb{G})$. Let $\mu \in N$ and K be a compact subset of $\mathbb{G}$. By Lemma 1.6.3, for any $n \in \mathbb{N}$, the collection $\mathcal{C}_n = \{B \in \mathcal{K}_n : B \cap K \neq \emptyset\}$ is finite. For each $n \in \mathbb{N}$, let C_n be the union of all sets in $\mathcal{C}_n$. Then $\{C_n : n \in \mathbb{N}\}$ is a decreasing sequence of relatively compact sets, each of which contains K, and $\mu(C_n) \in \mathbb{N}$. We claim that $K = \bigcap_{n\in\mathbb{N}} C_n$. Let d be a metric in $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$ making it separable and complete. Suppose $x \in \bigcap_{n\in\mathbb{N}} C_n$. For each $n \in \mathbb{N}$ there is $K_{n,j_n} \in \mathcal{C}_n$ such that $x \in K_{n,j_n}$. Choose $x_n \in K_{n,j_n} \cap K$. Observe that

$$d(x, x_n) \leq |K_{n,j_n}| \leq \sup_{j\in\mathbb{N}} |K_{n,j}| \underset{n\to\infty}{\to} 0,$$

where $|\cdot|$ denotes the diameter with respect to the metric d. It follows that $x \in K$. Therefore $\mu(K) = \lim_{n\to\infty} \mu(C_n) \in \mathbb{N}$. (ii) This follows from (i) and the obvious observation that $\mathbb{M}(\mathbb{G}) \subset \bar{\mathbb{M}}(\mathbb{G})$. □

Corollary 1.6.5. *Let $\mathbb{G}$ be a l.c.s.h. space. For any $a \in \mathbb{R}_+$, we have*

$$\{\mu \in \mathbb{M}(\mathbb{G}) : \exists x \in \mathbb{G}, \mu(\{x\}) \geq a\} \in \mathcal{M}(\mathbb{G}).$$

The above result holds true when $\mathbb{M}(\mathbb{G})$ and $\mathcal{M}(\mathbb{G})$ are replaced by $\bar{\mathbb{M}}(\mathbb{G})$ and $\bar{\mathcal{M}}(\mathbb{G})$ respectively.

Proof. By Lemma 1.1.4, $\mathbb{G}$ may be covered by a countable union of compact sets. Then it is enough to prove that for all compacts $B \in \mathcal{B}(\mathbb{G})$, the set

$$A := \{\mu \in \mathbb{M}(\mathbb{G}) : \exists x \in B, \mu(\{x\}) \geq a\}$$

is measurable. Let $\{B_{n,j}\}_{j=1}^{k_n}$ be a sequence of nested partitions of B as in Lemma 1.6.3. Let

$$C := \bigcap_{n\in\mathbb{N}} \bigcup_{j} \{\mu \in \mathbb{M}(\mathbb{G}) : \mu(B_{n,j}) \geq a\}.$$

It is clear that $A \subset C$. Now let $\mu \in C$. For any $n \in \mathbb{N}$, there exists some $j(n)$ such that $\mu\left(B_{n,j(n)}\right) \geq a$. Choosing arbitrary $x_n \in B_{n,j(n)}, n \in \mathbb{N}$, it is seen from the compactness of B that there exists some subsequence x_{σ_n} converging to some $x \in B$. Every open set G containing x contains eventually the $B_{\sigma_n, j(\sigma_n)}$ and therefore $\mu(G) \geq a$. This being true for all open sets G containing x, it follows that $\mu(\{x\}) \geq a$. Then $\mu \in A$, and therefore $C \subset A$. Therefore $A = C$ and the announced measurability follows. □

We are now ready to prove that the event considered in Equation (1.6.1) is measurable; indeed

$$\{\mu \in \mathbb{M}(\mathbb{G}) : \forall x \in \mathbb{G}, \mu(\{x\}) \leq 1\} = \mathbb{M}(\mathbb{G}) \setminus \{\mu \in \mathbb{M}(\mathbb{G}) : \exists x \in \mathbb{G}, \mu(\{x\}) \geq 2\},$$

which is measurable by Corollary 1.6.5.

Note that, in general, the simplicity of a point process Φ is not equivalent to the statement that $\forall x \in \mathbb{G}, \mathbf{P}(\Phi(\{x\}) \leq 1) = 1$ as shown in the following counterexample.

Example 1.6.6. Counterexample. *Let X be a real random variable having a probability density function and let $\Phi = 2\delta_X$. Clearly Φ is not simple whereas for all $x \in \mathbb{R}$, $\mathbf{P}(\Phi(\{x\}) \leq 1) = \mathbf{P}(X \neq x) = 1$.*

1.6.2 Enumeration of points

We first show that any counting measure can be decomposed into a weighted sum of Dirac measures over its atoms and that these atoms may be enumerated in a measurable way. To do so, we need some preliminary lemmas.

The set A of atoms of the locally finite measure μ on $\mathbb{G}$ is

$$A = \{x \in \mathbb{G} : \mu(\{x\}) > 0\}.$$

The restriction μ_a of μ to A is called the *atomic component* of μ. The measure μ can be decomposed as

$$\mu = \mu_a + \mu_d,$$

with μ_d the *diffuse component* of μ.

Lemma 1.6.7. *The diffuse component of a counting measure on a l.c.s.h. space $\mathbb{G}$ is the null measure.*

Proof. It is enough to show that a diffuse counting measure μ on $\mathbb{G}$ is the null measure. For any $x \in \mathbb{G}$, there exists a neighborhood $U_x \in \mathcal{B}(\mathbb{G})$ such that $\mu(U_x) = 0$. (Indeed, assume for the sake of a contradiction that for any neighborhood $U \in \mathcal{B}(\mathbb{G})$ of x, $\mu(U) \geq 1$. Consider a sequence of decreasing relatively compact neighborhoods $\{U_n\}_{n\in\mathbb{N}}$ of x such that $\bigcap_{n\in\mathbb{N}} U_n = \{x\}$. It follows from the continuity from above of measures [44, Theorem E p.38] that $\mu(\{x\}) = \lim_{n\to\infty} \mu(U_n) \geq 1$, which contradicts the assumption that μ is diffuse.) Let K be a compact of $\mathbb{G}$. From the covering $K \subset \bigcup_{x\in K} U_x$, we may extract a finite covering; say $K \subset \bigcup_{i=1}^n U_{x_i}$. Thus $\mu(K) \leq \sum_{i=1}^n \mu(U_{x_i}) = 0$. Taking a sequence of compacts $\{K_n\}_{n\in\mathbb{N}^*}$ increasing to $\mathbb{G}$ shows by continuity from below [44, Theorem D p.38] that

$$\mu(\mathbb{G}) = \mu\left(\lim_{n\to\infty} K_n\right) = \lim_{n\to\infty} \mu(K_n) = 0.$$

□

Lemma 1.6.8. *[52, Lemma 2.1] Let $\mathbb{G}$ be a l.c.s.h. space. Any measure $\mu \in \mathbb{M}(\mathbb{G})$ can be written*

$$\mu = \sum_{j=1}^{J} b_j \delta_{x_j}, \tag{1.6.3}$$

where $J \in \bar{\mathbb{N}}$, $(x_j)_{j=1,\ldots,J}$ is a sequence of points of $\mathbb{G}$ without accumulation point and $(b_j)_{j=1,\ldots,J}$ are positive integers. Moreover this decomposition is unique up to a permutation of the terms.

Proof. By the last lemma, μ has no diffuse component. By Lemma 1.1.4, $\mathbb{G}$ admits a countable partition into relatively compact sets, say $B_0, B_1, \ldots$. Since μ is locally finite, $\mu(B_i) < \infty$ for each $i \in \mathbb{N}$. Since μ is a counting measure, for each, $x \in B_i$, $\mu(\{x\}) \in \mathbb{N}$. The number of $x \in B_i$ such that $\mu(\{x\}) \geq 1$ is finite, say J_i. Let $x_{i,1}, \ldots, x_{i,J_i}$ be such points and $b_{i,l} = \mu(\{x_{i,l}\})$ for $l = 1, \ldots, J_i$. Then

$$\mu = \sum_{i \in \mathbb{N}} \sum_{l=1}^{J_i} b_{i,l} \delta_{x_{i,l}}.$$

which may be written in the form (1.6.3). Moreover, the above construction shows the uniqueness of the decomposition up to a permutation of the terms. Since μ is locally finite, then the sequence $(x_j)_{j=1,\ldots,J}$ has no accumulation point. □

Notation 1.6.9. *Sometimes we will write $x \in \mu$ to say that x is an* atom *of μ, i.e., $\mu(\{x\}) \geq 1$.*

For $\mu \in \mathbb{M}(\mathbb{G})$ and each $k \in \mathbb{N}^*$, we introduce a measure μ_k^* counting the atoms of μ with mass equal to k; that is

$$\mu_k^*(B) = \sum_{x \in B \cap \mu} \mathbf{1}\{\mu(\{x\}) = k\}, \quad B \in \mathcal{B}(\mathbb{G}). \tag{1.6.4}$$

The measure

$$\mu^*(B) = \sum_{x \in B \cap \mu} \mathbf{1}\{\mu(\{x\}) \geq 1\}, \quad B \in \mathcal{B}(\mathbb{G}) \tag{1.6.5}$$

is called the *support measure* associated to μ; that is μ^* is a simple measure having the same atoms as μ.

Lemma 1.6.10. *Let $\mathbb{G}$ be a l.c.s.h. space. The mappings $\mu \mapsto \mu^*$ given by (1.6.5) and $\mu \mapsto \mu_k^*$ given by (1.6.4), $k \in \mathbb{N}^*$ are measurable on $\mathbb{M}(\mathbb{G})$.*

Proof. Let $B \in \mathcal{B}_c(\mathbb{G})$ and $m \in \mathbb{N}$. We aim to prove that

$$\{\mu \in \mathbb{M}(\mathbb{G}) : \mu_k^*(B) = m\} \in \mathcal{M}(\mathbb{G}).$$

To do so, let $\{K_{n,j}\cap B\}_{j=1}^{k_n}$ be a sequence of nested partitions of B as in Lemma 1.6.3. Consider n sufficiently large so that $\{K_{n,j}\cap B\}_{j=1}^{k_n}$ separate the atoms of μ in B. Thus

$$\begin{aligned}\mu_k^*(B) &= \sum_{x\in B\cap\mu}\mathbf{1}\{\mu(\{x\})=k\}\\ &=\sum_{j=1}^{k_n}\sum_{x\in K_{n,j}\cap B\cap\mu}\mathbf{1}\{\mu(\{x\})=k\}=\sum_{j=1}^{k_n}\mathbf{1}\{\mu(K_{n,j}\cap B)=k\}.\end{aligned}$$

Thus $\{\mu\in\mathbb{M}(\mathbb{G}):\mu_k^*(B)=m\}$ is the union over all $m_1,\ldots,m_{k_n}\in\{0,1\}$ such that $m_1+\ldots+m_{k_n}=m$ of

$$\bigcap_{j=1}^{k_n}\{\mu\in\mathbb{M}(\mathbb{G}):\mathbf{1}\{\mu(K_{n,j}\cap B)=k\}=m_j\},$$

which is in $\mathcal{M}(\mathbb{G})$. The measurability of $\mu\mapsto\mu^*$ follows from the fact that the support measure is the sum of the μ_k^*, for $k\geq 1$. □

We are now ready to prove the existence of a measurable enumeration of the points of counting measures.

Proposition 1.6.11. Measurable enumeration of atoms *[52, Lemma 2.3]. Let $\mathbb{G}$ be a l.c.s.h. space. The decomposition (1.6.3) may be chosen in such a way that the mapping $\mu\mapsto J(\mu)$ defined on $\mathbb{M}(\mathbb{G})$ is measurable and for every $j\in\mathbb{N}^*$, the mappings $\mu\mapsto x_j(\mu)$ and $\mu\mapsto b_j(\mu)$ defined on $\{\mu\in\mathbb{M}(\mathbb{G}):J(\mu)\geq j\}$ are measurable.*

Proof. **Simple case**. We will firstly prove the announced result for the set $\mathbb{M}_{\mathrm{s}}(\mathbb{G})$ of simple counting measures on $\mathbb{G}$. Observe that $\mathbb{M}_{\mathrm{s}}(\mathbb{G})\in\mathcal{M}(\mathbb{G})$ by Corollary 1.6.5. The mapping $\mu\mapsto J(\mu)=\mu(\mathbb{G})$ defined on $\mathbb{M}_{\mathrm{s}}(\mathbb{G})$ is clearly $\mathcal{M}(\mathbb{G})$-measurable. Let $\mathcal{K}_n=\{K_{n,j}\}_{j\in\mathbb{N}}$ ($n\in\mathbb{N}$) be a sequence of nested partitions of $\mathbb{G}$ as in Lemma 1.6.3; in particular $K_{n,j}\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ for all $n,j\in\mathbb{N}$. This lemma ensures that, for all $B\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$, $n\in\mathbb{N}$, only finitely many sets in $\mathcal{K}_n$ intersect B. In particular, for any $n\in\mathbb{N}^*$, there are only finitely many $K_{n,j}$ in each $K_{n-1,i}$. Then we may choose the indexes j of $K_{n,j}$ in such a way that we begin by indexing the subsets of $K_{n-1,0}$, then those of $K_{n-1,1}$, then those of $K_{n-1,2}$, etc. Recall that condition (1.6.2) ensures that $\mathcal{K}_n$ separates eventually the points of $\mathbb{G}$; i.e., for all $x\neq y\in\mathbb{G}$, there exists n such that x and y belong to different sets in $\mathcal{K}_n$, say $K_{n,i}$ and $K_{n,j}$ respectively. We will say that $x\prec y$ whenever $i<j$. This order is well defined due to our convention on the choice of the indexes j of $K_{n,j}$. We will show that we may enumerate the atoms of any $\mu\in\mathbb{M}_{\mathrm{s}}(\mathbb{G})$ according to the order $\prec$ defined above. First, we classify the atoms of μ according to their occurrence in $K_{0,0},K_{0,1},\ldots$. For the atoms in $K_{0,j}$ (there are a finite number of them) order them according to the order $\prec$ defined above. It remains to show that for every $k\in\mathbb{N}^*$, the mapping $\mu\mapsto x_k(\mu)$ defined on $\{\mu\in\mathbb{M}_{\mathrm{s}}(\mathbb{G}):J(\mu)\geq k\}$ is measurable. It is seen by induction that

it is enough to consider $k = 1$. Thus we have to show that, for each $B \in \mathcal{B}(\mathbb{G})$, the set $\{\mu \in \mathbb{M}_s(\mathbb{G}) : J(\mu) \geq 1, x_1(\mu) \in B\}$ is measurable. Observe that this set is equal to

$$\{\mu \in \mathbb{M}_s(\mathbb{G}) : \mu(\mathbb{G}) = \mu(B) = 1\} \cup \{\mu \in \mathbb{M}_s(\mathbb{G}) : J(\mu) \geq 2, x_1(\mu) \in B\},$$

where the first set is clearly $\mathcal{M}(\mathbb{G})$-measurable. When $J(\mu) \geq 2$, assume that $x = x_1(\mu) \in B$ and let $y = x_2(\mu)$. Since $x \prec y$, there exists n such that x and y belong to different sets in $\mathcal{K}_n$. Let j be such that $x \in K_{n,j}$. Then

$$\mu(K_{n,i}) = 0, \quad i = 0, \ldots, j-1$$

and

$$\mu(K_{n,j}) = \mu(K_{n,j} \cap B) = 1.$$

Conversely, assume that for some $\mu \in \mathbb{M}_s(\mathbb{G})$ there exist some n, j such that the above two conditions hold true. Let x be the atom of μ in $K_{n,j} \cap B$. Any other (eventual) atom y of μ is in $K_{n,l}$ for some $l > j$. Then $x \prec y$ and therefore $x_1(\mu) = x$ which is in B. Thus $\{\mu \in \mathbb{M}_s(\mathbb{G}) : J(\mu) \geq 1, x_1(\mu) \in B\}$ is equal to

$$\bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left[\left(\bigcap_{i=0}^{j-1} \{\mu : \mu(K_{n,i}) = 0\} \right) \cap \{\mu : \mu(K_{n,j}) = \mu(K_{n,j} \cap B) = 1\} \right]$$

which is $\mathcal{M}(\mathbb{G})$-measurable. **General case**. We now prove the announced result for the whole set $\mathbb{M}(\mathbb{G})$ of counting measures on $\mathbb{G}$. Recall that for $\mu \in \mathbb{M}(\mathbb{G})$ and each $k \in \mathbb{N}^*$, we denote by μ_k^* the measure (1.6.4) counting the atoms of μ with mass equal to k. The set function $B \times \{k\} \mapsto \mu_k^*(B)$ defines a measure on $\mathbb{G} \times \mathbb{N}^*$ which we denote by $\widetilde{\mu}$. Clearly, $\widetilde{\mu} \in \mathbb{M}_s(\mathbb{G} \times \mathbb{N}^*)$. Then by the first part of the proof,

$$\widetilde{\mu} = \sum_{j=1}^{J} \delta_{(x_j, b_j)},$$

where the mapping $\widetilde{\mu} \mapsto J(\widetilde{\mu})$, defined on $\mathbb{M}_s(\mathbb{G} \times \mathbb{N}^*)$, is measurable, and for every $j \in \mathbb{N}^*$, the mappings $\widetilde{\mu} \mapsto x_j(\widetilde{\mu})$ and $\widetilde{\mu} \mapsto b_j(\widetilde{\mu})$, defined on $\{\widetilde{\mu} \in \mathbb{M}_s(\mathbb{G} \times \mathbb{N}^*) : J(\widetilde{\mu}) \geq j\}$, are measurable. One concludes the proof when observing that, by Lemma 1.6.10, the mapping $\mu \mapsto \widetilde{\mu}$, defined on $\mathbb{M}(\mathbb{G})$, is $\mathcal{M}(\mathbb{G})$-measurable. □

Corollary 1.6.12. *Let $\mathbb{G}$ be a l.c.s.h. space. Any measure $\mu \in \mathbb{M}(\mathbb{G})$ can be written*

$$\mu = \sum_{j=1}^{\mu(\mathbb{G})} \delta_{y_j}, \tag{1.6.6}$$

where $(y_j)_{j=1,\ldots,\mu(\mathbb{G})}$ is a sequence of points of $\mathbb{G}$ without accumulation point and the mappings $\mu \mapsto y_j(\mu)$ defined on $\{\mu \in \mathbb{M}(\mathbb{G}) : \mu(\mathbb{G}) \geq j\}$ are measurable.

Proof. The decomposition (1.6.3) may be writen in the form

$$\mu = \sum_{j=1}^{\mu(\mathbb{G})} \delta_{x_{\sigma(j)}},$$

where $\sigma(1) = \ldots = \sigma(b_1) = 1$, $\sigma(b_1+1) = \ldots = \sigma(b_1+b_2) = 2$, etc. It remains to show that the above enumeration preserves measurablity. To this end, observe that $x_j(\mu) = x_{\sigma(j)}(\mu^*)$ where μ^* is the support measure given by (1.6.5). Invoking Lemma 1.6.10 shows that $\mu \mapsto \mu^*$ is measurable. Finally, Proposition 1.6.11 implies that $\mu^* \mapsto x_j(\mu^*)$ is measurable, which concludes the proof. □

Notation 1.6.13. Enumeration of atoms. *In view of Corollary 1.6.12, we can write*

$$\mu = \sum_{j\in\mathbb{Z}:y_j\in\mu} \delta_{y_j}, \tag{1.6.7}$$

assuming implicitly that:

- *the mappings $\mu \mapsto y_j(\mu)$ are measurable;*
- *the index j ranges over a (possibly proper) subset of $\mathbb{Z}$;*
- *denoting the atoms of μ by y_j is "local" in the sense that we can also write $\nu = \sum_{j\in\mathbb{Z}:y_j\in\nu} \delta_{y_j}$ for another measure ν.*

In particular, with the above convention, we will often write for a measurable function f on $\mathbb{M}(\mathbb{G})$,

$$\int_{\mathbb{G}} f \mathrm{d}\mu = \sum_{j\in\mathbb{Z}:y_j\in\mu} f(y_j),$$

or even

$$\int_{\mathbb{G}} f \mathrm{d}\mu = \sum_{j\in\mathbb{Z}} f(y_j)$$

when the association $\mu = \sum_{j\in\mathbb{Z}} \delta_{y_j}$ is clear from the context.

The enumeration of points of counting measures described in the proof of Proposition 1.6.11 can be replaced by more explicit ones for particular cases of $\mathbb{G}$ as shown in the following examples.

Example 1.6.14. Enumeration of points in $\mathbb{R}$. *On $\mathbb{R}$, it is usual to enumerate the points of counting measures in the increasing order; that is*

$$\ldots x_{-1} \leq x_0 \leq 0 < x_1 \leq x_2 \ldots \tag{1.6.8}$$

The above enumeration is measurable. Indeed, for any $\mu \in \mathbb{M}(\mathbb{G}), k \in \mathbb{N}, t \in \mathbb{R}$,

$$x_k(\mu) \leq t \Leftrightarrow \begin{cases} \mu((0,t]) \geq k, & \text{for } k \geq 1, t > 0, \\ \mu((t,0]) \leq |k|, & \text{for } k \leq 0, t < 0. \end{cases}$$

Example 1.6.15. Enumeration of points in $\mathbb{R}^d$. *On $\mathbb{R}^d$ with $d \geq 2$, there is no natural way of enumerating the points of counting measures similar to that described in Example 1.6.14 on $\mathbb{R}$. Here are some examples of enumerations in $\mathbb{R}^d$:*

(i) Lexicographic order of polar coordinates. That is, we enumerate the points in the increasing order of their distances to the origin and in case of equality, we use the lexicographic order of the angular coordinates.

(ii) Enumerate the points in the order that a growing d-cube hit them and break the ties with the lexicographic order of Cartesian coordinates.

Example 1.6.16. Counterexample. *The* lexicographic order of the Cartesian coordinates *may not be used to enumerate the points of counting measures in $\mathbb{R}^d$. This is due to the fact that for $d = 1$, there may be points going to $-\infty$. For $d \geq 2$, even when we restrict ourselves to $\mathbb{R}_+^d$, the projection of the points of a counting measure μ on the first coordinate axis may have an accumulation point at 0 and therefore $x_0(\mu)$ is not well defined.*

Remark 1.6.17. *It will be often implicitly assumed that a particular way of point enumeration (1.6.7) is chosen. In general, the only required property of such enumeration of points on any space $\mathbb{G}$ is its measurability; this means that no particular enumeration is privileged. If so, it will be explicitly stated.*

1.6.3 Generating function

Note that the Laplace transform of a point process may be extended to measurable functions $f : \mathbb{G} \to \bar{\mathbb{R}}_+$ (i.e., $\mathbb{R}_+ \cup \{+\infty\}$) by the same formula as for nonnegative measurable functions; that is

$$\mathcal{L}_\Phi(f) = \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right],$$

where as usual the integral in the above expression vanishes when $\Phi(\omega)$ is the null measure whatever is the function f.

Definition 1.6.18. *Let $\mathbb{G}$ be a l.c.s.h. space and $\mathcal{V}(\mathbb{G})$ be the set of measurable functions $v : \mathbb{G} \to [0,1]$. The* generating function *of a point process Φ, denoted by $\mathcal{G}_\Phi$, is defined by*

$$\mathcal{G}_\Phi(v) = \mathcal{L}_\Phi(-\log v) = \mathbf{E}\left[\exp\left(\int_{\mathbb{G}} \log[v(x)]\,\Phi(\mathrm{d}x)\right)\right], \quad v \in \mathcal{V}(\mathbb{G}). \quad (1.6.9)$$

Example 1.6.19. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$. Taking $v \equiv 1$ in (1.6.9), we get $\mathcal{G}_\Phi(1) = 1$. Taking $v \equiv 0$, we get $\mathcal{G}_\Phi(0) = \mathbf{P}(\Phi = 0)$. More generally, taking $v = \mathbf{1}_B$ for some $B \in \mathcal{B}(\mathbb{G})$, we get $\mathcal{G}_\Phi(\mathbf{1}_B) = \mathbf{P}(\Phi(B^{\mathrm{c}}) = 0)$. In particular, the void probability equals*

$$\mathbf{P}(\Phi(B) = 0) = \mathcal{G}_\Phi(\mathbf{1}_{B^{\mathrm{c}}}) = \mathcal{L}_\Phi(f),$$

where

$$f(x) = \begin{cases} \infty, & x \in B, \\ 0, & x \in B^{\mathrm{c}}. \end{cases}$$

Note that

$$\mathcal{G}_{\Phi}(v) = \mathbf{E}\left[\prod_{X \in \Phi} v(X)\right], \quad v \in \mathcal{V}(\mathbb{G}),$$

with the convention that the product in the right-hand side of the above equality is equal to 1 when $\Phi(\omega)$ is the null measure.

Remark 1.6.20. *Let $\mathbb{G}$ be a l.c.s.h. space. If $v \in \mathcal{V}(\mathbb{G})$ is such that the support of $1 - v$ is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$, then $B = \{x \in \mathbb{G} : v(x) \neq 1\} \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$ and therefore the product*

$$\prod_{X \in \Phi} v(X) = \prod_{X \in \Phi \cap B} v(X)$$

comprises a finite number of terms.

1.6.4 Factorial powers and moment measures

Given a counting measure μ on a measurable space and $n \in \mathbb{N}^*$, recall Definition 14.E.1 of the n-th factorial power $\mu^{(n)}$. This extends to point processes as follows.

Definition 1.6.21. *For a point process on a l.c.s.h. space Φ on $\mathbb{G}$, we denote by $\Phi^{(n)}$ its* n-th factorial power *($n \in \mathbb{N}^*$). We call $M_{\Phi^{(n)}}$ the* n-th factorial moment measure.

The fact that $\Phi^{(n)}$ is itself a point process (measurability issue) may be proved in the same lines as Lemma 1.3.7.

Lemma 1.6.22. Second order moments. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and $A, B \in \mathcal{B}(\mathbb{G})$ such that $M_{\Phi}(A)$ and $M_{\Phi}(B)$ are finite. Then*

$$\operatorname{cov}(\Phi(A), \Phi(B)) = M_{\Phi^{(2)}}(A \times B) + M_{\Phi}(A \cap B) - M_{\Phi}(A) M_{\Phi}(B). \tag{1.6.10}$$

$$\operatorname{Var}(\Phi(B)) = M_{\Phi^{(2)}}(B \times B) + M_{\Phi}(B) - M_{\Phi}(B)^2. \tag{1.6.11}$$

Proof. Let $A, B \in \mathcal{B}(\mathbb{G})$ such that $M_{\Phi}(A)$ and $M_{\Phi}(B)$ are finite. Then

$$\begin{aligned} \operatorname{cov}(\Phi(A), \Phi(B)) &= \mathbf{E}[\Phi(A)\Phi(B)] - \mathbf{E}[\Phi(A)]\mathbf{E}[\Phi(B)] \\ &= M_{\Phi^2}(A \times B) - M_{\Phi}(A) M_{\Phi}(B) \\ &= M_{\Phi^{(2)}}(A \times B) + M_{\Phi}(A \cap B) - M_{\Phi}(A) M_{\Phi}(B), \end{aligned}$$

where the third equality follows from (14.E.5). Taking $A = B$ in the above equality, we get

$$\operatorname{Var}(\Phi(B)) = \operatorname{cov}(\Phi(B), \Phi(B)) = M_{\Phi^{(2)}}(B \times B) + M_{\Phi}(B) - M_{\Phi}(B)^2.$$

□

1.7 Exercises

Exercise 1.7.1. *Let $\mathbb{G}$ be a l.c.s.h. space. For any $C \in \mathcal{B}(\mathbb{G})$, show that the mapping $\mu \mapsto \mu_C$ defined on $\bar{\mathbb{M}}(\mathbb{G})$ by $\mu_C(B) = \mu(B \cap C)$ is $(\bar{\mathcal{M}}(\mathbb{G}), \bar{\mathcal{M}}(\mathbb{G}))$-measurable.*

Solution 1.7.1. *For any $B \in \mathcal{B}(\mathbb{G}), A \in \mathcal{B}(\mathbb{R})$,*

$$\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu_C(B) \in A\right\} = \left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B \cap C) \in A\right\} \in \bar{\mathcal{M}}(\mathbb{G}).$$

Since, by definition, $\bar{\mathcal{M}}(\mathbb{G})$ is generated by $\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B) \in A\right\}$, where $B \in \mathcal{B}(\mathbb{G}), A \in \mathcal{B}(\mathbb{R})$, then the announced measurability follows from [11, Theorem 13.1 p.182].

Exercise 1.7.2. Quadratic random measure. *Let $\{Z(x)\}_{x \in \mathbb{R}^d}$ be a real Gaussian stochastic process with continuous trajectories. For any $B \in \mathcal{B}(\mathbb{G})$, let*

$$\Phi(B) = \int_B Z(x)^2 \, dx.$$

1. *Show that Φ is a random measure.*
2. *Assume that Z is centered and that $\operatorname{cov}(Z(x), Z(y)) = C(x-y)$. Show that*

$$M_\Phi(B) = C(0)|B|.$$
$$\operatorname{cov}(\Phi(A), \Phi(B)) = 2\int_{A \times B} C(x-y)^2 \, dx dy.$$

Solution 1.7.2. *1. We check that Φ is a random measure. Since $x \mapsto Z(x, \omega)$ is continuous, then it is bounded on bounded sets and therefore the measure $\Phi(\omega)$ is locally finite. Then it follows from Proposition 1.4.1 that Φ is a random measure.*

2. For all $B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned} M_\Phi(B) &= \mathbf{E}[\Phi(B)] \\ &= \int_B \mathbf{E}\left[Z(x)^2\right] dx = C(0)|B|. \end{aligned}$$

moreover, for $A, B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned} \mathbf{E}[\Phi(A)\Phi(B)] &= \mathbf{E}\left[\int_A Z(x)^2 \, dx \int_B Z(y)^2 \, dy\right] \\ &= \int_{A \times B} \mathbf{E}\left[Z(x)^2 Z(y)^2\right] dx dy \\ &= \int_{A \times B} \left(2C(x-y)^2 + C(0)^2\right) dx dy, \end{aligned}$$

where we use Isserlis' theorem for the last equality. Combining the two above equations, one gets the announced expression of $\operatorname{cov}(\Phi(A), \Phi(B))$.

Exercise 1.7.3. Independent random measures. *Let $\mathbb{G}$ be a l.c.s.h. space. Recall that a measure μ on a l.c.s.h. space $\mathbb{G}$ is said to have an* atom *at $x \in \mathbb{G}$ when $\mu(\{x\}) > 0$. A measure μ is said to be* diffuse *when it has no atoms; i.e., $\mu(\{x\}) = 0$ for all $x \in \mathbb{G}$. Let Φ_1 and Φ_2 be two independent random measures such that either M_{Φ_1} or M_{Φ_2} is diffuse. Show that Φ_1 and Φ_2 have no common atoms.*

Solution 1.7.3. *Assume that M_{Φ_1} is diffuse. It follows from (1.4.2) that*

$$\mathbf{E}\left[\int_{\mathbb{G}} \Phi_1(\{x\}) \Phi_2(\mathrm{d}x)\right] = \int_{\mathbb{G}} M_{\Phi_1}(\{x\}) M_{\Phi_2}(\mathrm{d}x) = 0.$$

Thus $\int_{\mathbb{G}} \Phi_1(\{x\}) \Phi_2(\mathrm{d}x) = 0$ almost surely.

Exercise 1.7.4. Monte Carlo integration. *We aim to calculate numerically the integral $I = \int_B f \mathrm{d}x$ where $B \in \mathcal{B}(\mathbb{R}^d)$ and $f : \mathbb{R}^d \to \mathbb{R}$ is integrable on B. Let Φ be a random measure on $\mathbb{R}^d$ with mean measure $M_\Phi(\mathrm{d}x) = \mathbf{1}_B(x)\mathrm{d}x$, and defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Let $\Phi_1, \ldots, \Phi_n$ be independent realizations of Φ. Show that $\mathbf{P}$ a.s.,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\int_B f \mathrm{d}\Phi_j\right) = \int_B f \mathrm{d}x. \tag{1.7.1}$$

Solution 1.7.4. *By Theorem 1.2.5*

$$\mathbf{E}\left[\int_B f \mathrm{d}\Phi\right] = \int_B f \mathrm{d}x.$$

The result then follows from the law of large numbers.

Chapter 2

Basic models and operations

2.1 Poisson point processes

Definition 2.1.1. *Let Λ be a locally finite measure on a l.c.s.h space $\mathbb{G}$. A point process Φ is said to be* Poisson *with* intensity measure Λ *if for all pairwise disjoint sets $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G})$, the random variables $\Phi(B_1), \ldots, \Phi(B_k)$ are independent Poisson random variables with respective means $\Lambda(B_1), \ldots, \Lambda(B_k)$; i.e., $\forall n_1, \ldots, n_k \in \mathbb{N}$,*

$$\mathbf{P}(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k) = \prod_{i=1}^{k} e^{-\Lambda(B_i)} \frac{\Lambda(B_i)^{n_i}}{n_i!}.$$

Definition 2.1.2. Homogeneous Poisson point process on $\mathbb{R}^d$. *If Φ is a Poisson point process on $\mathbb{R}^d$ with intensity measure $\Lambda(\mathrm{d}x) = \lambda \times \mathrm{d}x$ where $\lambda \in \mathbb{R}_+^*$ and $\mathrm{d}x$ denotes the Lebesgue measure, then Φ is called a* homogeneous *Poisson point process of* intensity λ.

Example 2.1.3. Poisson point process on a discrete space. *Let $\mathbb{G}$ be a l.c.s.h. space. A Poisson point process with intensity measure $\Lambda = \alpha\delta_x$ for some fixed $x \in \mathbb{G}$ is equal to $\Phi = N\delta_x$ where N is a Poisson random variable with mean α. More generally, given $x_1, \ldots, x_n \in \mathbb{G}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}_+$, a Poisson point process with intensity measure $\Lambda = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ is equal to $\Phi = \sum_{i=1}^{n} N_i \delta_{x_i}$ where $N_1, \ldots, N_n$ are independent Poisson random variables with respective means $\alpha_1, \ldots, \alpha_n$.*

Observe that the mean measure of a Poisson point process Φ is equal to its intensity measure; that is $M_\Phi(B) = \mathbf{E}[\Phi(B)] = \Lambda(B), B \in \mathcal{B}(\mathbb{G})$ since, by the very definition, $\Phi(B)$ is a Poisson random variable of mean $\Lambda(B)$. The void probability of a Poisson point process is equal to $\nu_\Phi(B) = \mathbf{P}(\Phi(B) = 0) = e^{-\Lambda(B)}$.

It is also abvious from the very definition that the restriction of a Poisson point process Φ of intensity measure Λ to some $B \in \mathcal{B}(\mathbb{G})$, i.e., the point process $\Phi|_B(\cdot) = \Phi(\cdot \cap B)$, is a Poisson point process of intensity measure $\Lambda|_B(\cdot) = \Lambda(\cdot \cap B)$.

2.1.1 Laplace transform

Proposition 2.1.4. Poisson point process Laplace transform. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with intensity measure Λ. Then the Laplace transform of Φ is given by*

$$\mathcal{L}_\Phi(f) = \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right] = \exp\left(-\int_{\mathbb{G}}(1-e^{-f})\mathrm{d}\Lambda\right), \tag{2.1.1}$$

for all measurable functions $f : \mathbb{G} \to \bar{\mathbb{R}}_+$. Moreover, for any such function,

$$\mathbf{E}\left[\exp\left(\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right] = \exp\left(\int_{\mathbb{G}}(e^{f}-1)\mathrm{d}\Lambda\right). \tag{2.1.2}$$

Proof. Let Φ be a Poisson point process with intensity measure Λ. Consider first a simple function $f = \sum_{j=1}^n a_j \mathbf{1}_{B_j}$, where $a_1, \ldots, a_n \geq 0$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{G})$ are pairwise disjoint. Then

$$\begin{aligned}
\mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right] &= \mathbf{E}\left[\exp\left(-\sum_{j=1}^n a_j \Phi(B_j)\right)\right] \\
&= \prod_{j=1}^n \mathbf{E}\left[e^{-a_j\Phi(B_j)}\right] \\
&= \prod_{j=1}^n \exp\left[-\Lambda(B_j)(1-e^{-a_j})\right] \\
&= \exp\left[-\sum_{j=1}^n (1-e^{-a_j})\Lambda(B_j)\right] \\
&= \exp\left(-\int_{\mathbb{G}}(1-e^{-f})\mathrm{d}\Lambda\right),
\end{aligned}$$

where the second equality is due to the independence of the random variables $\Phi(B_j)$, $j = 1, \ldots, n$, and the third one follows from the fact that $\Phi(B_j)$ is Poisson with mean $\Lambda(B_j)$. If f is a general measurable nonnegative function on $\mathbb{G}$, there exists a nondecreasing sequence of simple functions converging to it; and therefore the monotone convergence theorem gives the announced identity (2.1.1). The proof of Equation (2.1.2) follows the same lines as that of (2.1.1). □

Corollary 2.1.5. *Let Φ be a Poisson point process on a l.c.s.h space $\mathbb{G}$ with intensity measure Λ. Its generating function (cf. Definition 1.6.18) is given by*

$$\mathcal{G}_\Phi(v) = \exp\left(-\int_{\mathbb{G}}(1-v)\mathrm{d}M_\Phi\right), \quad v \in \mathcal{V}(\mathbb{G}). \tag{2.1.3}$$

Proof. This follows from Equations (1.6.9) and (2.1.1). □

The following proposition proves the existence of Poisson point processes with finite intensity measures by constructing them explicitly. This construction also shows how to simulate such processes.

Proposition 2.1.6. Construction of a Poisson point process with finite intensity measure. *Let Λ be a finite measure on a l.c.s.h. space $\mathbb{G}$. Let N be a Poisson random variable with mean $\Lambda(\mathbb{G})$ and $X_1, X_2, \ldots$ be i.i.d. random variables with values in $\mathbb{G}$, independent of N, and such that $\mathbf{P}(X_1 \in \cdot) = \Lambda(\cdot)/\Lambda(\mathbb{G})$. Then $\Phi := \sum_{j=1}^{N} \delta_{X_j}$ is a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with intensity measure Λ.*

Proof. For any measurable $f : \mathbb{G} \to \mathbb{R}_+$, we have

$$\begin{aligned}
\mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right] &= \mathbf{E}\left[\prod_{j=1}^{N} e^{-f(X_j)}\right] \\
&= \mathbf{P}(N=0) + \sum_{n\geq 1} \mathbf{E}\left[1_{\{N=n\}} \prod_{j=1}^{n} e^{-f(X_j)}\right] \\
&= e^{-\Lambda(\mathbb{G})} + \sum_{n\geq 1} \mathbf{P}(N=n)\mathbf{E}\left[\prod_{j=1}^{n} e^{-f(X_j)}\right] \\
&= \sum_{n\geq 0} e^{-\Lambda(\mathbb{G})} \frac{\Lambda(\mathbb{G})^n}{n!} \left[\int_{\mathbb{G}} e^{-f(x)} \frac{1}{\Lambda(\mathbb{G})} \Lambda(\mathrm{d}x)\right]^n \\
&= \exp\left[-\Lambda(\mathbb{G}) + \int_{\mathbb{G}} e^{-f(x)} \Lambda(\mathrm{d}x)\right] \\
&= \exp\left(-\int_{\mathbb{G}} (1-e^{-f}) \mathrm{d}\Lambda\right).
\end{aligned}$$

The result follows from the fact that the Laplace transform characterizes the distribution of a point process; cf. Corollary 1.2.2. □

Definition 2.1.7. *We say that a random measure Φ on a l.c.s.h. space $\mathbb{G}$ has* a fixed atom *at some $x \in \mathbb{G}$ if $\mathbf{P}(\Phi(\{x\}) > 0) > 0$.*

It is easy to see that:

Lemma 2.1.8. *A Poisson point process Φ on a l.c.s.h. space $\mathbb{G}$ with intensity measure Λ has a fixed atom at some $x \in \mathbb{G}$ if and only if $\Lambda(\{x\}) > 0$.*

Proof. Since $\Phi(\{x\})$ is a Poisson random variable with mean $\Lambda(\{x\})$, then

$$\mathbf{P}(\Phi(\{x\}) > 0) = 1 - e^{-\Lambda(\{x\})}.$$

□

Proposition 2.1.9. Simplicity. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with intensity measure Λ. Then Φ is simple if and only if Λ is diffuse (i.e., has no atoms).*

Proof. **Direct part**. If Λ has an atom at some $x \in \mathbb{G}$, then $\Phi(\{x\})$ is a Poisson random variable with mean $\Lambda(\{x\})$, thus $\mathbf{P}(\Phi(\{x\}) \geq 2) > 0$. **Converse part**. Assume that Λ is diffuse. Since the space $\mathbb{G}$ may be partitioned into a countable collection of locally compact sets, it is enough to show that Φ is simple on any $W \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$. Recall that the truncation of a Poisson point process to W is also Poisson with intensity measure $\Lambda(\cdot \cap W)$. Moreover,

$$
\begin{aligned}
&\mathbf{P}(\Phi \text{ is simple in } W) \\
&= \sum_{n\geq 0} \mathbf{P}(\Phi(W) = n)\mathbf{P}(\Phi \text{ is simple in } W \mid \Phi(W) = n) \\
&= \mathbf{P}(\Phi(W) \leq 1) \\
&+ \sum_{n\geq 2} \mathbf{P}(\Phi(W) = n) \int_W \cdots \int_W \mathbf{1}_{\{\sum_{j=1}^n \delta_{x_j} \text{ is simple}\}} \frac{\Lambda(\mathrm{d}x_1)}{\Lambda(W)} \cdots \frac{\Lambda(\mathrm{d}x_n)}{\Lambda(W)} \\
&= \mathbf{P}(\Phi(W) \leq 1) \\
&+ \sum_{n\geq 2} \mathbf{P}(\Phi(W) = n) \int_W \int_{W\setminus\{x_n\}} \cdots \int_{W\setminus\{x_n,\ldots,x_2\}} \frac{\Lambda(\mathrm{d}x_1)}{\Lambda(W)} \cdots \frac{\Lambda(\mathrm{d}x_n)}{\Lambda(W)},
\end{aligned}
$$

where the second equality follows from the construction in Proposition 2.1.6. Since Λ is diffuse, i.e., $\Lambda(\{z\}) = 0$, for all $z \in \mathbb{G}$, we have

$$
\begin{aligned}
\int_W \int_{W\setminus\{x_n\}} \cdots \int_{W\setminus\{x_n,\ldots,x_2\}} & \frac{\Lambda(\mathrm{d}x_1)}{\Lambda(W)} \cdots \frac{\Lambda(\mathrm{d}x_n)}{\Lambda(W)} \\
&= \int_W \cdots \int_W \frac{\Lambda(\mathrm{d}x_1)}{\Lambda(W)} \cdots \frac{\Lambda(\mathrm{d}x_n)}{\Lambda(W)} = 1.
\end{aligned}
$$

Then

$$
\mathbf{P}(\Phi \text{ is simple in } W) = \mathbf{P}(\Phi(W) \leq 1) + \sum_{n\geq 2} \mathbf{P}(\Phi(W) = n) = 1.
$$

□

2.1.2 Characterizations

We have already shown that the distribution of a point process is characterized by its Laplace transform; cf. Corollary 1.2.2. We now give further characterizations.

Theorem 2.1.10. Rényi's theorem. *The probability distribution of a simple point process Φ on a l.c.s.h space $\mathbb{G}$ is characterized by its void probability function $\nu_\Phi(B) = \mathbf{P}(\Phi(B) = 0), B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$.*

Proof. Cf. [52, Theorem 3.3]. By Corollary 1.3.4, it is enough to shows that the finite-dimensional distributions of Φ on locally compact sets are characterized by $\nu_\Phi(B), B \in \mathcal{B}_c(\mathbb{G})$. To do so, we will proceed in several steps.

1. Firstly, we prove by induction on $k \in \mathbb{N}^*$ that for all $A_1, \ldots, A_k, B \in \mathcal{B}_c(\mathbb{G})$, $u_k = \mathbf{P}(\Phi(A_1) > 0, \ldots, \Phi(A_k) > 0, \Phi(B) = 0)$ can be computed from the void probability function ν_Φ. This follows from the fact that for $k = 1$,

$$\begin{aligned}\mathbf{P}(\Phi(A_1) > 0, \Phi(B) = 0) &= \mathbf{P}(\Phi(B) = 0) - \mathbf{P}(\Phi(B \cup A_1) = 0) \\ &= \nu_\Phi(B) - \nu_\Phi(B \cup A_1)\end{aligned}$$

   and the recursive relation

$$\begin{aligned}&\mathbf{P}(\Phi(A_1) > 0, \ldots, \Phi(A_k) > 0, \Phi(B) = 0) \\ &= \mathbf{P}(\Phi(A_1) > 0, \ldots, \Phi(A_{k-1}) > 0, \Phi(B) = 0) \\ &- \mathbf{P}(\Phi(A_1) > 0, \ldots, \Phi(A_{k-1}) > 0, \Phi(A_k \cup B) = 0).\end{aligned}$$

2. Let $B \in \mathcal{B}_c(\mathbb{G})$ and let $\{B_{n,j}\}_{j=1}^{k_n}$ be a sequence of nested partitions of B as in Lemma 1.6.3. For any $n \in \mathbb{N}$, let

$$H_n(B) = \sum_{j=1}^{k_n} H(B_{n,j}),$$

   where $H(A) = 1\{\Phi(A) > 0\}$, $A \in \mathcal{B}_c(\mathbb{G})$. Since the point process Φ is simple and the partitions $\{B_{n,j}\}$ eventually separate the points of Φ in B, we get as $n \to \infty$,

$$H_n(B) \uparrow \Phi(B), \quad \text{almost surely.} \tag{2.1.4}$$

3. We now prove that, for all $B_1, \ldots, B_k \in \mathcal{B}_c(\mathbb{G}), j_1, \ldots, j_k \in \mathbb{N}$, $\mathbf{P}(H_n(B_1) \leq j_1, \ldots, H_n(B_k) \leq j_k)$ can be expressed in terms of ν_Φ only. We begin by $\mathbf{P}(H_n(B) = j)$. Note that $H_n(B)$ counts the subsets $B_{n,j}$ comprising at least one point of Φ, then

$$\mathbf{P}(H_n(B) = j) = \sum_{\substack{i_0, \ldots, i_{k_n} \in \{0,1\} \\ i_0 + \cdots + i_{k_n} = j}} \mathbf{P}(H(B_{n,0}) = i_0, \ldots, H(B_{n,k_n}) = i_{k_n}).$$

   Moreover, for $i_0, \ldots, i_{k_n} \in \{0, 1\}$,

$$\begin{aligned}&\mathbf{P}(H(B_{n,0}) = i_0, \ldots, H(B_{n,k_n}) = i_{k_n}) \\ &= \mathbf{P}\left(\bigcap_{l : i_l = 1} \{\Phi(B_{n,l}) > 0\} \cap \left\{\Phi\left(\bigcup_{m : i_m = 0} B_{n,m}\right) = 0\right\}\right),\end{aligned}$$

which can be expressed in terms of ν_Φ by the result in Step 1. We may prove in the same lines as above that, for all $j_1, \ldots, j_k \in \mathbb{N}$,

$$\mathbf{P}(H_n(B_1) = j_1, \ldots, H_n(B_k) = j_k)$$

can be expressed in terms of ν_Φ, and, by summation, the same holds also true for

$$\mathbf{P}(H_n(B_1) \leq j_1, \ldots, H_n(B_k) \leq j_k).$$

4. Finally, it follows from (2.1.4) that almost surely

$$\{H_n(B_1) \leq j_1, \ldots, H_n(B_k) \leq j_k\} \downarrow \{\Phi(B_1) \leq j_1, \ldots, \Phi(B_k) \leq j_k\}$$

and therefore, by sequential continuity of probability, we get

$$\lim_{n\uparrow\infty} \mathbf{P}(H_n(B_1) \leq j_1, \ldots, H_n(B_k) \leq j_k) = \mathbf{P}(\Phi(B_1) \leq j_1, \ldots, \Phi(B_k) \leq j_k).$$

By Step 3 the left-hand side of the above equality is characterized by ν_Φ, then so is the finite-dimensional distribution in the right-hand side, which completes the proof.

□

Theorem 2.1.11. *Let Φ be a simple point process on a l.c.s.h. space $\mathbb{G}$. Then Φ is Poisson if and only if there exists a diffuse locally finite measure Λ on $\mathbb{G}$ such that, for all $A \in \mathcal{B}_c(\mathbb{G})$, $\mathbf{P}(\Phi(A) = 0) = e^{-\Lambda(A)}$.*

Proof. The direct part follows from Proposition 2.1.9. The converse part follows from Rényi's theorem 2.1.10. □

In particular:

Corollary 2.1.12. *If Φ is a simple point process on a l.c.s.h. space $\mathbb{G}$ such that, for all $A \in \mathcal{B}_c(\mathbb{G})$, $\Phi(A)$ is a Poisson random variables, then Φ is a Poisson point process.*

In the above corollary, the assumption that Φ is simple cannot be relaxed since one can construct two Poisson random variables N_1 and N_2, of parameters λ_1, λ_2, respectively, and such that $N_1 + N_2$ is Poisson of parameter $\lambda_1 + \lambda_2$, with N_1 and N_2 not being independent, cf. [94, §12.3].

Definition 2.1.13. *A random measure Φ on a l.c.s.h. space $\mathbb{G}$ is said to have the* complete independence *property if for all pairwise disjoint sets $B_1, \ldots, B_k \in \mathcal{B}_c(\mathbb{G})$, the random variables $\Phi(B_1), \ldots, \Phi(B_k)$ are independent.*

Theorem 2.1.14. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ without fixed atoms. Then Φ is Poisson if and only if Φ is simple and has the complete independence property.*

Proof. **Direct part**. Since Φ has no fixed atoms, Λ is diffuse by Lemma 2.1.8 and therefore Φ is simple by Proposition 2.1.9. The complete independence property follows from the definition of Poisson point processes. **Converse part**. Here is an outline of the proof; cf. [30, Lemma 2.4.IV] for the details. Let $Q(A) = -\log(\mathbf{P}(\Phi(A) = 0))\,, A \in \mathcal{B}(\mathbb{G})$. One checks that:

- Q is obviously nonnegative. Moreover, Q is additive by the complete independence property.
- Q is countably additive. Indeed, let $\{A_n\}$ be a sequence of sets in $\mathcal{B}(\mathbb{G})$ such that $A_n \downarrow \emptyset$. Then the events $\{\Phi(A_n) = 0\}$ increase to
$$\bigcup_n \{\Phi(A_n) = 0\} = \{\Phi(\textstyle\bigcap_n A_n) = 0\} = \Omega.$$
Thus $\mathbf{P}(\Phi(A_n) = 0) \to 1$; that is $Q(A_n) \to 0$.
- Q is locally finite; i.e., $\mathbf{P}(\Phi(A) = 0) > 0$ for any $A \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$. Indeed assume for the sake of a contradiction that $\mathbf{P}(\Phi(A) = 0) = 0$. Consider a sequence of nested partitions of A as given by Lemma 1.6.3. Deduce that Φ has a fixed atom which contradicts the first assumption in the theorem.

□

The above theorem justifies the fact that the Poisson point process is often considered when one does not assume (is not expecting) any "interactions" between points.

2.2 Operations on random measures and point processes

We now study operations on point processes such as superposition, thinning, independent displacement and marking. We will in particular show that many of these operations preserve the Poisson character; i.e., the transformation of a Poisson point process is also Poisson.

2.2.1 Superposition

The *superposition* of point processes consists in their sum in the measure theoretic sense. If they are simple and with disjoint supports, the support of the superposition is the set-theoretic union of their supports. This may be defined formally as follows.

Proposition 2.2.1. *Let $\Phi_0, \Phi_1, \ldots$ be a sequence of point processes on a l.c.s.h. space $\mathbb{G}$ defined on the same probability space. Let $B_0, B_1, \ldots$ be a sequence of relatively compact open sets whose union covers $\mathbb{G}$ (cf. Lemma 1.1.4). If*

$$\sum_{k\in\mathbb{N}} \mathbf{P}(\Phi_k(B_j) \neq 0) < \infty, \quad j \in \mathbb{N}, \tag{2.2.1}$$

then $\Phi = \sum_{k\in\mathbb{N}} \Phi_k$ *is a point process. The above condition is necessary when* $\Phi_1, \Phi_2, \ldots$ *are independent.*

Proof. Sufficiency. By the first Borel-Cantelli Lemma [11, Theorem 4.3 p.59], Equation (2.2.1) implies

$$\mathbf{P}\left(\Phi_k(B_j) \neq 0 \text{ infinitely often for } k \in \mathbb{N}\right) = 0, \quad j \in \mathbb{N}.$$

Then, $\mathbf{P}$-almost surely, for all $j \in \mathbb{N}$, $\Phi(B_j) = \sum_{k\in\mathbb{N}} \Phi_k(B_j) < \infty$. Since any $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$ may be covered by a finite union of the B_j, it follows that $\Phi(B) < \infty$. Invoking Proposition 1.1.7(iii) allows one to conclude. *Necessity.* Assume that $\Phi_0, \Phi_1, \ldots$ are independent and that $\Phi = \sum_{k\in\mathbb{N}} \Phi_k$ is a point process. Then for all $j \in \mathbb{N}$, $\sum_{k\in\mathbb{N}} \Phi_k(B_j) = \Phi(B_j) < \infty$. Then, since each $\Phi_k(B_j)$ is a nonnegative integer,

$$\mathbf{P}\left(\Phi_k(B_j) \neq 0 \text{ infinitely often for } k \in \mathbb{N}\right) = 0, \quad j \in \mathbb{N},$$

which implies (2.2.1) by the second Borel-Cantelli Lemma [11, Theorem 4.4 p.60]. □

Corollary 2.2.2. *A sufficient condition for (2.2.1) to hold is that the measure* $\sum_{k\in\mathbb{N}} \mathbf{E}\left[\Phi_k(\cdot)\right]$ *is locally finite on* $\mathbb{G}$.

Proof. This follows from the fact that $\mathbf{P}\left(\Phi_k(B_j) \neq 0\right) = \mathbf{E}\left[\mathbf{1}\{\Phi_k(B_j) \neq 0\}\right] \leq \mathbf{E}\left[\Phi_k\left(B_j\right)\right]$. □

Corollary 2.2.3. Superposition of Poisson point processes. *Let* $\Phi_0, \Phi_1, \ldots$ *be a sequence of independent Poisson point processes on a l.c.s.h. space* $\mathbb{G}$ *with intensity measures* $\Lambda_0, \Lambda_1, \ldots$. *Then their superposition* $\Phi = \sum_{k\in\mathbb{N}} \Phi_k$ *is Poisson if and only if the measure* $\Lambda = \sum_{k\in\mathbb{N}} \Lambda_k$ *is locally finite on* $\mathbb{G}$.

Proof. Necessity. Assume that $\Phi = \sum_{k\in\mathbb{N}} \Phi_k$ is Poisson. Then, for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,

$$\Lambda(B) = \sum_{k\in\mathbb{N}} \Lambda_k(B) = \sum_{k\in\mathbb{N}} \mathbf{E}\left[\Phi_k(B)\right] = \mathbf{E}\left[\Phi(B)\right] < \infty,$$

by definition of a Poisson point process. *Sufficiency.* Assume that $\Lambda = \sum_{k\in\mathbb{N}} \Lambda_k$ is locally finite. By Corollary 2.2.2, $\Phi = \sum_{k\in\mathbb{N}} \Phi_k$ is a point process. We calculate its Laplace transform

$$\begin{aligned}
\mathbf{E}\left[e^{-\sum_{k\in\mathbb{N}}\int_{\mathbb{G}} f(x)\Phi_k(\mathrm{d}x)}\right] &= \mathbf{E}\left[\prod_{k\in\mathbb{N}} e^{-\int_{\mathbb{G}} f(x)\Phi_k(\mathrm{d}x)}\right] \\
&= \prod_{k\in\mathbb{N}} \mathbf{E}\left[e^{-\int_{\mathbb{G}} f(x)\Phi_k(\mathrm{d}x)}\right] \\
&= \prod_{k\in\mathbb{N}} \exp\left[-\int_{\mathbb{G}} (1-e^{f(x)})\Lambda_k(\mathrm{d}x)\right] \\
&= \exp\left[-\int_{\mathbb{G}} (1-e^{f(x)})\Lambda(\mathrm{d}x)\right].
\end{aligned}$$

□

Proposition 2.2.4. *Given a locally finite measure Λ on a l.c.s.h. space $\mathbb{G}$, there exists a Poisson point process on $\mathbb{G}$ with intensity measure Λ.*

Proof. Construction of a Poisson point process on $\mathbb{G}$. Let $B_0, B_1, \ldots \in \mathcal{B}_c(\mathbb{G})$ be a countable partition of $\mathbb{G}$ and, for all $k \in \mathbb{N}$, let Φ_k be a Poisson point process on $\mathbb{G}$ with intensity measure $\Lambda_k(\cdot) = \Lambda(\cdot \cap B_k)$ (which may be constructed as in Proposition 2.1.6). Then Corollary 2.2.3 shows that $\Phi = \sum_{k\in\mathbb{N}} \Phi_k$ is Poisson with intensity measure $\Lambda = \sum_{k\in\mathbb{N}} \Lambda_k$. □

2.2.2 Thinning of points

The thinning of a point process consists in suppressing some subset of its points. In this section, we will consider *independent thinnings* where the decision to suppress or keep each point is taken independently from the others. More precisely:

Definition 2.2.5. *Let $\Phi = \sum_{k\in\mathbb{N}} \delta_{X_k}$ be a point process on a l.c.s.h. space $\mathbb{G}$ and $p : \mathbb{G} \to [0,1]$ be some measurable function called the* retention function. *Let $U_0, U_1, \ldots$ be a sequence of i.i.d. random variables independent of Φ, uniformly distributed in $[0,1]$. The* thinning *of Φ with* retention function *p is defined as*

$$\tilde{\Phi} = \sum_{k\in\mathbb{N}} \mathbf{1}\{U_k \leq p(X_k)\} \delta_{X_k}.$$

In other words: Given Φ, each point $X \in \Phi$ is erased with probability $1 - p(X)$ independently from the other points.

Proposition 2.2.6. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$, $p : \mathbb{G} \to [0,1]$ be some measurable function, and let $\tilde{\Phi}$ be the thinning of Φ with retention function p. Then $\tilde{\Phi}$ is a point process with mean measure*

$$M_{\tilde{\Phi}}(\mathrm{d}x) = p(x) M_{\Phi}(\mathrm{d}x).$$

Moreover, its Laplace transform is given by

$$\mathcal{L}_{\tilde{\Phi}}(f) = \mathcal{L}_{\Phi}\left(\tilde{f}\right), \tag{2.2.2}$$

for all measurable $f : \mathbb{G} \to \mathbb{R}_+$, where

$$\tilde{f}(x) = -\log\left[1 - p(x)\left(1 - e^{-f(x)}\right)\right].$$

Proof. We have to show the measurability of $\tilde{\Phi} : \Omega \to \mathbb{M}(\mathbb{G})$, which is equivalent to showing that, for all $B \in \mathcal{B}(\mathbb{G})$,

$$\tilde{\Phi}(B) = \sum_{k\in\mathbb{N}} \mathbf{1}\{U_k \leq p(X_k)\} \mathbf{1}\{X_k \in B\}$$

is a random variable; cf. Proposition 1.1.7(iii). This follows from the fact that a series of nonnegative random variables is a random variable. Moreover,

$$\begin{aligned}
\mathbf{E}\left[\tilde{\Phi}(B)\right] &= \mathbf{E}\left[\mathbf{E}\left[\tilde{\Phi}(B) \mid \Phi\right]\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{N}} \mathbf{P}\left(U_k \le p(X_k) \mid \Phi\right) \mathbf{1}\{X_k \in B\}\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{N}} p(X_k)\, \mathbf{1}\{X_k \in B\}\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}} p(x)\, \mathbf{1}\{x \in B\}\, \Phi(\mathrm{d}x)\right] \\
&= \int_{\mathbb{G}} p(x)\, \mathbf{1}\{x \in B\}\, M_\Phi(\mathrm{d}x),
\end{aligned}$$

where the last equality follows from Theorem 1.2.5. For all measurable $f : \mathbb{G} \to \mathbb{R}_+$, the Laplace transform of $\tilde{\Phi}$ at f is given by

$$\begin{aligned}
\mathcal{L}_{\tilde{\Phi}}(f) &= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f \mathrm{d}\tilde{\Phi}\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{N}} \mathbf{1}\{U_k \le p(X_k)\} f(X_k)\right)\right] \\
&= \mathbf{E}\left[\prod_{k\in\mathbb{N}} e^{-\mathbf{1}\{U_k \le p(X_k)\} f(X_k)}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\prod_{k\in\mathbb{N}} e^{-\mathbf{1}\{U_k \le p(X_k)\} f(X_k)} \middle| \Phi\right]\right] \\
&= \mathbf{E}\left[\prod_{k\in\mathbb{N}} \left(1 - p(X_k) + p(X_k)\, e^{-f(X_k)}\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} \tilde{f} \mathrm{d}\Phi\right)\right] = \mathcal{L}_\Phi\left(\tilde{f}\right).
\end{aligned}$$

□

Corollary 2.2.7. *The thinning of a Poisson point process on a l.c.s.h. space $\mathbb{G}$ of intensity measure Λ with retention function p is a Poisson point process of intensity measure $\tilde{\Lambda}$ defined by*

$$\tilde{\Lambda}(B) = \int_B p(x)\, \Lambda(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{G}).$$

Proof. If follows from Proposition 2.2.6 that the Laplace transform of the thinning process is

$$
\begin{aligned}
\mathcal{L}_{\tilde{\Phi}}(f) &= \mathcal{L}_{\Phi}\left(\tilde{f}\right) \\
&= \exp\left(-\int_{\mathbb{G}} (1 - e^{-\tilde{f}})\mathrm{d}\Lambda\right) \\
&= \exp\left(-\int_{\mathbb{G}} \left(1 - e^{-f(x)}\right) p(x)\,\Lambda(\mathrm{d}x)\right) \\
&= \exp\left(-\int_{\mathbb{G}} \left(1 - e^{-f(x)}\right) \tilde{\Lambda}(\mathrm{d}x)\right),
\end{aligned}
$$

which is the Laplace transform of a Poisson point process of intensity measure $\tilde{\Lambda}$. The characterization of a Poisson point process by its Laplace transform (Proposition 2.1.4) completes the proof. □

2.2.3 Image

Definition 2.2.8. *Let $\mathbb{G}$ and $\mathbb{G}'$ be two l.c.s.h spaces equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{G})$ and $\mathcal{B}(\mathbb{G}')$ respectively. Let $g : \mathbb{G} \to \mathbb{G}'$ be a measurable function such that $g^{-1}(B) \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$ for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G}')$. For any locally finite measure μ on $\mathbb{G}$, $\mu \circ g^{-1}$ is a locally finite measure on $\mathbb{G}'$ called the* image *of μ by g.*

The image $\Phi \circ g^{-1}$ of a random measure Φ on $\mathbb{G}$ is a random measure on $\mathbb{G}'$. Note that the image of a point process $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ by a function g is $\sum_{k\in\mathbb{Z}} \delta_{g(X_k)}$; i.e., the image of a point process consists in the deterministic displacement of all its points by g.

Proposition 2.2.9. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ of intensity measure Λ and let $g : \mathbb{G} \to \mathbb{G}'$ be a measurable function such that $g^{-1}(B) \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$ for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G}')$. Then $\Phi \circ g^{-1}$ is a Poisson point process of intensity measure $\Lambda \circ g^{-1}$.*

Proof. By Corollary 1.2.2, the distribution of a Poisson point process is characterized by its Laplace transform given by Equation (2.1.1). Let Φ be a Poisson point process on $\mathbb{G}$ of intensity measure Λ and let $\Phi' = \Phi \circ g^{-1}$. Since Λ is locally finite by definition of Poisson point processes, then $\Lambda \circ g^{-1}$ is locally finite. Moreover, the Laplace transform of Φ' is given for all measurable $f : \mathbb{G}' \to \mathbb{R}_+$

by

$$\begin{aligned}
\mathcal{L}_{\Phi'}(f) &= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}'} f(y)\,\Phi'(\mathrm{d}y)\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}'} f(y)\,\Phi\circ g^{-1}(\mathrm{d}y)\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f(g(x))\,\Phi(\mathrm{d}x)\right)\right] \\
&= \exp\left(-\int_{\mathbb{G}} (1-e^{-f(g(x))})\Lambda(\mathrm{d}x)\right) \\
&= \exp\left(-\int_{\mathbb{G}'} (1-e^{-f(y)})\Lambda\circ g^{-1}(\mathrm{d}x)\right),
\end{aligned}$$

where the third and fifth equalities are due to the change of variable theorem for measures. □

Example 2.2.10. Change of coordinates in $\mathbb{R}^n$. *Let U be an open subset of $\mathbb{R}^n$ and $g: U \to \mathbb{R}^n$ be injective, with continuously differentiable coordinates such that $|\det J_g(x)| > 0$ for all $x \in \mathbb{R}^n$. Then, by the change of variable formula for integrals [92, Theorem 3-13], for all measurable functions $f: U \to \mathbb{R}^n$,*

$$\int_U f(x)\,\mathrm{d}x = \int_{g(U)} f(g^{-1}(y))\left|\det J_{g^{-1}}(y)\right|\mathrm{d}y.$$

Let Φ be a homogeneous Poisson point process on U of intensity measure $\Lambda(\mathrm{d}x) = \lambda(x)\,\mathrm{d}x$ for some nonnegative measurable function λ. Assume that $g^{-1}(B) \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^n)$ for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^n)$. Then, by Proposition 2.2.9, the image $\Phi' = \Phi \circ g^{-1}$ is a Poisson point process of intensity measure $\Lambda' = \Lambda \circ g^{-1}$. By the change of variable theorem for measures,

$$\begin{aligned}
\int f\left(g^{-1}(y)\right)\Lambda'(\mathrm{d}y) &= \int f(x)\,\Lambda(\mathrm{d}x) \\
&= \int f(x)\,\lambda(x)\,\mathrm{d}x \\
&= \int f\left(g^{-1}(y)\right)\lambda\left(g^{-1}(y)\right)\left|\det J_{g^{-1}}(y)\right|\mathrm{d}y.
\end{aligned}$$

Then

$$\Lambda'(\mathrm{d}y) = \lambda\left(g^{-1}(y)\right)\left|\det J_{g^{-1}}(y)\right|\mathrm{d}y. \tag{2.2.3}$$

Example 2.2.11. Homogenization. *This is a continuation of the above example. In the particular case $\lambda(x) = |\det J_g(x)|$, we see that $\Lambda'(\mathrm{d}y) = \mathrm{d}y$. Then given $\lambda(\cdot)$, finding a function g such that $|\det J_g(x)| = \lambda(x)$ allows one to transform Φ into a* homogeneous *Poisson point process $\Phi \circ g^{-1}$. In particular, on $\mathbb{R}$, we may take $g(x) = \int_0^x \lambda(t)\,\mathrm{d}t$.*

Example 2.2.12. Polar coordinates in $\mathbb{R}^2$. *Let $\mathbb{G} = \mathbb{R}^2 \backslash \{(x,0) : x \geq 0\}$ and $\mathbb{G}' = (0,\infty) \times (0, 2\pi)$ and $g : \mathbb{G} \to \mathbb{G}'; (x,y) \mapsto (r,\theta)$ be the polar coordinate function, i.e.,*

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Let Φ be a homogeneous Poisson point process on $\mathbb{G}$ of intensity measure $\Lambda(\mathrm{d}x \times \mathrm{d}y) = \lambda \mathrm{d}x\mathrm{d}y$. Then, by Proposition 2.2.9, its polar coordinates image $\tilde{\Phi} = \Phi \circ g^{-1}$ is a Poisson point process of intensity measure $\tilde{\Lambda} = \Lambda \circ g^{-1}$. By the change of variable theorem for measures, for all $f : \mathbb{G} \to \mathbb{R}_+$,

$$\begin{aligned} \int f(r\cos\theta, r\sin\theta)\,\tilde{\Lambda}(\mathrm{d}r \times \mathrm{d}\theta) &= \int f(x,y)\,\Lambda(\mathrm{d}x \times \mathrm{d}y) \\ &= \lambda \int f(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= \lambda \int f(r\cos\theta, r\sin\theta)\, r\mathrm{d}r\mathrm{d}\theta, \end{aligned}$$

where the last equality is due to the change of variable formula for integrals. Thus

$$\tilde{\Lambda}(\mathrm{d}r \times \mathrm{d}\theta) = \lambda r \mathrm{d}r\mathrm{d}\theta. \tag{2.2.4}$$

Therefore

$$\tilde{\Lambda}((0,r) \times (0,\theta)) = \frac{1}{2}\lambda r^2 \theta.$$

2.2.4 Independent displacement of points

We aim to transform the points of a point process Φ independently from each other; the point $X \in \Phi$ being transformed into some Y randomly and independently from the other points of Φ. The construction is formalized in the following definition.

Definition 2.2.13. *Let $\mathbb{G}$ and $\mathbb{G}'$ be two l.c.s.h. spaces equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{G})$ and $\mathcal{B}(\mathbb{G}')$, respectively. Let Φ be a point process on $\mathbb{G}$ and let $p(\cdot,\cdot)$ be a* probability kernel *from $\mathbb{G}$ to $\mathbb{G}'$ (cf. Definition 14.D.1). Consider a measurable enumeration $X_0, X_1, \ldots$ of the points of Φ and let $\{Y_k\}_{k\in\mathbb{N}}$ be, conditionally to Φ, an independent sequence such that*

$$\mathbf{P}(Y_k \in \cdot \mid \Phi) = p(X_k, \cdot), \quad \forall k \in \mathbb{N}.$$

Then

$$\tilde{\Phi} = \sum_{k\in\mathbb{N}} \delta_{Y_k}$$

is called an independent displacement *of the point process Φ* by the kernel *p.*

Lemma 2.2.14. *In the context of Definition 2.2.13, we have* *(i)* $\tilde{\Phi}(B)$ *is a random variable for any* $B \in \mathcal{B}(\mathbb{G}')$*. Moreover,*

$$\mathbf{E}\left[\tilde{\Phi}(B)\right] = \int_{\mathbb{G}} p(x, B)\, M_{\Phi}(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{G}'). \tag{2.2.5}$$

(ii) If $\tilde{\Phi}(\omega)$ *is locally finite for* $\mathbf{P}$*-almost all* ω*, then* $\tilde{\Phi}$ *is a point process with mean measure given by the above equation.*

(iii) If the measure defined by the right-hand side of (2.2.5) is locally finite, then $\tilde{\Phi}(\omega)$ *is locally finite for* $\mathbf{P}$*-almost all* ω*.*

Proof. (i) Observe first that for any $B \in \mathcal{B}(\mathbb{G}')$, $\tilde{\Phi}(B) = \sum_{k\in\mathbb{N}} \mathbf{1}\{Y_k \in B\}$ is a random variable. Moreover, for all $B \in \mathcal{B}(\mathbb{G}')$,

$$\begin{aligned}
\mathbf{E}\left[\tilde{\Phi}(B)\right] &= \mathbf{E}\left[\mathbf{E}\left[\tilde{\Phi}(B) \mid \Phi\right]\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\sum_{k\in\mathbb{N}} \mathbf{1}\{Y_k \in B\} \mid \Phi\right]\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{N}} \mathbf{E}\left[\mathbf{1}\{Y_k \in B\} \mid \Phi\right]\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{N}} p(X_k, B)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}} p(x, B)\Phi(\mathrm{d}x)\right] = \int_{\mathbb{G}} p(x, B) M_{\Phi}(\mathrm{d}x),
\end{aligned}$$

where we use Theorem 1.2.5 for the last equality. (ii) This follows from Corollary 1.1.8. (iii) Assume now that the measure defined by (2.2.5) is locally finite. Then, for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G}')$, $\mathbf{E}\left[\tilde{\Phi}(B)\right] < \infty$, which implies that $\tilde{\Phi}$ is almost surely a locally finite measure (this may be proved by recalling that, by Lemma 1.1.4, $\mathbb{G}'$ may be covered by a countable union of compact sets). Therefore $\tilde{\Phi}$ is a point process on $\mathbb{G}'$. □

Example 2.2.15. I.i.d. shifts of points. *An example of independent displacement of points of Definition 2.2.13 is provided by* i.i.d shifts of points *in the Euclidean space, which are obtained when* $Y_k = X_k + Z_k$*, where the* Z_k*'s are i.i.d. and independent of* Φ*.*

Proposition 2.2.16. *The Laplace transform of the independent displacement* $\tilde{\Phi}$ *of the point process* Φ *on a l.c.s.h. space* $\mathbb{G}$ *by the kernel* p *is given by*

$$\mathcal{L}_{\tilde{\Phi}}(f) = \mathcal{L}_{\Phi}\left(\tilde{f}\right), \tag{2.2.6}$$

for all measurable $f : \mathbb{G} \to \mathbb{R}_+$*, where*

$$\tilde{f}(x) = -\log\left[\int_{\mathbb{G}'} e^{-f(y)} p(x, \mathrm{d}y)\right].$$

Proof. The Laplace transform of $\tilde{\Phi}$ is given for all measurable $f : \mathbb{G} \to \mathbb{R}_+$ by

$$\begin{aligned}
\mathcal{L}_{\tilde{\Phi}}(f) &= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{N}} f(Y_k)\right)\right] \\
&= \mathbf{E}\left[\prod_{k\in\mathbb{N}} e^{-f(Y_k)}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\prod_{k\in\mathbb{N}} e^{-f(Y_k)}\middle|\Phi\right]\right] \\
&= \mathbf{E}\left[\prod_{k\in\mathbb{N}} \int_{\mathbb{G}'} e^{-f(y)} p(X_k, \mathrm{d}y)\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{N}} \tilde{f}(X_k)\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} \tilde{f}\mathrm{d}\Phi\right)\right] = \mathcal{L}_{\Phi}\left(\tilde{f}\right).
\end{aligned}$$

□

Theorem 2.2.17. Displacement theorem. *Let $\tilde{\Phi}$ be the independent displacement of a Poisson point process Φ on a l.c.s.h. space $\mathbb{G}$ by some kernel p such that the measure defined by (2.2.5) is locally finite. Then $\tilde{\Phi}$ is a Poisson point process with intensity measure given by (2.2.5).*

Proof. By Proposition 2.2.16 the Laplace transform of $\tilde{\Phi}$ is given for all measurable $f : \mathbb{G} \to \mathbb{R}_+$ by

$$\begin{aligned}
\mathcal{L}_{\tilde{\Phi}}(f) &= \mathcal{L}_{\Phi}\left(\tilde{f}\right) \\
&= \exp\left(-\int_{\mathbb{G}}\left(1 - e^{-\tilde{f}(x)}\right) M_{\Phi}(\mathrm{d}x)\right) \\
&= \exp\left[-\int_{\mathbb{G}}\left(1 - \int_{\mathbb{G}'} e^{-f(y)} p(x, \mathrm{d}y)\right) M_{\Phi}(\mathrm{d}x)\right] \\
&= \exp\left[-\int_{\mathbb{G}}\int_{\mathbb{G}'}\left(1 - e^{-f(y)}\right) p(x, \mathrm{d}y)\, M_{\Phi}(\mathrm{d}x)\right] \\
&= \exp\left[-\int_{\mathbb{G}'}\left(1 - e^{-f(y)}\right) M_{\tilde{\Phi}}(\mathrm{d}y)\right],
\end{aligned}$$

where $M_{\tilde{\Phi}}$ is the measure defined by (2.2.5). The characterization of a Poisson point process by its Laplace transform completes the proof. □

2.2.5 Independent marking of points

We aim now to associate to each point $X \in \Phi$ a mark Z such that the marks of the different points are independent from each other. This is formalized in the following definition.

Definition 2.2.18. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h spaces equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{G})$ and $\mathcal{B}(\mathbb{K})$ respectively. Let Φ be a point process on $\mathbb{G}$ and let $\tilde{p}$ be a* probability kernel *from $\mathbb{G}$ to $\mathbb{K}$ called a* mark kernel. *Consider a measurable enumeration $X_0, X_1, \ldots$ of the points of Φ and let $\{Z_k\}_{k\in\mathbb{N}}$ be, conditionally to Φ, an independent sequence such that*

$$\mathbf{P}(Z_k \in \cdot \mid \Phi) = \tilde{p}(X_k, \cdot), \quad \forall k \geq 0.$$

Then

$$\tilde{\Phi} = \sum_{k\in\mathbb{N}} \delta_{(X_k, Z_k)} \tag{2.2.7}$$

is a point process on $\mathbb{G} \times \mathbb{K}$ (with the corresponding product Borel σ-algebra) called an independently marked point process *with* ground point process Φ.

If the kernel $\tilde{p}(\cdot,\cdot)$ does not depend on its first coordinate, then $\tilde{\Phi}$ is called an i.i.d. marked point process. *In this case, $F(\cdot) := \mathbf{P}(Z_k \in \cdot \mid \Phi)$ is called the* mark distribution.

Let $\mathbb{G}' = \mathbb{G} \times \mathbb{K}$ with the associated product σ-algebra. In order to check that $\tilde{\Phi}$ is a point process indeed, the key observation is that $\tilde{\Phi}$ may be obtained as transformation of the process Φ by the displacement kernel

$$p(x, B \times K) = \delta_x(B)\tilde{p}(x, K), \quad B \in \mathcal{B}(\mathbb{G}), K \in \mathcal{B}(\mathbb{K})$$

from $\mathbb{G}$ to $\mathbb{G}'$, and to observe that for all $C \in \mathcal{B}_c(\mathbb{G})$, and $K \in \mathcal{B}(\mathbb{K})$, we have

$$\tilde{\Phi}(C \times K) \leq \Phi(C) < \infty.$$

Since the projection of a relatively compact of a product space into one of the component spaces is relatively compact, $\tilde{\Phi}(\omega)$ is locally finite for $\mathbf{P}$-almost all ω. Then Lemma 2.2.14 allows one to conclude.

Lemma 2.2.19. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h spaces and $\tilde{\Phi}$ be the independently marked point process associated to Φ through the probability kernel $\tilde{p}$ from $\mathbb{G}$ to $\mathbb{K}$. Then the mean measure of $\tilde{\Phi}$ on $\mathbb{G} \times \mathbb{K}$ is*

$$M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z) = \tilde{p}(x, \mathrm{d}z) M_\Phi(\mathrm{d}x). \tag{2.2.8}$$

Moreover, if M_Φ is locally finite then so is $M_{\tilde{\Phi}}$.

Proof. The formula follows directly from (2.2.5). If M_Φ is locally finite, then for all $C \in \mathcal{B}_c(\mathbb{G})$, and $K \in \mathcal{B}(\mathbb{K})$, we have

$$M_{\tilde{\Phi}}(C \times K) \leq M_\Phi(C) < \infty.$$

□

Proposition 2.2.20. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces and $\tilde{\Phi}$ be the independently marked point process associated to Φ through the probability kernel $\tilde{p}$ from $\mathbb{G}$ to $\mathbb{K}$. Then the Laplace transform of $\tilde{\Phi}$ is given by*

$$\mathcal{L}_{\tilde{\Phi}}(f) = \mathcal{L}_\Phi\left(\tilde{f}\right) \tag{2.2.9}$$

for all measurable $f : \mathbb{G} \times \mathbb{K} \to \mathbb{R}_+$, *where*

$$\tilde{f}(x) = -\log\left[\int_{\mathbb{K}} e^{-f(x,z)}\tilde{p}(x, \mathrm{d}z)\right].$$

Proof. The result follows from Proposition 2.2.16 and

$$\tilde{f}(x) = -\log\left[\int_{\mathbb{G}\times\mathbb{K}} e^{-f(y,z)}\delta_x(\mathrm{d}y)\tilde{p}(y, \mathrm{d}z)\right] = -\log\left[\int_{\mathbb{K}} e^{-f(x,z)}\tilde{p}(x, \mathrm{d}z)\right].$$

□

The above proposition shows that the distribution of the independently marked point process $\tilde{\Phi}$ doesn't depend on the enumeration of the points of Φ. Moreover, we may extend Definition 2.2.18 by calling independently marked point process with ground process Φ any point process whose Laplace tranform is given by (2.2.9). In doing so, the independent displacement of a point process is seen as an operation on its distribution. Similar observations may be made for thinning and independent displacements of points.

Theorem 2.2.21. *Let* $\mathbb{G}$ *and* $\mathbb{K}$ *be two l.c.s.h spaces and* $\tilde{\Phi}$ *be an independently marked point process associated to a Poisson point process* Φ *through the probability kernel* $\tilde{p}$ *from* $\mathbb{G}$ *to* $\mathbb{K}$*. Then* $\tilde{\Phi}$ *is itself Poisson on* $\mathbb{G} \times \mathbb{K}$ *with Laplace transform*

$$\mathcal{L}_{\tilde{\Phi}}(f) = \exp\left(-\int_{\mathbb{G}}\left[1 - \int_{\mathbb{K}} \exp(-f(x,z))\tilde{p}(x, \mathrm{d}z)\right] M_{\Phi}(\mathrm{d}x)\right).$$

Proof. Immediate from the displacement theorem 2.2.17. □

Example 2.2.22. Polar coordinates in $\mathbb{R}^2$. *Continuing Example 2.2.12, in polar coordinates, a homogeneous Poisson point process* Φ *on* $\mathbb{R}^2$ *becomes an inhomogeneous Poisson point process* $\tilde{\Phi}$ *with intensity (2.2.4)*

$$\tilde{\Lambda}(\mathrm{d}r \times \mathrm{d}\theta) = 2\pi\lambda r \mathrm{d}r \frac{\mathrm{d}\theta}{2\pi}.$$

Comparing the above formula with (2.2.8), we see that $\tilde{\Phi}$ *may be obtained as a Poisson point process on* $\mathbb{R}_+^*$ *with intensity* $\Lambda(\mathrm{d}r) = 2\pi\lambda r \mathrm{d}r$ independently marked *with marks uniformly distributed on the interval* $(0, 2\pi)$.

2.2.6 Marked random measures

Recall that we introduced independently marked point processes on $\mathbb{R}^d \times \mathbb{K}$ in Definition 2.2.18. We will now define a general class of marked random measures.

To do so, let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{G})$ and $\mathcal{B}(\mathbb{K})$, respectively. Let $\tilde{\mathbb{M}}(\mathbb{G} \times \mathbb{K})$ be the space of measures $\tilde{\mu}$ on $(\mathbb{G} \times \mathbb{K}, \mathcal{B}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{K}))$ such that $\tilde{\mu}(B \times \mathbb{K}) < \infty$ for all $B \in \mathcal{B}_c(\mathbb{G})$. Let $\tilde{\mathcal{M}}(\mathbb{G} \times \mathbb{K})$ be the σ-algebra on $\tilde{\mathbb{M}}(\mathbb{G} \times \mathbb{K})$ generated by the mappings $\tilde{\mu} \mapsto \tilde{\mu}(B \times K), B \in \mathcal{B}(\mathbb{G}), K \in \mathcal{B}(\mathbb{K})$.

Definition 2.2.23. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces. A random measure $\tilde{\Phi}$ on $\mathbb{G} \times \mathbb{K}$ with values in $\tilde{\mathbb{M}}(\mathbb{G} \times \mathbb{K})$ is called a* marked random measure *on $\mathbb{G}$ with marks in $\mathbb{K}$. Its projection on $\mathbb{G}$, that is $\Phi(\cdot) = \tilde{\Phi}(\cdot \times \mathbb{K})$, is called the* ground random measure. *If $\tilde{\Phi}$ is a point process, then we say that it is a* marked point process *and its projection Φ is called the* ground process.

Example 2.2.24. *An independently marked point process on $\mathbb{R}^d \times \mathbb{K}$ in the sense of Definition 2.2.18 is a marked random measure on $\mathbb{R}^d$ with marks in $\mathbb{K}$.*

2.2.7 Mixtures

We begin by recalling a general way to mix different probability measures in order to get a new one.

Definition 2.2.25. *Let $(\mathbb{X}, \mathcal{X}, \lambda_{\mathbb{X}})$ be a probability space and let $\{\Phi_x\}_{x\in\mathbb{X}}$ be a family of random measures on a l.c.s.h. space $\mathbb{G}$ such that $\mathbf{P}_{\Phi_x}(L)$ is measurable in x for every $L \in \bar{\mathcal{M}}(\mathbb{G})$. Then the* mixture *of $\{\Phi_x\}_{x\in\mathbb{X}}$ with respect to $\lambda_{\mathbb{X}}$ is the random measure on $\mathbb{G}$ whose distribution is the mixture of $\{\mathbf{P}_{\Phi_x}\}_{x\in\mathbb{X}}$ with respect to $\lambda_{\mathbb{X}}$; cf. Theorem 14.D.4.*

Lemma 2.2.26. *Let $(\mathbb{X}, \mathcal{X}, \lambda_{\mathbb{X}})$ be a probability space and let $\{\Phi_x\}_{x\in\mathbb{X}}$ be a family of random measures on a l.c.s.h. space $\mathbb{G}$ such that $\mathbf{P}(\Phi_x(B_1) \le t_1, \ldots, \Phi_x(B_k) \le t_k)$ is measurable in x for every $k \in \mathbb{N}^*, B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G}), t_1, \ldots, t_k \in \mathbb{R}$. Then $\mathbf{P}_{\Phi_x}(L)$ is measurable in x for every $L \in \bar{\mathcal{M}}(\mathbb{G})$.*

Proof. Let $\mathcal{I}$ be the class of subsets of $\bar{\mathcal{M}}(\mathbb{G})$ of the form

$$\left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B_1) \le t_1, \ldots, \mu(B_k) \le t_k\right\},$$

for some $k \in \mathbb{N}^* = \mathbb{N}\setminus\{0\}, B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G}), t_1, \ldots, t_k \in \mathbb{R}$. Note that $\mathcal{I}$ is a π-system (i.e. closed with respect to finite intersections). Let

$$\mathcal{D} = \left\{L \in \bar{\mathcal{M}}(\mathbb{G}) : \mathbf{P}_{\Phi_x}(L) \text{ is measurable in } x\right\},$$

Observe that $\mathcal{D}$ is a Dynkin system on $\bar{\mathcal{M}}(\mathbb{G})$; i.e.

$$\bar{\mathcal{M}}(\mathbb{G}) \in \mathcal{D},$$

$$L \in \mathcal{D} \Rightarrow L^{\mathrm{c}} \in \mathcal{D},$$

and for every nondecreasing sequence $\{L_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{D}$, we have

$$\lim_{n\to\infty} L_n \in \mathcal{D}.$$

Since $\mathcal{D}$ contains the π-system $\mathcal{I}$, then $\mathcal{D}$ contains $\sigma(\mathcal{I})$ by the Dynkin's theorem [11, Theorem 3.2 p.42]. $\square$

We may mix random measures under the following less stringent conditions.

Lemma 2.2.27. Random measure mixture. *Let $(\mathbb{X}, \mathcal{X}, \lambda_{\mathbb{X}})$ be a probability space and let $\{\Phi_x\}_{x\in\mathbb{X}}$ be a family of random measures on a l.c.s.h. space $\mathbb{G}$. If $\mathcal{L}_{\Phi_x}(f)$ is measurable in x for every $f \in \mathfrak{F}_{\mathrm{c}}(\mathbb{G})$, then the* mixture *of $\mathbf{P}_{\Phi_x}$ with respect to $\lambda_{\mathbb{X}}$, that is $\lambda(\mathrm{d}x \times \mathrm{d}\mu) = \mathbf{P}_{\Phi_x}(\mathrm{d}\mu)\lambda_{\mathbb{X}}(\mathrm{d}x)$, is a well defined probability on $\mathbb{X} \times \bar{\mathbb{M}}(\mathbb{G})$. The projection of λ on $\bar{\mathbb{M}}(\mathbb{G})$ is the probability distribution of some random measure, say Φ, whose mean measure is*

$$M_\Phi(B) = \mathbf{E}[\Phi_X(B)], \quad B \in \mathcal{B}(\mathbb{G}) \tag{2.2.10}$$

and Laplace transform of Φ is

$$\mathcal{L}_\Phi(f) = \mathbf{E}[\mathcal{L}_{\Phi_X}(f)], \quad f \in \mathfrak{F}_+(\mathbb{G}), \tag{2.2.11}$$

where X is a random variable with distribution $\lambda_{\mathbb{X}}$.

Proof. Cf. [52, Lemma 1.7] for the existence of the mixture probability $\lambda(\mathrm{d}\mu \times \mathrm{d}x) = \mathbf{P}_{\Phi_x}(\mathrm{d}\mu)\lambda_{\mathbb{X}}(\mathrm{d}x)$. The projection of λ on $\bar{\mathbb{M}}(\mathbb{G})$ is the probability distribution of some random measure, say Φ; that is

$$\mathbf{P}_\Phi(\mathrm{d}\mu) = \int_{\mathbb{X}} \mathbf{P}_{\Phi_x}(\mathrm{d}\mu)\lambda_{\mathbb{X}}(\mathrm{d}x).$$

The mean measure of Φ is

$$\begin{aligned} M_\Phi(B) &= \int_{\bar{\mathbb{M}}(\mathbb{G})} \mu(B)\mathbf{P}_\Phi(\mathrm{d}\mu) \\ &= \int_{\mathbb{X}}\int_{\bar{\mathbb{M}}(\mathbb{G})} \mu(B)\mathbf{P}_{\Phi_x}(\mathrm{d}\mu)\lambda_{\mathbb{X}}(\mathrm{d}x) \\ &= \int_{\mathbb{X}} M_{\Phi_x}(B)\lambda_{\mathbb{X}}(\mathrm{d}x) = \mathbf{E}[\Phi_X(B)], \end{aligned}$$

where $X \overset{\text{dist.}}{\sim} \lambda_{\mathbb{X}}$. Similarly, the Laplace transform of Φ is

$$\begin{aligned} \mathcal{L}_\Phi(f) &= \mathbf{E}\left[e^{-\Phi f}\right] \\ &= \int_{\bar{\mathbb{M}}(\mathbb{G})} e^{-\mu f}\mathbf{P}_\Phi(\mathrm{d}\mu) \\ &= \int_{\mathbb{X}}\int_{\bar{\mathbb{M}}(\mathbb{G})} e^{-\mu f}\mathbf{P}_{\Phi_x}(\mathrm{d}\mu)\lambda_{\mathbb{X}}(\mathrm{d}x) \\ &= \int_{\mathbb{X}} \mathcal{L}_{\Phi_x}(f)\lambda_{\mathbb{X}}(\mathrm{d}x) = \mathbf{E}[\mathcal{L}_{\Phi_X}(f)]. \end{aligned}$$

□

Example 2.2.28. Mixed Binomial point process. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with values in a l.c.s.h. space $\mathbb{G}$. For all $n \in \mathbb{N}$, $\Phi_n = \sum_{k=1}^n \delta_{X_k}$ is called a Binomial point process. We have shown in Example 1.2.4 that*

$$M_{\Phi_n}(B) = n\mathbf{P}_{X_1}(B), \quad B \in \mathcal{B}(\mathbb{G})$$

and

$$\mathcal{L}_{\Phi_n}(f) = \left[\mathcal{L}_{f(X_1)}(1)\right]^n, \quad f \in \mathfrak{F}_+(\mathbb{G}).$$

Let N be a random variable with values in $\mathbb{N}$ independent of $\{X_k\}_{k\geq 1}$. By Lemma 2.2.27, $\Phi := \Phi_N = \sum_{k=1}^N \delta_{X_k}$ is a random measure called a mixed Binomial point process *with mean measure*

$$\begin{aligned} M_\Phi(B) &= \mathbf{E}[\Phi_N(B)] \\ &= \mathbf{E}[\mathbf{E}[\Phi_N(B) \mid N]] \\ &= \mathbf{E}[N \times \mathbf{P}_{X_1}(B)] \\ &= \mathbf{E}[N] \times \mathbf{P}_{X_1}(B), \quad B \in \mathcal{B}(\mathbb{G}). \end{aligned} \tag{2.2.12}$$

Note that M_Φ is locally finite which implies that Φ is almost surely locally finite. Moreover, the Laplace transform of Φ is

$$\begin{aligned} \mathcal{L}_\Phi(f) &= \mathbf{E}[\mathcal{L}_{\Phi_N}(f)] \\ &= \mathbf{E}[\mathbf{E}[\mathcal{L}_{\Phi_N}(f) \mid N]] \\ &= \mathbf{E}\left[\left[\mathcal{L}_{f(X_1)}(1)\right]^N\right] \\ &= \mathcal{G}_N\left(\mathcal{L}_{f(X_1)}(1)\right), \quad f \in \mathfrak{F}_+(\mathbb{G}), \end{aligned} \tag{2.2.13}$$

where $\mathcal{G}_N(z) = \mathbf{E}[z^N]$ is the generating function of N.

Example 2.2.29. Poisson point process. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with values in a l.c.s.h. space $\mathbb{G}$ and N be a Poisson random variable of mean θ independent of $\{X_k\}_{k\geq 1}$. The generating function of a Poisson random variable is*

$$\mathcal{G}_N(z) = e^{\theta(z-1)}.$$

Then the Laplace transform of the mixed Binomial point process $\Phi := \sum_{k=1}^N \delta_{X_k}$ is

$$\begin{aligned} \mathcal{L}_\Phi(f) &= \mathcal{G}_N\left(\mathcal{L}_{f(X_1)}(1)\right) \\ &= \exp\left[\theta(\mathcal{L}_{f(X_1)}(1) - 1)\right] \\ &= \exp\left[\theta(\mathbf{E}\left[e^{-f(X_1)}\right] - 1)\right] \\ &= \exp\left[\theta\left(\int_{\mathbb{G}} e^{-f(x)}\mathbf{P}_{X_1}(\mathrm{d}x) - 1\right)\right] \\ &= \exp\left[\int_{\mathbb{G}}\left(e^{-f(x)} - 1\right)\theta\mathbf{P}_{X_1}(\mathrm{d}x)\right]. \end{aligned}$$

Thus, as expected, Φ is a Poisson point process of intensity measure $\theta\mathbf{P}_{X_1}(\mathrm{d}x)$.

Example 2.2.30. Compound process. *Let $X_1, X_2, \ldots$ be fixed points in a l.c.s.h space $\mathbb{G}$ and let $\beta_1, \beta_2, \ldots$ be independent and identically distributed random* *variables in* $\mathbb{R}_+$. *Then* $\Phi = \sum_{k=1}^n \beta_k \delta_{X_k}$ *is a random measure on* $\mathbb{G}$, *with Laplace transform*

$$\begin{aligned}\mathbf{E}\left[\exp\left(-\sum_{k=1}^n \beta_k f(X_k)\right)\right] &= \prod_{k=1}^n \mathbf{E}\left[\exp\left(-\beta_k f(X_k)\right)\right] = \prod_{k=1}^n \mathcal{L}_{\beta_1}(f(X_k)) \\ &= \exp\left(-\sum_{k=1}^n \log\left[\mathcal{L}_{\beta_1}(f(X_k))\right]\right) \\ &= \exp\left(\int_{\mathbb{G}} \log\left[\mathcal{L}_{\beta_1}(f(x))\right]\mu(\mathrm{d}x)\right), \quad f \in \mathfrak{F}_+(\mathbb{G}),\end{aligned}$$

where $\mu = \sum_{k=1}^n \delta_{X_k}$. *Note that for all* $f \in \mathfrak{F}_c(\mathbb{G})$, *the right-hand side of the above equation is measurable in* μ. *Then by Lemma 2.2.27, we can mix with respect to* $\mu = \Phi$ *considered now as a point process on* $\mathbb{G}$, *thus obtaining a* β-compound of Φ, *with Laplace transform*

$$\mathbf{E}\left[\exp\left(\int_{\mathbb{G}} \log\left[\mathcal{L}_{\beta_1}(f(x))\right]\Phi(\mathrm{d}x)\right)\right] = \mathcal{L}_\Phi\left[\log\left(\mathcal{L}_{\beta_1} \circ f\right)\right], \quad f \in \mathfrak{F}_+(\mathbb{G}).$$

In the particular case when β_1 *is Bernoulli with parameter* p, *we get the* thinning *of* Φ *with retention function* p *which has the Laplace transform*

$$\mathcal{L}_\Phi\left(-\log\left[pe^{-f} + (1-p)\right]\right) = \mathcal{L}_\Phi\left(-\log\left[1 - p\left(1 - e^{-f}\right)\right]\right), \quad f \in \mathfrak{F}_+(\mathbb{G}),$$

which is a particular case of thinning (cf. Proposition 2.2.6).

2.3 Constructing new models

2.3.1 Cox point processes

A Cox point process (also called a doubly stochastic Poisson point process) may be viewed as a Poisson point process whose intensity measure is random. It may be defined formally with the help of Lemma 2.2.27 as follows.

For any $\mu \in \bar{\mathbb{M}}(\mathbb{G})$, denote by Φ_μ a Poisson point process of intensity measure μ. Note that for all $f \in \mathfrak{F}_c(\mathbb{G})$,

$$\mathcal{L}_{\Phi_\mu}(f) = \exp\left(-\int_{\mathbb{G}} (1 - e^{-f(t)})\mu(\mathrm{d}t)\right)$$

is measurable in μ. Then, by Lemma 2.2.27, given a random measure Λ on $\mathbb{G}$, the mixture Φ_Λ with respect to Λ is a point process on $\mathbb{G}$.

Definition 2.3.1. *Given a random measure* Λ *on a l.c.s.h. space* $\mathbb{G}$, *the mixture* Φ_Λ *with respect to* Λ *is called a* Cox point process *(also known as* doubly stochastic Poisson point process*) directed by* Λ.

Corollary 2.3.2. *Let Φ be a Cox point process directed by a random measure Λ on a l.c.s.h. space $\mathbb{G}$. Its mean measure is*

$$M_{\Phi}(B) = M_{\Lambda}(B), \quad B \in \mathcal{B}(\mathbb{G}). \tag{2.3.1}$$

Its Laplace transform is

$$\mathcal{L}_{\Phi}(f) = \mathcal{L}_{\Lambda}\left(1 - e^{-f}\right), \quad f \in \mathfrak{F}_{+}(\mathbb{G}). \tag{2.3.2}$$

Its generating function is

$$\mathcal{G}_{\Phi}(v) = \mathcal{L}_{\Lambda}(1 - v), \quad v \in \mathcal{V}(\mathbb{G}). \tag{2.3.3}$$

Proof. The mean measure follows from (2.2.10)

$$\begin{aligned} M_{\Phi}(B) &= \mathbf{E}[\Phi_{\Lambda}(B)] \\ &= \mathbf{E}[\mathbf{E}[\Phi_{\Lambda}(B) \mid \Lambda]] \\ &= \mathbf{E}[\Lambda(B)] = M_{\Lambda}(B), \quad B \in \mathcal{B}(\mathbb{G}). \end{aligned}$$

The Laplace transform is deduced from (2.2.11)

$$\begin{aligned} \mathcal{L}_{\Phi}(f) &= \mathbf{E}[\mathcal{L}_{\Phi_{\Lambda}}(f)] \\ &= \mathbf{E}[\mathbf{E}[\mathcal{L}_{\Phi_{\Lambda}}(f) \mid \Lambda]] \\ &= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}}\left[1 - e^{-f(t)}\right]\Lambda(\mathrm{d}t)\right)\right] \\ &= \mathcal{L}_{\Lambda}\left(1 - e^{-f}\right), \quad f \in \mathfrak{F}_{+}(\mathbb{G}). \end{aligned}$$

The generating function follows from (1.6.9)

$$\mathcal{G}_{\Phi}(v) = \mathcal{L}_{\Phi}(-\log v) = \mathcal{L}_{\Lambda}(1 - v), \quad v \in \mathcal{V}(\mathbb{G}).$$

□

Definition 2.3.3. *Let Φ be a Cox point process directed by a random measure Λ on a l.c.s.h. space $\mathbb{G}$. The random measure Λ is called the* directing measure *of the Cox point process Φ. In case when the directing measure Λ has an integral representation $\Lambda(B) = \int_B \xi(x)\,\mathrm{d}x$, with $\{\xi(x)\}_{x\in\mathbb{R}^d}$ being a locally integrable and measurable stochastic process called the* directing process *of the Cox point process Φ.*

Definition 2.3.4. Stationary point processes. *A point process Φ on $\mathbb{R}^d$ is called* stationary *when its distribution is invariant with respect to translations; i.e., $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ and $S_t\Phi := \sum_{k\in\mathbb{Z}} \delta_{X_k - t}$ have the same distribution for all $t \in \mathbb{R}^d$.*

Such processes will be studied in details in Chapter 6. In particular, the above definition will be generalized to random measures in Definition 6.1.2.

Lemma 2.3.5. *Let Φ be a Cox point process on $\mathbb{R}^d$ directed by Λ. Then Φ is stationary iff Λ is so.*

Proof. This follows from the fact that for all $t \in \mathbb{R}^d$, $S_t\Phi$ is a Cox point process directed by $S_t\Lambda$. □

Example 2.3.6. Mixed Poisson point process. *Consider a Cox point process Φ directed by $\Lambda = X\mu$, where X is a nonnegative random variable and μ is a measure on a l.c.s.h. space $\mathbb{G}$ assumed to be locally finite. Then Φ is called a* mixed Poisson point process. *Its generating function follows from (2.3.3), for all $v \in \mathcal{V}(\mathbb{G})$,*

$$\begin{aligned}\mathcal{G}_{\Phi}(v) &= \mathcal{L}_{\Lambda}(1-v)\\ &= \mathbf{E}\left[\exp\left(-X\int_{\mathbb{G}}[1-v(t)]\,\mu(\mathrm{d}t)\right)\right] = \mathcal{L}_X\left(\int_{\mathbb{G}}[1-v(t)]\,\mu(\mathrm{d}t)\right).\end{aligned} \tag{2.3.4}$$

Example 2.3.7. Mixed Poisson point process leading to negative binomial distributions. *Let Φ be a mixed Poisson point process as in Example 2.3.6; that is a Cox point process Φ directed by $\Lambda = X\mu$. Assume that X has the gamma probability distribution with shape α and scale λ; that is with Laplace transform*

$$\mathcal{L}_X(t) = (1+\lambda t)^{-\alpha}.$$

Then the generating function of Φ follows from (2.3.4): for all $v \in \mathcal{V}(\mathbb{G})$,

$$\mathcal{G}_{\Phi}(v) = \left(1+\lambda\int_{\mathbb{G}}[1-v(t)]\,\mu(\mathrm{d}t)\right)^{-\alpha}. \tag{2.3.5}$$

In particular, for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,

$$\mathcal{G}_{\Phi(B)}(x) = \mathcal{G}_{\Phi}(1-(1-x)\,1_B) = (1+\lambda\mu(B)(1-x))^{-\alpha},$$

which, compared to (13.A.29), shows that $\Phi(B)$ is has a negative binomial probability distribution with parameters α and $\rho = \frac{\lambda\mu(B)}{1+\lambda\mu(B)}$. Thus $\Phi(B)$ has the same probability distribution as the point process in Example 2.3.22. Nevertheless the two point processes have different distributions since they have different generating functions.

Example 2.3.8. Independently marked Cox point process. *Let Φ be a Cox point processes on a l.c.s.h. space $\mathbb{G}$ directed by Λ. Let $\tilde{\Phi}$ be the corresponding independently marked point process with marks in some l.c.s.h. space $\mathbb{K}$ and with the mark kernel $\tilde{p}(\cdot,\cdot)$ (cf. Defintion 2.2.18). Then $\tilde{\Phi}$ is a Cox point process with directing measure $\tilde{\Lambda}$ given by*

$$\tilde{\Lambda}(B) = \int_B \tilde{p}(x,\mathrm{d}z)\Lambda(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{G}\times\mathbb{K}).$$

Indeed, by Proposition 2.2.20 the Laplace transform of $\tilde{\Phi}$ is given by, for all measurable $f : \mathbb{G} \times \mathbb{K} \to \mathbb{R}_+$,

$$\mathcal{L}_{\tilde{\Phi}}(f) = \mathcal{L}_{\Phi}\left(\tilde{f}\right),$$

where $\tilde{f}(x) = -\log\left[\int_{\mathbb{K}} e^{-f(x,z)}\tilde{p}(x, \mathrm{d}z)\right]$. Using (2.3.2) it follows that

$$\begin{aligned}
\mathcal{L}_{\tilde{\Phi}}(f) &= \mathcal{L}_{\Lambda}\left(1 - e^{-\tilde{f}}\right) \\
&= \mathcal{L}_{\Lambda}\left(\int_{\mathbb{K}}\left(1 - e^{-f(\cdot,z)}\right)\tilde{p}(\cdot, \mathrm{d}z)\right) \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}\times\mathbb{K}}\left(1 - e^{-f(x,z)}\right)\tilde{p}(x, \mathrm{d}z)\Lambda(\mathrm{d}x)\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}\times\mathbb{K}}\left(1 - e^{-f(x,z)}\right)\tilde{\Lambda}(\mathrm{d}x \times \mathrm{d}z)\right)\right] \\
&= \mathcal{L}_{\tilde{\Lambda}}\left(1 - e^{-f}\right).
\end{aligned}$$

Comparing the above expression with Equation (2.3.2), we recognize the Laplace transform of a Cox point process with directing measure $\tilde{\Lambda}$.

2.3.2 Gibbs point processes

A Gibbs process is a point process whose probability distribution has a density with respect to the probability distribution of a Poisson point process. More formally:

Definition 2.3.9. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ and let $f : \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{R}}_+$ be some measurable function such that $\mathbf{E}[f(\Phi)] = 1$. A point process $\tilde{\Phi}$ with probability distribution*

$$\mathbf{P}_{\tilde{\Phi}}(\mathrm{d}\mu) = f(\mu)\,\mathbf{P}_{\Phi}(\mathrm{d}\mu) \tag{2.3.6}$$

is called a Gibbs point process *with* density f *with respect to Φ (called* weight process*).*

Observe that for any measurable function $h : \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{R}}_+$,

$$\begin{aligned}
\mathbf{E}\left[h\left(\tilde{\Phi}\right)\right] &= \int_{\mathbb{M}(\mathbb{G})} h(\mu)\,\mathbf{P}_{\tilde{\Phi}}(\mathrm{d}\mu) \\
&= \int_{\mathbb{M}(\mathbb{G})} h(\mu) f(\mu)\,\mathbf{P}_{\Phi}(\mathrm{d}\mu) \\
&= \mathbf{E}[h(\Phi) f(\Phi)],
\end{aligned} \tag{2.3.7}$$

where the second equality follows from (2.3.6).

Remark 2.3.10. *For all measurable functions $g : \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{R}}_+$ satisfying $0 < \mathbf{E}[g(\Phi)] < \infty$, let $f := \frac{g}{\mathbf{E}[g(\Phi)]}$. Since $\mathbf{E}[f(\Phi)] = 1$, we may consider a Gibbs point process with density f with respect to Φ.*

2.3.3 Cluster point processes

Definition 2.3.11. *Let* $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k,\Phi_k)}$ *be an independently marked point process on* $\mathbb{R}^d$ *(cf. Definition 2.2.18) with ground process* $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ *and marks in* $\mathbb{M}\left(\mathbb{R}^d\right)$ *with kernel*

$$\mathbf{P}\left(\Phi_k \in \cdot \mid \Phi\right) = \tilde{p}\left(X_k, \cdot\right).$$

Let $\bar{\Phi}$ *be defined by*

$$\bar{\Phi}\left(B\right) = \sum_{k\in\mathbb{Z}} \Phi_k\left(B - X_k\right), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right) \tag{2.3.8}$$

and assume that $\mathbf{P}$*-almost surely*

$$\bar{\Phi}\left(B\right) < \infty, \quad B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right). \tag{2.3.9}$$

Then $\bar{\Phi}$ *is called a* cluster point process *on* $\mathbb{R}^d$*. The process* Φ *is called the* parent process*, whereas the* Φ_k*'s are the* descendant processes*.*

The fact that $\bar{\Phi}$ is a point process follows from the fact that a countable sum of measures is again a measure, from the assumption that $\bar{\Phi}$ is $\mathbf{P}$-almost surely locally finite, and finally invoking Proposition 1.1.7(iii).

Proposition 2.3.12. *The mean measure of the cluster point process* $\bar{\Phi}$ *defined by (2.3.8) is*

$$M_{\bar{\Phi}}\left(B\right) = \int_{\mathbb{R}^d} M_{\Phi_x}\left(B - x\right) M_{\Phi}\left(\mathrm{d}x\right), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right), \tag{2.3.10}$$

where, for all $x \in \mathbb{R}^d$*,* Φ_x *is a Point process on* $\mathbb{R}^d$ *with probability distribution* $\mathbf{P}\left(\Phi_x \in \cdot\right) = \tilde{p}\left(x, \cdot\right)$*. Its Laplace transform is*

$$\mathcal{L}_{\bar{\Phi}}\left(f\right) = \mathcal{L}_{\Phi}\left(\bar{f}\right), \quad f \in \mathfrak{F}_+\left(\mathbb{R}^d\right), \tag{2.3.11}$$

where

$$\bar{f}\left(x\right) = -\log \mathcal{L}_{\Phi_x}\left(S_x f\right), \quad x \in \mathbb{R}^d,$$

where $S_x f : t \mapsto f\left(t + x\right)$*.*

Proof. The mean measure of $\bar{\Phi}$ is, for $B \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\begin{aligned}
M_{\bar{\Phi}}(B) &= \mathbf{E}\left[\bar{\Phi}(B)\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{Z}} \Phi_k (B - X_k)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d\times\mathbb{M}(\mathbb{R}^d)} \mu(B-x)\,\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}\mu)\right] \\
&= \int_{\mathbb{R}^d\times\mathbb{M}(\mathbb{R}^d)} \mu(B-x)\, M_{\tilde{\Phi}}(\mathrm{d}x\times\mathrm{d}\mu) \\
&= \int_{\mathbb{R}^d\times\mathbb{M}(\mathbb{R}^d)} \mu(B-x)\,\tilde{p}(x,\mathrm{d}\mu)\, M_{\Phi}(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d\times\mathbb{M}(\mathbb{R}^d)} \mathbf{E}\left[\Phi_x(B-x)\right] M_{\Phi}(\mathrm{d}x) = \int_{\mathbb{R}^d} M_{\Phi_x}(B-x)\, M_{\Phi}(\mathrm{d}x)\,,
\end{aligned}$$

where the fourth equality follows from the Campbell averaging theorem 1.2.5, and the fifth equality is due to Lemma 2.2.19. The Laplace transform of $\bar{\Phi}$ is, for $f \in \mathfrak{F}_+\left(\mathbb{R}^d\right)$,

$$\begin{aligned}
\mathcal{L}_{\bar{\Phi}}(f) &= \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f\mathrm{d}\bar{\Phi}\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d} f(x)\,\Phi_k(\mathrm{d}x - X_k)\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{Z}} \tilde{f}(X_k,\Phi_k)\right)\right] = \mathcal{L}_{\tilde{\Phi}}\left(\tilde{f}\right),
\end{aligned}$$

where

$$\tilde{f}(y,\mu) = \int_{\mathbb{R}^d} f(x)\,\mu(\mathrm{d}x - y)\,.$$

It follows from Proposition 2.2.20 that

$$\mathcal{L}_{\tilde{\Phi}}\left(\tilde{f}\right) = \mathcal{L}_{\Phi}\left(\bar{f}\right),$$

where

$$\begin{aligned}
\bar{f}(y) &= -\log\left[\int_{\mathbb{M}(\mathbb{R}^d)} e^{-\tilde{f}(y,\mu)}\tilde{p}(y,\mathrm{d}\mu)\right] \\
&= -\log\left[\int_{\mathbb{M}(\mathbb{R}^d)} \exp\left(-\int_{\mathbb{R}^d} f(x)\,\mu(\mathrm{d}x - y)\right)\tilde{p}(y,\mathrm{d}\mu)\right] \\
&= -\log\left[\int_{\mathbb{M}(\mathbb{R}^d)} \exp\left(-\int_{\mathbb{R}^d} S_y f\mathrm{d}\mu\right)\tilde{p}(y,\mathrm{d}\mu)\right] = -\log\mathcal{L}_{\Phi_y}(S_y f)
\end{aligned}$$

□

Example 2.3.13. Cox cluster point process *[64]. Consider a cluster point process $\bar{\Phi}$ defined by (2.3.8). If the descendant processes Φ_x are Poisson, then we say that $\bar{\Phi}$ is a* Cox cluster point process. *Its Laplace transform is given by Equation (2.3.11) where*

$$\begin{aligned}\bar{f}(x) &= -\log \mathcal{L}_{\Phi_x}(S_x f)\\ &= \int_{\mathbb{R}^d} (1-e^{-S_x f})\mathrm{d}M_{\Phi_x}\\ &= \int_{\mathbb{R}^d} (1-e^{-f(u+x)})M_{\Phi_x}(\mathrm{d}u)\\ &= \int_{\mathbb{R}^d} (1-e^{-f(t)})M_{\Phi_x}(\mathrm{d}t-x)\,.\end{aligned}$$

General Cox point processes will be defined in Section 2.3.1 below.

Example 2.3.14. Generalized Neyman-Scott process. *Consider a cluster point process $\bar{\Phi}$ defined by (2.3.8). If the parent process Φ is Poisson, then we say that $\bar{\Phi}$ is a* Generalized Neyman-Scott process. *Its Laplace transform of $\bar{\Phi}$ follows from Equation (2.3.11)*

$$\begin{aligned}\mathcal{L}_{\bar{\Phi}}(f) &= \mathcal{L}_{\Phi}\left(\bar{f}\right)\\ &= \exp\left(-\int_{\mathbb{R}^d}(1-e^{-\bar{f}})\mathrm{d}M_{\Phi}\right)\\ &= \exp\left(-\int_{\mathbb{R}^d}\left[1-\mathcal{L}_{\Phi_x}(S_x f)\right]M_{\Phi}(\mathrm{d}x)\right).\end{aligned}$$

I.i.d. cluster point processes

We define now a subclass of cluster point processes where the descendant processes are i.i.d. marks.

Definition 2.3.15. *Let $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k,\Phi_k)}$ be an i.i.d. marked point process on $\mathbb{R}^d$ with marks Φ_k in $\mathbb{M}\left(\mathbb{R}^d\right)$, let $\Phi = \sum_{k\in\mathbb{Z}}\delta_{X_k}$ be the associated ground process, and assume that*

$$\int_{\mathbb{R}^d} \mathbf{P}\left(\Phi_0(B-x)\neq 0\right)M_{\Phi}(\mathrm{d}x) < \infty, \quad B\in\mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right). \tag{2.3.12}$$

Then $\bar{\Phi}$ defined by

$$\bar{\Phi}(B) = \sum_{k\in\mathbb{Z}} \Phi_k(B-X_k)\,, \quad B\in\mathcal{B}\left(\mathbb{R}^d\right) \tag{2.3.13}$$

is called an i.i.d. cluster point process *on $\mathbb{R}^d$ (the fact that $\bar{\Phi}$ is indeed a point process is proved in Proposition 2.3.16 below).*

Proposition 2.3.16. *With notations and conditions in Definition 2.3.15 (in particular Condition (2.3.12)), the following results hold.*

(i) $\bar{\Phi}$ *defined by (2.3.13) is a point process.*

(ii) Its mean measure is

$$M_{\bar{\Phi}}(B) = \int_{\mathbb{R}^d} M_{\Phi_0}(B - x)\, M_{\Phi}(\mathrm{d}x), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right).$$

(iii) Its Laplace transform is

$$\mathcal{L}_{\bar{\Phi}}(f) = \mathcal{L}_{\Phi}\left(\bar{f}\right), \quad f \in \mathfrak{F}_+\left(\mathbb{R}^d\right), \tag{2.3.14}$$

where

$$\bar{f}(x) = -\log \mathcal{L}_{\Phi_0}(S_x f), \quad x \in \mathbb{R}^d,$$

where $S_x f : t \mapsto f(t + x)$.

Proof. (i) In order to simplify the notation, we introduce for all $\mu = \sum_{k\in\mathbb{Z}} \delta_{x_k} \in \mathbb{M}\left(\mathbb{R}^d\right)$,

$$\bar{\Phi}_\mu(B) = \sum_{k\in\mathbb{Z}} \Phi_k(B - x_k), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right),$$

which may be viewed as a superposition of the point processes $\{\Phi_k(\cdot - x_k)\}_{k\in\mathbb{Z}}$. It follows from Proposition 2.2.1 that $\bar{\Phi}_\mu$ is a point process on $\mathbb{R}^d$ when

$$\sum_{k\in\mathbb{Z}} \mathbf{P}(\Phi_k(B - x_k) \neq 0) < \infty,$$

for all $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$. Thus, if for all $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$,

$$\sum_{k\in\mathbb{Z}} \mathbf{P}(\Phi_0(B - x_k) \neq 0) < \infty, \quad \text{for } \mathbf{P}_\Phi\text{-almost all } \mu = \sum_{k\in\mathbb{Z}} \delta_{x_k}, \tag{2.3.15}$$

then $\bar{\Phi}_\mu$ is a point process on $\mathbb{R}^d$ for $\mathbf{P}_\Phi$-almost all μ. In this case, for all $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$,

$$\begin{aligned}
\mathbf{P}\left(\bar{\Phi}(B) < \infty\right) &= \mathbf{E}\left[\mathbf{P}\left(\bar{\Phi}(B) < \infty \mid \Phi\right)\right] \\
&= \int_{\mathbb{M}(\mathbb{R}^d)} \mathbf{P}\left(\bar{\Phi}_\mu(B) < \infty\right) \mathbf{P}_\Phi(\mathrm{d}\mu) \\
&= \int_{\mathbb{M}(\mathbb{R}^d)} 1 \times \mathbf{P}_\Phi(\mathrm{d}\mu) = 1,
\end{aligned}$$

and therefore $\bar{\Phi}$ is a point process (checking the sigma-additivity property is straightforward). It remains to show that (2.3.12) implies (2.3.15). For all $\mu = \sum_{k\in\mathbb{Z}} \delta_{x_k} \in \mathbb{M}\left(\mathbb{R}^d\right)$, let

$$g(\mu) := \sum_{k\in\mathbb{Z}} \mathbf{P}(\Phi_0(B - x_k) \neq 0) = \int_{\mathbb{R}^d} \mathbf{P}(\Phi_0(B - x) \neq 0)\, \mu(\mathrm{d}x).$$

It follows from the Campbell averaging theorem 1.2.5 that

$$\int g(\mu)\,\mathbf{P}_\Phi(\mathrm{d}\mu) = \int_{\mathbb{R}^d} \mathbf{P}(\Phi_0(B-x)\neq 0)\, M_\Phi(\mathrm{d}x)\,.$$

Thus (2.3.12) says that the above quantity is finite, which implies that $g(\mu)$ is finite for $\mathbf{P}_\Phi$-almost all μ, which is precisely (2.3.15). (ii) The mean measure of the cluster point process is

$$\begin{aligned}
M_{\bar\Phi}(B) &= \mathbf{E}\left[\sum_{k\in\mathbb{Z}} \Phi_k(B-X_k)\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\sum_{k\in\mathbb{Z}} \Phi_k(B-X_k)\middle|\,\Phi\right]\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{Z}} \mathbf{E}\left[\Phi_k(B-X_k)|\,X_k\right]\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{Z}} M_{\Phi_0}(B-X_k)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} M_{\Phi_0}(B-x)\,\Phi(\mathrm{d}x)\right] = \int_{\mathbb{R}^d} M_{\Phi_0}(B-x)\, M_\Phi(\mathrm{d}x)\,,
\end{aligned}$$

where $B \in \mathcal{B}(\mathbb{R}^d)$, and where the last equality follows from the Campbell averaging theorem 1.2.5. (iii) The Laplace transform of $\bar\Phi$ given Φ equals, for $f\in\mathfrak{F}_+(\mathbb{R}^d)$,

$$\begin{aligned}
\mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f\mathrm{d}\bar\Phi\right)\middle|\,\Phi\right] &= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d} f(x)\,\Phi_k(\mathrm{d}x-X_k)\right)\middle|\,\Phi\right] \\
&= \mathbf{E}\left[\prod_{k\in\mathbb{Z}}\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi_k(\mathrm{d}x-X_k)\right)\middle|\,\Phi\right] \\
&= \prod_{k\in\mathbb{Z}}\mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi_k(\mathrm{d}x-X_k)\right)\middle|\,X_k\right] \\
&= \prod_{k\in\mathbb{Z}}\mathcal{L}_{\Phi_k}(S_{X_k}f)\,,
\end{aligned}$$

then

$$\begin{aligned}
\mathcal{L}_{\bar\Phi}(f) &= \mathbf{E}\left[\prod_{k\in\mathbb{Z}}\mathcal{L}_{\Phi_k}(S_{X_k}f)\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{Z}} -\log\mathcal{L}_{\Phi_k}(S_{X_k}f)\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{Z}} -\log\mathcal{L}_{\Phi_0}(S_{X_k}f)\right)\right] = \mathcal{L}_\Phi(\bar f)\,,
\end{aligned}$$

where $\bar f(x) = -\log\mathcal{L}_{\Phi_0}(S_x f)$. $\square$

Example 2.3.17. Matérn cluster point process‘. *Consider an i.i.d. cluster point process $\bar{\Phi}$ defined by (2.3.13). Assume that the parent process Φ is a homogeneous Poisson point processon $\mathbb{R}^2$ with intensity λ, and that the descendant process Φ_0 is a Poisson point process on $\mathbb{R}^2$ with intensity measure*

$$M_{\Phi_0}(\mathrm{d}y) = \frac{\mu}{\pi r^2} 1_{B(0,r)}(y)\,\mathrm{d}y.$$

Then the mean measure of the cluster point process $\bar{\Phi}$ is, for $B \in \mathcal{B}(\mathbb{R}^2)$,

$$\begin{aligned}
M_{\bar{\Phi}}(B) &= \lambda \int_{\mathbb{R}^2} M_{\Phi_0}(B - x)\,\mathrm{d}x \\
&= \frac{\lambda\mu}{\pi r^2} \int_{\mathbb{R}^2} \left(\int_B 1_{B(0,r)}(y - x)\,\mathrm{d}y \right) \mathrm{d}x \\
&= \frac{\lambda\mu}{\pi r^2} \int_B \left(\int_{\mathbb{R}^2} 1_{B(0,r)}(y - x)\,\mathrm{d}x \right) \mathrm{d}y \\
&= \frac{\lambda\mu}{\pi r^2} \int_B \pi r^2 \mathrm{d}y = \lambda\mu\,|B|\,.
\end{aligned}$$

In the particular case of a stationary ground process, we have the following result.

Example 2.3.18. Stationary i.i.d. cluster process. *Let $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k,\Phi_k)}$ be an i.i.d. marked point process on $\mathbb{R}^d$ with marks Φ_k in $\mathbb{M}(\mathbb{R}^d)$. Assume that $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ is stationary with mean measure $\lambda\mathrm{d}x$ for some $\lambda \in \mathbb{R}_+$ and that $M_{\Phi_0}(\mathbb{R}^d) < \infty$. Then $\bar{\Phi}$ defined by*

$$\bar{\Phi}(B) = \sum_{k\in\mathbb{Z}} \Phi_k(B - X_k), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

is a well defined point process. Moreover, it is stationary with mean measure $\lambda M_{\Phi_0}(\mathbb{R}^d)\,\mathrm{d}x$.

Indeed, for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathbf{P}(\Phi_0(B - x) \neq 0)\, M_\Phi(\mathrm{d}x) &= \mathbf{E}\left[\int_{\mathbb{R}^d} 1\{\Phi_0(B - x) \neq 0\}\, M_\Phi(\mathrm{d}x) \right] \\
&\leq \mathbf{E}\left[\int_{\mathbb{R}^d} \Phi_0(B - x)\, M_\Phi(\mathrm{d}x) \right] \\
&= \lambda \mathbf{E}\left[\int_{\mathbb{R}^d} \Phi_0(B - x)\,\mathrm{d}x \right] \\
&= \lambda \mathbf{E}\left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbf{1}\{s \in B - x\}\, \Phi_0(\mathrm{d}s) \right) \mathrm{d}x \right] \\
&= \lambda \mathbf{E}\left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbf{1}\{s \in B - x\}\,\mathrm{d}x \right) \Phi_0(\mathrm{d}s) \right] \\
&= \lambda\,|B|\,\mathbf{E}\left[\int_{\mathbb{R}^d} \Phi_0(\mathrm{d}s) \right] = \lambda\,|B|\,M_{\Phi_0}(\mathbb{R}^d) < \infty.
\end{aligned}$$

Then Condition (2.3.12) is fulfilled. It follows from Proposition 2.3.16 that $\bar{\Phi}$ is a well defined point process and that its mean measure equals

$$\begin{aligned} M_{\bar{\Phi}}(B) &= \int_{\mathbb{R}^d} M_{\Phi_0}(B-x)\, M_{\Phi}(\mathrm{d}x) \\ &= \lambda \int_{\mathbb{R}^d} M_{\Phi_0}(B-x)\,\mathrm{d}x \\ &= \lambda \mathbf{E}\left[\int_{\mathbb{R}^d} \Phi_0(B-x)\,\mathrm{d}x\right] = \lambda |B|\, M_{\Phi_0}\left(\mathbb{R}^d\right), \end{aligned}$$

where the last equality is proved in the previous computations.

It remains to prove that $\bar{\Phi}$ is stationary. Indeed, let $B_1, \ldots, B_n \in \mathcal{B}\left(\mathbb{R}^d\right)$ and $t \in \mathbb{R}^d$ and note that

$$\bar{\Phi}(B_i + t) = \sum_{k \in \mathbb{Z}} \Phi_k(B_i - (X_k - t)), \quad i \in \{1, \ldots, n\}.$$

Observe that $\Phi' = \sum_{k\in\mathbb{Z}} \delta_{X_k - t}$ has the same probability distribution as Φ by the stationarity of the latter. Then $\left(\bar{\Phi}(B_1+t), \ldots, \bar{\Phi}(B_n+t)\right)$ has the same probability distribution as $\left(\bar{\Phi}(B_1), \ldots, \bar{\Phi}(B_n)\right)$, which proves that $\bar{\Phi}$ is stationary.

Example 2.3.19. Stationary renewal cluster point process *[28]. Let $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a stationary point process on $\mathbb{R}$ with mean measure $\lambda \mathrm{d}x$ for some $\lambda \in \mathbb{R}_+$. Moreover, for each $k \in \mathbb{Z}$, assume given an i.i.d. sequence of positive valued random variables $\left\{S_n^k\right\}_{n\in\mathbb{N}^*}$ and an integer-valued random variable L_k independent of $\{S_n^k\}_{n\in\mathbb{N}^*}$. Assume that $\left\{\left(L_k, \{S_n^k\}_{n\in\mathbb{N}^*}\right)\right\}_{k\in\mathbb{Z}}$ are i.i.d. and let*

$$T_0^k = 0, \quad T_n^k = S_1^k + \ldots + S_n^k, \quad n \in \mathbb{N}^*, k \in \mathbb{Z}$$

and

$$\Phi_k = \sum_{n=0}^{L_k} \delta_{T_n^k}, \quad k \in \mathbb{Z}.$$

Assume that $\mathbf{E}[L_0] < \infty$. Then the cluster point process

$$\bar{\Phi} = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{L_n} \delta_{X_k + T_n^k}$$

has mean measure $\lambda \mathbf{E}[L_0]\mathrm{d}x$. This process is called a renewal cluster point process.

Generating function

Lemma 2.3.20. Generating function for cluster point processes. *Let $\bar{\Phi}$ be a cluster point process on $\mathbb{R}^d$ with parent process Φ and descendant processes $\{\Phi_x\}_{x\in\mathbb{R}^d}$. Its generating function (cf. Definition 1.6.18) is given by*

$$\mathcal{G}_{\bar{\Phi}}(v) = \mathcal{G}_{\Phi}\left(\mathcal{G}_{\Phi_x}(S_x v)\right), \quad v \in \mathcal{V}\left(\mathbb{R}^d\right), \tag{2.3.16}$$

(which may be seen as the analogous of (13.A.18) for compound random variables).

Proof. The Laplace transform of $\bar{\Phi}$ is given by (2.3.11), then its generating function follows from (1.6.9)

$$\begin{aligned}\mathcal{G}_{\bar{\Phi}}(v) &= \mathcal{L}_{\bar{\Phi}}(-\log v) \\ &= \mathcal{L}_{\Phi}\left(-\log \mathcal{L}_{\Phi_x}\left(S_x(-\log v)\right)\right) \\ &= \mathcal{L}_{\Phi}\left(-\log \mathcal{L}_{\Phi_x}\left(-\log S_x v\right)\right) \\ &= \mathcal{G}_{\Phi}\left(\mathcal{G}_{\Phi_x}\left(S_x v\right)\right), \quad v \in \mathcal{V}\left(\mathbb{R}^d\right).\end{aligned}$$

□

Example 2.3.21. Compound Poisson point process. *Let $\bar{\Phi}$ be a cluster point process on $\mathbb{R}^d$ with parent process Φ and descendant processes $\{\Phi_x\}_{x\in\mathbb{R}^d}$. Assume that the parent process Φ is Poisson ($\bar{\Phi}$ is then called a Poisson cluster point process). It follows from Equations (2.3.16) and (2.1.3) that*

$$\mathcal{G}_{\bar{\Phi}}(v) = \exp\left[-\int_{\mathbb{R}^d}\left(1-\mathcal{G}_{\Phi_x}\left(S_x v\right)\right) M_{\Phi}(\mathrm{d}x)\right].$$

Assume moreover that $\Phi_x = Z_x\delta_0$ where Z_x is an integer valued random variable ($\bar{\Phi}$ is then called a compound Poisson *point process) representing the* cluster size, *then*

$$\mathcal{G}_{\Phi_x}(v) = \mathbf{E}\left[\prod_{Y\in\Phi_x} v(Y)\right] = \mathbf{E}\left[v(0)^{Z_x}\right] = \mathcal{G}_{Z_x}(v(0)).$$

Thus

$$\mathcal{G}_{\bar{\Phi}}(v) = \exp\left[-\int_{\mathbb{R}^d}\left(1-\mathcal{G}_{Z_x}(v(x))\right) M_{\Phi}(\mathrm{d}x)\right], \quad v \in \mathcal{V}\left(\mathbb{R}^d\right). \tag{2.3.17}$$

Example 2.3.22. Compound Poisson leading to negative binomial distribution. *Let $\bar{\Phi}$ be a compound Poisson point process as in Example 2.3.21 with Poisson parent process Φ and descendant processes $\{\Phi_x\}_{x\in\mathbb{R}^d}$ with $\Phi_x = Z_x\delta_0$ where the cluster size Z_x has the logarithmic probability distribution with parameter $\rho \in (0,1)$; that is*

$$\mathbf{P}(Z_x = n) = -\log(1-\rho)\frac{\rho^n}{n}, \quad n \in \mathbb{N}^*,$$

or, equivalently,

$$\mathcal{G}_{Z_x}(y) = \frac{\log(1-\rho x)}{\log(1-\rho)}.$$

If follows from Example 13.A.29 that, for all $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$, $\bar{\Phi}(B)$ has the negative Binomial *probability distribution with parameters*

$$\alpha = -\frac{M_{\Phi}(B)}{\log(1-\rho)}$$

and ρ. Then $\bar{\Phi}$ is called a negative Binomial point process. *Its generating function follows from (2.3.17)*

$$\begin{aligned}\mathcal{G}_{\bar{\Phi}}(v) &= \exp\left[-\int_{\mathbb{R}^d}\left(1-\mathcal{G}_{Z_x}(v(x))\right)M_\Phi(\mathrm{d}x)\right] \\ &= \exp\left[-\int_{\mathbb{R}^d}\left(1-\frac{\log(1-\rho v(x))}{\log(1-\rho)}\right)M_\Phi(\mathrm{d}x)\right], \quad v\in\mathcal{V}\left(\mathbb{R}^d\right).\end{aligned}$$

2.3.4 Powers and factorial powers

Given a point process Φ, recall the notation Φ^n of its n-th power and $\Phi^{(n)}$ of its n-th factorial power ($n \in \mathbb{N}^*$).

Lemma 2.3.23. Moments measures and simplicity. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$.*

(i) If Φ is simple, then $M_{\Phi^{(n)}}\left(\{(x_1,\ldots,x_n)\in\mathbb{G}^n : x_i = x_j \text{ for some } i\neq j\}\right) = 0$.

(ii) Φ is simple iff $M_{\Phi^{(2)}}\left(\{(x,x) : x\in\mathbb{G}\}\right) = 0$.

(iii) If Φ is simple, then the restriction of M_{Φ^n} to the set

$$\mathbb{G}^{(n)} = \{(x_1,\ldots,x_n)\in\mathbb{G}^n : x_i\neq x_j \text{ for any } i\neq j\}$$

is equal to $M_{\Phi^{(n)}}$.

Proof. (i) Let

$$A_n = \{(x_1,\ldots,x_n)\in\mathbb{G}^n : x_i = x_j \text{ for some } i\neq j\}.$$

Since Φ is simple, then $\Phi = \sum_{j\in\mathbb{Z}}\delta_{X_j}$ where the X_j's are almost surely pairwise distinct. Therefore,

$$\begin{aligned}M_{\Phi^{(n)}}(A_n) &= \mathbf{E}\left[\Phi^{(n)}(A_n)\right] \\ &= \mathbf{E}\left[\sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}}\mathbf{1}\{(X_{j_1},\ldots,X_{j_n})\in A_n\}\right] = 0,\end{aligned}$$

where $\mathbb{Z}^{(n)} = \{(j_1,\ldots,j_n)\in\mathbb{Z}^n : j_k\neq j_l, \text{ for any } l\neq k\}$.. (ii) Necessity follows from (i). We now prove sufficiency. Let Φ be a point process on $\mathbb{G}$ such that $M_{\Phi^{(2)}}\left(\{(x,x) : x\in\mathbb{G}\}\right) = 0$. We have to prove the simplicity condition (1.6.1); or, equivalently,

$$\mathbf{P}\left(\exists x\in\mathbb{G} : \Phi(\{x\})\geq 2\right) = 0.$$

Indeed,

$$\begin{aligned}
\mathbf{P}\left(\exists x \in \mathbb{G} : \Phi(\{x\}) \geq 2\right) &= \mathbf{P}\left(\exists x \in \mathbb{G} : \left(\Phi - \delta_x\right)\left(\{x\}\right) \geq 1\right) \\
&= \mathbf{E}\left[\mathbf{1}\left\{\exists x \in \mathbb{G} : \left(\Phi - \delta_x\right)\left(\{x\}\right) \geq 1\right\}\right] \\
&\leq \mathbf{E}\left[\int_{\mathbb{G}} \mathbf{1}\left\{\left(\Phi - \delta_x\right)\left(\{x\}\right) \geq 1\right\} \ \Phi\left(\mathrm{d}x\right)\right] \\
&\leq \mathbf{E}\left[\int_{\mathbb{G}} \left(\Phi - \delta_x\right)\left(\{x\}\right) \Phi\left(\mathrm{d}x\right)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}^2} \mathbf{1}\left\{x = y\right\}\left(\Phi - \delta_x\right)\left(\mathrm{d}y\right) \ \Phi\left(\mathrm{d}x\right)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}^2} \mathbf{1}\left\{x = y\right\}\left(\mathrm{d}y\right) \ \Phi^{(2)}\left(\mathrm{d}x \times \mathrm{d}y\right)\right] \\
&= M_{\Phi^{(2)}}\left(\{(x, x) : x \in \mathbb{G}\}\right) = 0,
\end{aligned}$$

where the last but one equality follows from (14.E.7) (iii) Since Φ is simple, then $\Phi = \sum_{j\in\mathbb{Z}} \delta_{X_j}$ where the X_j's are almost surely pairwise distinct. Thus

$$\Phi^{(n)} = \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} \delta_{(X_{j_1},\ldots,X_{j_n})} = \sum_{\left(X_{j_1},\ldots,X_{j_n}\right)\in\mathbb{G}^{(n)}} \delta_{(X_{j_1},\ldots,X_{j_n})},$$

and

$$\Phi^n = \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^n} \delta_{(X_{j_1},\ldots,X_{j_n})}.$$

For any $B \in \mathcal{B}\left(\mathbb{G}^n\right)$ such that $B \subset \mathbb{G}^{(n)}$, we have

$$\begin{aligned}
\Phi^n\left(B\right) &= \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^n} \mathbf{1}\left\{\left(X_{j_1}, \ldots, X_{j_n}\right) \in B\right\} \\
&= \sum_{\left(X_{j_1},\ldots,X_{j_n}\right)\in\mathbb{G}^{(n)}} \mathbf{1}\left\{\left(X_{j_1}, \ldots, X_{j_n}\right) \in B\right\} = \Phi^{(n)}\left(B\right).
\end{aligned}$$

□

Proposition 2.3.24. Factorial moment measures of thinning processes. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$, $p : \mathbb{G} \to [0, 1]$ be some measurable function, and let $\tilde{\Phi}$ be the thinning of Φ with retention function p. Then*

$$M_{\tilde{\Phi}^{(n)}}\left(B\right) = \int_B p\left(x_1\right)\ldots p\left(x_n\right) M_{\Phi^{(n)}}\left(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n\right), \quad B \in \mathcal{B}\left(\mathbb{G}^n\right).$$

Proof. Let $\Phi = \sum_{k\in\mathbb{N}} \delta_{X_k}$. Recall that $\tilde{\Phi}$ is defined as

$$\tilde{\Phi} = \sum_{k\in\mathbb{N}} \mathbf{1}\left\{U_k \leq p\left(X_k\right)\right\} \delta_{X_k},$$

where $U_0, U_1, \ldots$ is a sequence of i.i.d. random variables independent of Φ, uniformly distributed in $[0,1]$. Then for any $B \in \mathcal{B}(\mathbb{G}^n)$,

$$\tilde{\Phi}^{(n)}(B) = \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} \mathbf{1}\{U_{j_1} \le p(X_{j_1}), \ldots, U_{j_n} \le p(X_{j_n})\}\, \delta_{(X_{j_1},\ldots,X_{j_n})}.$$

Thus

$$\begin{aligned}
\mathbf{E}\left[\tilde{\Phi}^{(n)}(B)\right] &= \mathbf{E}\left[\mathbf{E}\left[\tilde{\Phi}^{(n)}(B) \mid \Phi\right]\right] \\
&= \mathbf{E}\left[\sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} p(X_{j_1})\ldots p(X_{j_n})\, \mathbf{1}\{(X_{j_1},\ldots,X_{j_n}) \in B\}\right] \\
&= \mathbf{E}\left[\int_B p(x_1)\ldots p(x_n)\, \Phi^{(n)}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n)\right] \\
&= \int_B p(x_1)\ldots p(x_n)\, M_{\Phi^{(n)}}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n),
\end{aligned}$$

where the last equality is due to the Campbell averaging formula. □

Proposition 2.3.25. Poisson factorial moment measures. *For a Poisson point process Φ on a l.c.s.h. space $\mathbb{G}$,*

$$M_{\Phi^{(n)}} = (M_\Phi)^n, \tag{2.3.18}$$

that is the n-th factorial moment measure equals the n-th power of the mean measure.

Proof. For any $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,

$$\begin{aligned}
M_{\Phi^{(n)}}(B^n) &= \mathbf{E}\left[\Phi^{(n)}(B^n)\right] \\
&= \mathbf{E}\left[\Phi(B)(\Phi(B)-1)\ldots(\Phi(B)-n+1)^+\right] = g^{(n)}(1),
\end{aligned}$$

where g is the generation function of $\Phi(B)$; that is $g(z) = \mathbf{E}\left[z^{\Phi(B)}\right]$. Since $\Phi(B)$ is a Poisson random variable $g(z) = e^{-M_\Phi(B)(1-z)}$, we may continue the above equalities as follows

$$M_{\Phi^{(n)}}(B^n) = [M_\Phi(B)]^n = (M_\Phi)^n(B^n).$$

Consider now pairwise disjoint sets $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G})$, and $n, n_1, \ldots, n_k \in \mathbb{N}$

such that $n_1 + \cdots + n_k = n$. Writing $\Phi = \sum_j \delta_{X_j}$, then

$$
\begin{aligned}
&M_{\Phi^{(n)}}\left(B_1^{n_1} \times \ldots \times B_k^{n_k}\right)\\
&= \mathbf{E}\left[\Phi^{(n)}\left(B_1^{n_1} \times \ldots \times B_k^{n_k}\right)\right]\\
&= \mathbf{E}\left[\sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} \mathbf{1}\left\{X_{j_1},\ldots,X_{j_{n_1}} \in B_1^{n_1}\right\}\ldots\mathbf{1}\left\{X_{j_{n-n_k+1}},\ldots,X_{j_n} \in B_k^{n_k}\right\}\right]\\
&= \mathbf{E}\left[\sum_{\left(j_1,\ldots,j_{n_1}\right)\in\mathbb{Z}^{(n_1)}} \mathbf{1}\left\{X_{j_1},\ldots,X_{j_{n_1}} \in B_1^{n_1}\right\}\ldots\sum_{\left(j_{n-n_k+1},\ldots,j_n\right)\in\mathbb{Z}^{(n_k)}}\mathbf{1}\left\{X_{j_{n-n_k+1}},\ldots,X_{j_n} \in B_k^{n_k}\right\}\right]\\
&= \mathbf{E}\left[\prod_{i=1}^{k}\Phi^{(n_i)}\left(B_i^{n_i}\right)\right]\\
&= \prod_{i=1}^{k}\mathbf{E}\left[\Phi^{(n_i)}\left(B_i^{n_i}\right)\right]\\
&= \prod_{i=1}^{k}\left(M_\Phi\right)^{n_i}\left(B_i^{n_i}\right) = \left(M_\Phi\right)^{n}\left(B_1^{n_1}\times\ldots\times B_k^{n_k}\right).
\end{aligned}
$$

Recall that the product measure $(M_\Phi)^n$ is characterized by its value on rectangles (i.e., sets of the forms $A_1 \times \ldots \times A_k$); cf. [44, Theorem 35.B]. Then

$$M_{\Phi^{(n)}} = \left(M_\Phi\right)^n .$$

□

Example 2.3.26. *Let Φ be a Poisson point process on $\mathbb{R}^d$, then*

$$
\begin{aligned}
M_{\Phi^2}(A\times B) &= M_{\Phi^{(2)}}(A\times B) + M_\Phi(A\cap B)\\
&= \left(M_\Phi\right)^2(A\times B) + M_\Phi(A\cap B)\\
&= M_\Phi(A)\,M_\Phi(B) + M_\Phi(A\cap B),
\end{aligned}
$$

where the first equality is due to (14.E.5), the second one follows from Proposition 2.3.25, and the third equality is due to (14.E.2). For instance, if Φ is a homogeneous Poisson point process with intensity λ, then

$$M_{\Phi^2}(A\times B) = \lambda^2\left|A\right|\left|B\right| + \lambda\left|A\cap B\right|,$$

where the first term is proportional to the Lebesgue measure on $\mathbb{R}^d\times\mathbb{R}^d$, whereas the second term corresponds to a positive mass on the diagonal of $\mathbb{R}^d\times\mathbb{R}^d$.

Example 2.3.27. Cox point process moment measures. *The n-th factorial moment measure of a Cox point process Φ on a l.c.s.h. space $\mathbb{G}$ equals the n-th* *moment measure of the directing measure* Λ. *Indeed, for* $B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned} M_{\Phi^{(n)}}(B) &= \mathbf{E}\left[\Phi^{(n)}(B)\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\Phi_{\Lambda}^{(n)}(B) \mid \Lambda\right]\right] \\ &= \mathbf{E}\left[\Lambda^{n}(B)\right] = M_{\Lambda^{n}}(B), \end{aligned}$$

where the third equality follows from Proposition 2.3.25. If follows from Equation (1.6.11) that

$$\begin{aligned} \operatorname{Var}(\Phi(B)) &= M_{\Phi^{(2)}}(B \times B) - M_{\Phi}(B)^{2} + M_{\Phi}(B) \\ &= M_{\Lambda^{2}}(B \times B) - M_{\Lambda}(B)^{2} + M_{\Lambda}(B) \\ &= \mathbf{E}\left[\Lambda(B)^{2}\right] - \mathbf{E}\left[\Lambda(B)\right]^{2} + M_{\Lambda}(B) \\ &= \operatorname{Var}(\Lambda(B)) + M_{\Lambda}(B) \geq M_{\Lambda}(B). \end{aligned}$$

Note that for a Poisson point process $\tilde{\Phi}$ *with intensity measure* $M_{\tilde{\Phi}}(B) = M_{\Lambda}(B)$, *we have* $\operatorname{Var}\left(\tilde{\Phi}(B)\right) = M_{\tilde{\Phi}}(B)$. *We deduce from the above inequality that* $\operatorname{Var}(\Phi(B)) \geq \operatorname{Var}\left(\tilde{\Phi}(B)\right)$; *thus a Cox point process is overdispersed (i.e., has greater variability) than a Poisson point process with the same mean measure.*

It follows from (14.E.5) that for all $A, B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned} M_{\Phi^{2}}(A \times B) &= M_{\Phi^{(2)}}(A \times B) + M_{\Phi}(A \cap B) \\ &= M_{\Lambda^{2}}(A \times B) + M_{\Lambda}(A \cap B). \end{aligned} \tag{2.3.19}$$

In particular, if A *and* B *are disjoint, then*

$$M_{\Phi^{2}}(A \times B) = M_{\Phi^{(2)}}(A \times B) + M_{\Phi}(A \cap B) = M_{\Lambda^{2}}(A \times B).$$

Thus a Cox point process does not have in general the independence property.

2.4 Shot-noise

Definition 2.4.1. *Let* Φ *be a random measure on a l.c.s.h. space* $\mathbb{G}$, $k \in \mathbb{N}^*$ *and* $f : \mathbb{G} \to \bar{\mathbb{C}}^k$ *be a measurable function which is either real nonnegative or in* $L^1_{\bar{\mathbb{C}}^k}(M_\Phi, \mathbb{G})$, *then*

$$\Phi(f) = \int_{\mathbb{G}} f(x)\, \Phi(\mathrm{d}x)$$

is called a cumulative shot-noise.

By Theorem 1.2.5, a cumulative shot-noise is a well defined random variable and

$$\mathbf{E}[\Phi(f)] = \int_{\mathbb{G}} f(x)\, M_\Phi(\mathrm{d}x).$$

Definition 2.4.2. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$, $k \in \mathbb{N}^*$ and $f : \mathbb{G} \to \bar{\mathbb{R}}^k$ be a measurable function, the* extremal shot-noise *is defined by*

$$V(f) := \sup_{X \in \Phi} f(X)$$

coordinatewise.

2.4.1 Laplace transform

Proposition 2.4.3. Cumulative shot-noise Laplace transform. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ and $f : \mathbb{G} \to \bar{\mathbb{R}}_+^k$ be measurable, then the Laplace transform of the cumulative shot-noise $\Phi(f) = \sum_{X \in \Phi} f(X)$ is given*

$$\mathcal{L}_{\Phi(f)}(t) = \mathcal{L}_{\Phi}(tf), \quad t \in \mathbb{R}_+^k,$$

where the inner product of two vectors $u, v \in \mathbb{R}^k$ is denoted by uv. If Φ is a Poisson point process, then

$$\mathcal{L}_{\Phi(f)}(t) = \exp\left[-\int_{\mathbb{G}} \left(1 - e^{-tf(x)}\right) M_{\Phi}(\mathrm{d}x)\right], \quad t \in \mathbb{R}_+^k.$$

If moreover $k = 1$, then

$$\mathbf{P}(\Phi(f) = 0) = \exp\left(-\int_{\mathbb{G}} \mathbf{1}\{f(x) > 0\} M_{\Phi}(\mathrm{d}x)\right).$$

Proof. By the linearity of the mapping $f \mapsto \Phi(f)$, we get

$$\mathbf{E}\left[e^{-t\Phi(f)}\right] = \mathbf{E}\left[e^{-\Phi(tf(x))}\right] = \mathcal{L}_{\Phi}(tf).$$

If Φ is a Poisson point process, then the expression (2.1.1) of its Laplace transform shows the expression of $\mathcal{L}_{\Phi(f)}(t)$. Assume moreover that $k = 1$. Noting that for all nonnegative random variables X,

$$\mathbf{P}(X = 0) = \lim_{t \uparrow \infty} \mathbf{E}\left[e^{-tX}\right],$$

we get

$$\begin{aligned}
\mathbf{P}(\Phi(f) = 0) &= \lim_{t \uparrow \infty} \mathbf{E}\left[e^{-t\Phi(f)}\right] \\
&= \exp\left\{-\int_{\mathbb{G}} \left(1 - \lim_{t \uparrow \infty} \mathbf{E}\left[e^{-tf(x)}\right]\right) M_{\Phi}(\mathrm{d}x)\right\} \\
&= \exp\left(-\int_{\mathbb{G}} \left(1 - \mathbf{1}\{f(x) = 0\}\right) M_{\Phi}(\mathrm{d}x)\right) \\
&= \exp\left(-\int_{\mathbb{G}} \mathbf{1}\{f(x) > 0\} M_{\Phi}(\mathrm{d}x)\right).
\end{aligned}$$

□

Proposition 2.4.4. Extremal shot-noise distribution. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and let $f : \mathbb{G} \to \bar{\mathbb{R}}^k$ be measurable, then the cumulative distribution function of the extremal shot-noise $V(f) = \sup_{X \in \Phi} f(X)$ is given by*

$$\mathbf{P}\left(V_j(f) \le u_j, \forall j = 1, \ldots, k\right) = \mathcal{L}_\Phi\left(\tilde{f}\right),$$

where

$$\tilde{f}(x) = -\log\left(\prod_{j=1}^{k} \mathbf{1}\left\{f_j(x) \le u_j\right\}\right). \tag{2.4.1}$$

In this relation, we adopt the convention $\exp(-\infty) = 0$*. If Φ is Poisson with intensity measure Λ, then*

$$\mathbf{P}\left(V_j(f) \le u_j, \forall j = 1, \ldots, k\right) = \exp\left[-\int_{\mathbb{G}}\left(1 - \prod_{j=1}^{k} \mathbf{1}\left\{f_j(x) \le u_j\right\}\right)\Lambda(\mathrm{d}x)\right].$$

In the particular case $k = 1$,

$$\mathbf{P}\left(V(f) \le u\right) = \exp\left[-\int_{\mathbb{G}} \mathbf{1}\left\{f(x) > u\right\}\Lambda(\mathrm{d}x)\right].$$

Proof.

$$\begin{aligned}
\mathbf{P}\left(V_j(f) \le u_j, \forall j = 1, \ldots, k\right) &= \mathbf{E}\left[\mathbf{1}\left\{V_j(f) \le u_j, \forall j = 1, \ldots, k\right\}\right] \\
&= \mathbf{E}\left[\prod_{j=1}^{k} \mathbf{1}\left\{V_j(f) \le u_j\right\}\right] \\
&= \mathbf{E}\left[\prod_{j=1}^{k} \mathbf{1}\left\{f(X) \le u_j, \forall X \in \Phi\right\}\right] \\
&= \mathbf{E}\left[\prod_{j=1}^{k} \prod_{X \in \Phi} \mathbf{1}\left\{f_j(X) \le u_j\right\}\right] \\
&= \mathbf{E}\left[\prod_{X \in \Phi} \prod_{j=1}^{k} \mathbf{1}\left\{f_j(X) \le u_j\right\}\right] \\
&= \mathbf{E}\left[\exp \sum_{X \in \Phi} \log\left(\prod_{j=1}^{k} \mathbf{1}\left\{f_j(X) \le u_j\right\}\right)\right] \\
&= \mathcal{L}_\Phi\left(\tilde{f}\right).
\end{aligned}$$

The result for Poisson follows from the expression of its Laplace transform. $\square$

Proposition 2.4.5. Joint probability distribution of extremal and cumulative shot-noise. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and $f : \mathbb{G} \to \bar{\mathbb{R}}_+^k$ be measurable, then the joint probability distribution of the extremal shot-noise*

$V(f) = \sup_{X\in\Phi} f(X)$ and the additive noise $\Phi(f) = \sum_{X\in\Phi} f(X)$ is given by, for all $u, z \in \mathbb{R}_+^k$,

$$\mathbf{E}\left[\mathbf{1}\left\{V_j(f) \le u_j, \forall j = 1,\dots,k\right\} e^{-z\Phi(f)}\right] = \mathcal{L}_\Phi\left(\tilde{f} - zf\right),$$

where $\tilde{f}$ is given by (2.4.1). If Φ is Poisson with intensity measure Λ, then

$$\begin{aligned}&\mathbf{E}\left[\mathbf{1}\left\{V_j(f) \le u_j, \forall j = 1,\dots,k\right\} e^{-z\Phi(f)}\right]\\ &= \exp\left[-\int_{\mathbb{G}}\left(1 - e^{-zf(x)}\prod_{j=1}^{k}\mathbf{1}\left\{f_j(x) \le u_j\right\}\right)\Lambda(\mathrm{d}x)\right].\end{aligned}$$

Proof. In the same lines as the proof of Proposition 2.4.4 we have

$$\begin{aligned}\mathbf{E}\left[\mathbf{1}\left\{V_j(f) \le u_j, \forall j = 1,\dots,k\right\} e^{-z\Phi(f)}\right] &= \mathbf{E}\left[e^{-z\Phi(f)}\prod_{j=1}^{k}\mathbf{1}\left\{V_j(f) \le u_j\right\}\right]\\ &= \mathbf{E}\left[e^{-z\Phi(f)}e^{-\Phi(\tilde{f})}\right] = \mathcal{L}_\Phi\left(\tilde{f} - zf\right).\end{aligned}$$

The result for Poisson follows from the expression of its Laplace transform. □

2.4.2 Second order moments

Case of Poisson shot-noise

Proposition 2.4.6. Cumulative shot-noise covariance. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ with mean measure Λ. Then for all measurable functions $f : \mathbb{G} \to \bar{\mathbb{C}}$ which is either nonnegative (i.e., with values in $\bar{\mathbb{R}}_+$) or in $L^1_{\bar{\mathbb{C}}}(\Lambda, \mathbb{G})$, the shot-noise $\Phi(f)$ is a well defined random variable, and has expectation*

$$\mathbf{E}[\Phi(f)] = \Lambda(f).$$

Assume now that Φ is Poisson. Then for all measurable functions $f, g : \mathbb{G} \to \bar{\mathbb{R}}_+$,

$$\mathbf{E}[\Phi(f)\Phi(g)] = \Lambda(fg) + \Lambda(f)\Lambda(g). \tag{2.4.2}$$

Moreover, for all functions $f, g \in L^1_{\bar{\mathbb{C}}}(\Lambda, \mathbb{G}) \cap L^2_{\bar{\mathbb{C}}}(\Lambda, \mathbb{G})$,

$$\operatorname{cov}(\Phi(f), \Phi(g)) = \Lambda(fg^*), \tag{2.4.3}$$

where $\operatorname{cov}(X, Y) = \mathbf{E}[XY^*] - \mathbf{E}[X]\mathbf{E}[Y^*]$ *is the covariance of two complex-valued random variables X, Y.*

Proof. The first part follows from the Campbell averaging theorem 1.2.5. Assuming now that Φ is Poisson. We will show (2.4.2) for all measurable functions $f, g : \mathbb{G} \to \bar{\mathbb{R}}_+$. We first show the above result for simple functions and then invoke the monotone convergence theorem for the general case. Let f and g be two simple functions which may be written, without loss of generality, as

$$f = \sum_{i=1}^{n} a_i 1_{B_i}, \quad g = \sum_{j=1}^{n} b_j 1_{B_j},$$

where $a_i, b_i \geq 0$ and the $B_i \in \mathcal{B}(\mathbb{G})$ are pairwise disjoint. Since when $i \neq j$, the random variables $\Phi(B_i)$ and $\Phi(B_j)$ are independent, we get

$$\begin{aligned}
\mathbf{E}[\Phi(f)\Phi(g)] &= \sum_{i,j=1}^{n} a_i b_j \mathbf{E}[\Phi(B_i)\Phi(B_j)] \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{n} a_i b_j \mathbf{E}[\Phi(B_i)\Phi(B_j)] + \sum_{j=1}^{n} a_j b_j \mathbf{E}[\Phi(B_j)^2] \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{n} a_i b_j \mathbf{E}[\Phi(B_i)]\mathbf{E}[\Phi(B_j)] + \sum_{j=1}^{n} a_j b_j \mathbf{E}[\Phi(B_j)^2] \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{n} a_i b_j \Lambda(B_i)\Lambda(B_j) + \sum_{j=1}^{k} a_j b_j [\Lambda(B_j) + \Lambda(B_j)^2] \\
&= \sum_{i,j=1}^{n} a_i b_j \Lambda(B_i)\Lambda(B_j) + \sum_{j=1}^{k} a_j b_j \Lambda(B_j) = \Lambda(f)\Lambda(g) + \Lambda(fg),
\end{aligned}$$

where, for the fourth equality, we used the fact that a Poisson random variable has equal mean and variance. For all measurable functions $f, g : \mathbb{G} \to \bar{\mathbb{R}}_+$, there exist nondecreasing sequences of simple functions $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ in $\mathfrak{F}_+(\mathbb{G})$ such that $f_n \uparrow f$ and $g_n \uparrow g$ as $n \to \infty$ pointwise. We have

$$\mathbf{E}[\Phi(f_n)\Phi(g_n)] = \Lambda(f_n)\Lambda(g_n) + \Lambda(f_n g_n).$$

Letting $n \to \infty$ in the above equality and invoking the monotone convergence theorem proves (2.4.2). We deduce that, for all $f, g \in L^1_{\bar{\mathbb{R}}_+}(\Lambda, \mathbb{G})$.

$$\operatorname{cov}(\Phi(f), \Phi(g)) = \Lambda(fg). \tag{2.4.4}$$

Let $f, g \in L^1_{\mathbb{R}}(\Lambda, \mathbb{G}) \cap L^2_{\mathbb{R}}(\Lambda, \mathbb{G})$. We decompose each of them into positive and negative parts; that is $f = f^+ - f^-$ and $g = g^+ - g^-$ where f^+, f^-, g^+, g^- are nonnegative. Using (2.4.4) we get

$$\mathbf{E}\left[\Phi(f^\pm)\Phi(g^\pm)\right] = \Lambda(f^\pm g^\pm) < \infty$$

by the Cauchy-Schwarz inequality (since $f^\pm$ and $g^\pm$ are square-integrable with respect to Λ). Moreover,

$$\begin{aligned}
\mathbf{E}[\Phi(f)\Phi(g)] &= \mathbf{E}\left[\left(\Phi(f^+)-\Phi(f^-)\right)\left(\Phi(g^+)-\Phi(g^-)\right)\right] \\
&= \mathbf{E}\left[\Phi(f^+)\Phi(g^+)+\Phi(f^-)\Phi(g^-)-\Phi(f^+)\Phi(g^-)-\Phi(f^-)\Phi(g^+)\right] \\
&= \left[\Lambda(f^+g^+)+\Lambda(f^+)\Lambda(g^+)\right]+\left[\Lambda(f^-g^-)+\Lambda(f^-)\Lambda(g^-)\right] \\
&\quad -\left[\Lambda(f^+g^-)+\Lambda(f^+)\Lambda(g^-)\right]-\left[\Lambda(f^-g^+)+\Lambda(f^-)\Lambda(g^+)\right] \\
&= \Lambda(f)\Lambda(g)+\Lambda(fg),
\end{aligned}$$

which gives the announced result. The complex case follows from the real case by similar manipulations. □

Example 2.4.7. Shot-noise with marks. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces and $\tilde{\Phi}=\sum_{n\in\mathbb{Z}}\delta_{(X_n,Z_n)}$ an independently marked point process on $\mathbb{G}\times\mathbb{K}$ with kernel $\tilde{p}(x,\mathrm{d}z)$ (cf. Definition 2.2.18) and ground process $\Phi=\sum_{n\in\mathbb{Z}}\delta_{X_n}$. Given some measurable function $f:\mathbb{G}\times\mathbb{K}\to\bar{\mathbb{C}}$, we aim to study the properties of the shot-noise*

$$\tilde{\Phi}(f)=\int_{\mathbb{G}\times\mathbb{K}} f(x,z)\,\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}z)=\sum_{n\in\mathbb{Z}} f(X_n,Z_n)$$

called a cumulative shot-noise with marks*. Note first that, by Lemma 2.2.19, the mean measure of $\tilde{\Phi}$ is given by*

$$M_{\tilde{\Phi}}(\mathrm{d}x\times\mathrm{d}z)=\tilde{p}(x,\mathrm{d}z)M_{\Phi}(\mathrm{d}x).$$

(i) If f is either nonnegative (i.e., with values in $\bar{\mathbb{R}}_+$) or in $L^1_{\mathbb{C}}(M_{\tilde{\Phi}},\mathbb{G}\times\mathbb{K})$, the shot-noise $\tilde{\Phi}(f)$ is a well defined random variable, and has expectation

$$\mathbf{E}\left[\tilde{\Phi}(f)\right]=\int_{\mathbb{R}^d\times\mathbb{K}} f(x,z)\,\tilde{p}(x,\mathrm{d}z)M_{\Phi}(\mathrm{d}x)=\int_{\mathbb{R}^d}\mathbf{E}\left[f(x,Z(x))\right]M_{\Phi}(\mathrm{d}x),$$

where the first equality follows from theorem 1.2.5 and for the second equality we introduce a stochastic process $\{Z(x)\}_{x\in\mathbb{G}}$ with values in $\mathbb{K}$ such that

$$\mathbf{P}(Z(x)\in K)=\tilde{p}(x,K),\quad K\in\mathcal{B}(\mathbb{K}).$$

(ii) If the ground process Φ is Poisson, then it follows from Proposition 2.4.6 that for all functions $f,g:\mathbb{G}\times\mathbb{K}\to\bar{\mathbb{C}}$ is integrable and square integrable with respect to $M_{\tilde{\Phi}}$,

$$\begin{aligned}
\mathrm{cov}(\tilde{\Phi}(f),\tilde{\Phi}(g)) &= M_{\tilde{\Phi}}(fg^*) \\
&= \int_{\mathbb{R}^d\times\mathbb{K}} f(x,z)\,g^*(x,z)\,\tilde{p}(x,\mathrm{d}z)M_{\Phi}(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d}\mathbf{E}\left[f(x,Z(x))\,g^*(x,Z(x))\right]M_{\Phi}(\mathrm{d}x).
\end{aligned}$$

Moreover, the above equality holds for all functions $f,g:\mathbb{G}\times\mathbb{K}\to\bar{\mathbb{R}}_+$ integrable with respect to $M_{\tilde{\Phi}}$.

(iii) *If the ground process* Φ *is Poisson, then it follow from Proposition 2.4.3 that for all measurable functions* $f : \mathbb{G} \times \mathbb{K} \to \bar{\mathbb{R}}_+$,

$$\mathbf{E}\left[e^{-\theta\tilde{\Phi}(f)}\right] = \exp\left[-\int_{\mathbb{G}}\left(1 - \mathbf{E}\left[e^{-\theta f(x, Z(x))}\right]\right) M_\Phi(\mathrm{d}x)\right], \quad \theta \in \mathbb{R}_+$$

and

$$\mathbf{P}\left(\tilde{\Phi}(f) = 0\right) = \exp\left[-\int_{\mathbb{G}} \mathbf{P}(f(x, Z(x)) > 0) M_\Phi(\mathrm{d}x)\right].$$

Example 2.4.8. Random filtering. *Let* $\mathbb{K}$ *be a l.c.s.h. space and* $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(X_n, Z_n)}$ *an independently marked point process on* $\mathbb{R}^d \times \mathbb{K}$ *with kernel* $\tilde{p}(x, \mathrm{d}z)$ *(cf. Definition 2.2.18) with ground process* $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$. *Given some measurable function* $h : \mathbb{R}^d \times \mathbb{K} \to \bar{\mathbb{C}}$, *we aim to study*

$$Y(t) = \int_{\mathbb{R}^d\times\mathbb{K}} h(t - x, z)\, \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z) = \sum_{n\in\mathbb{Z}} h(t - X_n, Z_n), \quad t \in \mathbb{R}^d \tag{2.4.5}$$

called the random filtering *of the point process* Φ *with* impulse response h. *Given* $t \in \mathbb{R}^d$, *letting* $f(x, z) = h(t - x, z), x \in \mathbb{R}^d, z \in \mathbb{K}$, *we get*

$$Y(t) = \sum_{n\in\mathbb{Z}} f(X_n, Z_n) = \tilde{\Phi}(f).$$

By arguments similar to those used in Example 2.4.7, we have:

(i) *Let* $h : \mathbb{R}^d \times \mathbb{K} \to \bar{\mathbb{R}}$ *be a measurable function which is either nonnegative or such that*

$$\int_{\mathbb{R}^d} \mathbf{E}[|h(t - x, Z(x))|] M_\Phi(\mathrm{d}x) < \infty, \quad t \in \mathbb{R}^d, \tag{2.4.6}$$

where $\{Z(x)\}_{x\in\mathbb{G}}$ *is a stochastic process with values in* $\mathbb{K}$ *such that*

$$\mathbf{P}(Z(x) \in K) = \tilde{p}(x, K), \quad K \in \mathcal{B}(\mathbb{K}).$$

Then the random filtering $Y(t)$ *given by (2.4.5) is a well defined random variable, and has expectation*

$$\mathbf{E}[Y(t)] = \int_{\mathbb{R}^d} \mathbf{E}[h(t - x, Z(x))] M_\Phi(\mathrm{d}x), \quad t \in \mathbb{R}^d.$$

(ii) *Assume that the ground process* Φ *is Poisson. For any measurable function* $h : \mathbb{R}^d \times \mathbb{K} \to \bar{\mathbb{C}}$ *satisfying (2.4.6) and*

$$\int_{\mathbb{R}^d} \mathbf{E}\left[|h(t - x, Z(x))|^2\right] M_\Phi(\mathrm{d}x) < \infty, \quad t \in \mathbb{R}^d,$$

we have

$$\operatorname{cov}\left(Y\left(t\right),Y\left(t+\tau\right)\right)=\int_{\mathbb{R}^{d}}\mathbf{E}\left[h\left(t-x,Z\left(x\right)\right)h^{*}\left(t+\tau-x,Z\left(x\right)\right)\right]M_{\Phi}\left(\mathrm{d}x\right),$$

for all $t,\tau\in\mathbb{R}^{d}$. *Moreover, the above equality holds for all measurable functions* $h:\mathbb{R}^{d}\times\mathbb{K}\to\bar{\mathbb{R}}_{+}$ *satisfying (2.4.6).*

(iii) Assume that the ground process Φ *is Poisson. Then for all measurable functions* $h:\mathbb{R}^{d}\times\mathbb{K}\to\bar{\mathbb{R}}_{+}$,

$$\mathbf{E}\left[e^{-\theta Y(t)}\right]=\exp\left(-\int_{\mathbb{R}^{d}}(1-\mathbf{E}\left[e^{-\theta h(t-x,Z(x))}\right])M_{\Phi}\left(\mathrm{d}x\right)\right),\quad\theta\in\mathbb{R}_{+}\tag{2.4.7}$$

and

$$\mathbf{P}\left(Y\left(t\right)=0\right)=\exp\left(-\int_{\mathbb{R}^{d}}\mathbf{P}\left(h\left(t-x,Z\left(x\right)\right)>0\right)M_{\Phi}\left(\mathrm{d}x\right)\right).$$

General case

Definition 2.4.9. *Let* Φ *be a random measure on a l.c.s.h. space* $\mathbb{G}$. *We define* L_{Φ}^{2} *as the set of measurable functions* $f:\mathbb{G}\to\bar{\mathbb{C}}$ *such that*

$$\mathbf{E}[\Phi(|f|)^{2}]<\infty.$$

Example 2.4.10. L_{Φ}^{2} for Poisson. *Let* Φ *be a Poisson point process on a l.c.s.h. space* $\mathbb{G}$ *with intensity measure* Λ. *For any measurable* $f:\mathbb{G}\to\bar{\mathbb{C}}$, *we deduce from (2.4.2) that*

$$\mathbf{E}[\Phi(|f|)^{2}]=\Lambda(|f|^{2})+\Lambda(|f|)^{2}.$$

Therefore,

$$L_{\Phi}^{2}=L_{\mathbb{C}}^{1}\left(\Lambda,\mathbb{G}\right)\cap L_{\mathbb{C}}^{2}\left(\Lambda,\mathbb{G}\right).$$

Lemma 2.4.11. *For any random measure* Φ *on a l.c.s.h. space* $\mathbb{G}$, $L_{\Phi}^{2}\subset L_{\mathbb{C}}^{1}\left(M_{\Phi},\mathbb{G}\right)$. *In particular, for all* $f\in L_{\Phi}^{2}$, *the shot-noise* $\Phi\left(f\right)$ *is well defined. If* Φ *is a point process, then*

$$L_{\Phi}^{2}\subset L_{\mathbb{C}}^{1}\left(M_{\Phi},\mathbb{G}\right)\cap L_{\mathbb{C}}^{2}\left(M_{\Phi},\mathbb{G}\right).$$

Proof. For any $f\in L_{\Phi}^{2}$, using (13.A.4) we deduce that $\mathbf{E}[\Phi(|f|)]<\infty$ and therefore, by Theorem 1.2.5, $\int|f|\mathrm{d}M_{\Phi}<\infty$; i.e., $f\in L_{\mathbb{C}}^{1}\left(M_{\Phi},\mathbb{G}\right)$, and the shot-noise $\Phi\left(f\right)$ is well defined. Assume now that Φ is a point process. Then

$$\Phi(|f|^{2})=\sum_{n\in\mathbb{Z}}\left|f\left(X_{n}\right)\right|^{2}\leq\left(\sum_{n\in\mathbb{Z}}\left|f\left(X_{n}\right)\right|\right)^{2}=\Phi(|f|)^{2}.$$

Thus for all $f\in L_{\Phi}^{2}$, $\mathbf{E}\left[\Phi(|f|^{2})\right]\leq\mathbf{E}[\Phi(|f|)^{2}]<\infty$, and since by Theorem 1.2.5 $\mathbf{E}\left[\Phi(|f|^{2})\right]=M_{\Phi}\left(|f|^{2}\right)$, it follows that $f\in L_{\mathbb{C}}^{2}\left(M_{\Phi},\mathbb{G}\right)$. $\square$

Example 2.4.12. L^2_Φ for the stochastic integral . *Let $\{\lambda(x)\}_{x\in\mathbb{R}^d}$ be a nonnegative measurable stochastic process assumed wide sense stationary; i.e., for all $x,t\in\mathbb{R}^d$, $\mathbf{E}\left[|\lambda(x)|^2\right]<\infty$, $\mathbf{E}[\lambda(x)]=\mathbf{E}[\lambda(0)]$ and $\operatorname{cov}(\lambda(x+t),\lambda(x))=\operatorname{cov}(\lambda(t),\lambda(0))$. Assume moreover that, for almost all $\omega\in\Omega$, the function $x\mapsto\lambda(x,\omega)$ is locally integrable. Consider the stochastic integral*

$$\Lambda(B)=\int_B\lambda(x)\,\mathrm{d}x,\quad B\in\mathcal{B}\left(\mathbb{R}^d\right).$$

It follows from Proposition 1.4.1 that Λ is a random measure on $\mathbb{R}^d$, and from Proposition 1.4.2 that its mean measure equals

$$\mathbf{E}[\Lambda(B)]=\int_B\mathbf{E}[\lambda(x)]\,\mathrm{d}x=\lambda\ell^d(B),$$

where $\lambda=\mathbf{E}[\lambda(0)]$ and ℓ^d is the Lebesgue measure on $\mathbb{R}^d$. Then, by Theorem 1.2.5, for all $f\in L^1_{\mathbb{C}}\left(\ell^d,\mathbb{R}^d\right)$, the shot-noise $\Lambda(|f|)$ is well defined. Moreover,

$$\begin{aligned}\mathbf{E}[\Lambda(|f|)^2]&=\mathbf{E}\left[\left(\int_{\mathbb{R}^d}|f(x)|\,\lambda(x)\,\mathrm{d}x\right)^2\right]\\&=\mathbf{E}\left[\int_{\mathbb{R}^d}|f(x)|\,|f(y)|\,\lambda(x)\,\lambda(y)\,\mathrm{d}x\mathrm{d}y\right]\\&=\int_{\mathbb{R}^d}|f(x)|\,|f(y)|\,\mathbf{E}[\lambda(x)\,\lambda(y)]\,\mathrm{d}x\mathrm{d}y\\&=\int_{\mathbb{R}^d}|f(x)|\,|f(y)|\operatorname{cov}(\lambda(x),\lambda(y))\,\mathrm{d}x\mathrm{d}y+\lambda^2\int_{\mathbb{R}^d}|f(x)|\,|f(y)|\,\mathrm{d}x\mathrm{d}y\\&=\int_{\mathbb{R}^d}|f(x)|\,|f(y)|\,\Gamma_\lambda(x-y)\,\mathrm{d}x\mathrm{d}y+\lambda^2\int_{\mathbb{R}^d}|f(x)|\,|f(y)|\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where $\Gamma_\lambda(t)=\operatorname{cov}(\lambda(x+t),\lambda(x))$ is the autocovariance function of $\{\lambda(x)\}_{x\in\mathbb{R}^d}$. By the Cauchy-Schwarz inequality,

$$\operatorname{cov}(\lambda(x+t),\lambda(x))\leq\operatorname{var}(\lambda(x+t))^{1/2}\operatorname{var}(\lambda(x))^{1/2}=\Gamma_\lambda(0)<\infty.$$

Then

$$\mathbf{E}[\Lambda(|f|)^2]\leq\left(\Gamma_\lambda(0)+\lambda^2\right)\left(\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^d}|g(y)|\,\mathrm{d}y\right)<\infty.$$

Thus $L^1_{\mathbb{C}}\left(\ell^d,\mathbb{R}^d\right)\subset L^2_\Lambda$. On the other hand, by lemma 2.4.11 $L^2_\Lambda\subset L^1_{\mathbb{C}}\left(\ell^d,\mathbb{R}^d\right)$, therefore

$$L^2_\Lambda=L^1_{\mathbb{C}}\left(\ell^d,\mathbb{R}^d\right).$$

Proposition 2.4.13. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$. Then for all measurable functions $f,g:\mathbb{G}\to\bar{\mathbb{C}}$ which are either nonnegative (i.e., with values in $\bar{\mathbb{R}}_+$) or in L^2_Φ,*

$$\mathbf{E}\left[\Phi(f)\,\Phi(g)^*\right]=\int_{\mathbb{G}^2}f(x)\,g(y)^*\,M_{\Phi^2}(\mathrm{d}x\times\mathrm{d}y),\tag{2.4.8}$$

where Φ^2 *is the square of the random measure* Φ *and* z^* *is the complex-conjugate of* $z \in \bar{\mathbb{C}}$.

Proof. For any nonnegative measurable functions $f, g : \mathbb{G} \to \bar{\mathbb{R}}$,

$$\begin{aligned}
\mathbf{E}\left[\Phi(f)\Phi(g)\right] &= \mathbf{E}\left[\int_{\mathbb{G}^2} f(x) g(y) \Phi(\mathrm{d}x) \Phi(\mathrm{d}y)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}^2} f(x) g(y) \Phi^2(\mathrm{d}x \times \mathrm{d}y)\right] \\
&= \int_{\mathbb{G}^2} f(x) g(y) M_{\Phi^2}(\mathrm{d}x \times \mathrm{d}y),
\end{aligned}$$

where the last equality follows from the Campbell averaging theorem 1.2.5 applied to the random measure Φ^2. Let now $f, g \in L^2_\Phi$. By the above equality

$$\begin{aligned}
\int_{\mathbb{G}^2} |f(x)| |g(y)| M_{\Phi^2}(\mathrm{d}x \times \mathrm{d}y) &= \mathbf{E}\left[\Phi(|f|)\Phi(|g|)\right] \\
&\leq \mathbf{E}\left[\Phi(|f|)^2\right]^{1/2} \mathbf{E}\left[\Phi(|g|)^2\right]^{1/2} < \infty,
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality. Then the function $\mathbb{G}^2 \to \bar{\mathbb{C}}; (x, y) \mapsto f(x) g^*(y)$ is integrable with respect to M_{Φ^2}. Thus, again by Theorem 1.2.5 applied to Φ^2,

$$\begin{aligned}
\int_{\mathbb{G}^2} f(x) g^*(y) M_{\Phi^2}(\mathrm{d}x \times \mathrm{d}y) &= \mathbf{E}\left[\int_{\mathbb{G}^2} f(x) g(y)^* \Phi^2(\mathrm{d}x \times \mathrm{d}y)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}^2} f(x) g(y)^* \Phi(\mathrm{d}x) \Phi(\mathrm{d}y)\right] \\
&= \mathbf{E}\left[\Phi(f)\Phi(g)^*\right].
\end{aligned}$$

□

2.4.3 U-statistics

Definition 2.4.14. *Let* Φ *be a random measure on a l.c.s.h. space* $\mathbb{G}$, $k, n \in \mathbb{N}^*$ *and let* $f : \mathbb{G}^n \to \bar{\mathbb{C}}^k$ *be a measurable function which is either real nonnegative or in* $L^1_{\bar{\mathbb{C}}^k}(M_{\Phi^{(n)}}, \mathbb{G}^n)$, *then*

$$\Phi^{(n)}(f) = \int_{\mathbb{G}^n} f(x) \Phi^{(n)}(\mathrm{d}x)$$

is called U-statistics *of order* n.

Observe that a U-statistics of order n is indeed a cumulative shot-noise with respect to the n-th factorial power of Φ. Then all the previous results for shot-noise apply for the U-statistics. In particular, by Theorem 1.2.5, a U-statistics is well defined random variable with expectation $\mathbf{E}\left[\Phi^{(n)}(f)\right] = \int_{\mathbb{G}^n} f(x) M_{\Phi^{(n)}}(\mathrm{d}x)$.

2.5 Sigma-finite random measures

It is sometimes useful to consider random measures which are not necessarily locally finite, but more generally σ-finite.

Definition 2.5.1. *Let $\tilde{\mathbb{M}}(\mathbb{G})$ be the space of σ-finite measures on a l.c.s.h. space $\mathbb{G}$ and $\tilde{\mathcal{M}}(\mathbb{G})$ be the σ-algebra on $\tilde{\mathbb{M}}(\mathbb{G})$ generated by the mappings $\mu \mapsto \mu(B), B \in \mathcal{B}(\mathbb{G})$. A* random measure *on $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$ is a measurable mapping $\Phi : \Omega \to \tilde{\mathbb{M}}(\mathbb{G})$ which is uniformly σ-finite; i.e., there is a countable family of sets $B_1, B_2, \ldots \in \mathcal{B}(\mathbb{G})$ covering $\mathbb{G}$ such that, for all $n \in \mathbb{N}^*$, $\Phi(B_n) < \infty$ a.s. A* point process *is a random measure such that $\Phi(B) \in \bar{\mathbb{N}}$ for all $B \in \mathcal{B}(\mathbb{G})$.*

Several concepts and results established earlier in the locally finite case, extend to the present σ-finite case; for example the measurability results in Lemma 1.1.5 and Proposition 1.1.7, the mean measure, void measure and Laplace transform in Definition 1.2.1, the distribution characterization by the finite-dimensional distributions in Lemma 1.1.12 and by the Laplace transform in Corollary 1.2.2, and also the Campbell's averaging theorem 1.2.5.

The definition of a Poisson point processmay be extended as follows.

Definition 2.5.2. *Let Λ be a σ-finite measure on a l.c.s.h. space $\mathbb{G}$. A point process Φ is said to be* Poisson *with intensity measure Λ if*

(i) for all pairwise disjoint sets $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G})$, the random variables

$$\Phi(B_1), \ldots, \Phi(B_k)$$

are independent;

(ii) and for all $B \in \mathcal{B}(\mathbb{G})$,

$$\mathbf{P}(\Phi(B) = n) = e^{-\Lambda(B)} \frac{\Lambda(B)^n}{n!}, \quad n \in \mathbb{N},$$

with the convention $\infty^n e^{-\infty} := 0$ for all $n \in \mathbb{N}$.

The Laplace transform of a Poisson point process is given by Proposition 2.1.4. Moreover, we have the following characterization:

Proposition 2.5.3. *Let Λ be a σ-finite measure on a l.c.s.h. space $\mathbb{G}$. A point process Φ is Poisson with intensity measure Λ iff*

$$\mathcal{L}_\Phi(f) = \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right] = \exp\left(-\int_{\mathbb{G}} (1 - e^{-f}) \mathrm{d}\Lambda\right),$$

for all $f \in \mathfrak{F}_+(\mathbb{G})$.

We have also the following superposition result extending Corollary 2.2.3.

Proposition 2.5.4. *Let $\Phi_0, \Phi_1, \ldots$ be a sequence of independent Poisson point processes on a l.c.s.h. space $\mathbb{G}$ with σ-finite intensity measures $\Lambda_0, \Lambda_1, \ldots$. If the measure $\Lambda = \sum_{k \in \mathbb{N}} \Lambda_k$ is σ-finite, then $\Phi = \sum_{k \in \mathbb{N}} \Phi_k$ is Poisson.*

Proof. The proof follows the same lines as the sufficiency part of Corollary 2.2.3. □

The above proposition allows one to construct a Poisson point process with given σ-finite intensity measure Λ.

Proposition 2.5.5. *Given a σ-finite measure Λ on a l.c.s.h. space $\mathbb{G}$, there exists a Poisson point process on $\mathbb{G}$ with intensity measure Λ.*

Proof. Construction of a Poisson point process on $\mathbb{G}$. Let $B_0, B_1, \ldots \in \mathcal{B}(\mathbb{G})$ be a countable partition of $\mathbb{G}$ such that $\Lambda(B_k) < \infty$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, let Φ_k be a Poisson point processes on $\mathbb{G}$ with intensity measures $\Lambda_k(\cdot) = \Lambda(\cdot \cap B_k)$ (which may be constructed as in Proposition 2.1.6). Then Proposition 2.5.4 shows that $\Phi = \sum_{k\in\mathbb{N}} \Phi_k$ is Poisson with intensity measure $\Lambda = \sum_{k\in\mathbb{N}} \Lambda_k$. □

Remark 2.5.6. Intensity measure of a Poisson point process. *If a Poisson point process with σ-*finite *intensity measure Λ is considered in the remaing part of the book, this will be explicitely stated. Otherwise, the intensity measure of a Poisson point process is assumed implicitely to be* locally finite.

2.6 Further examples

2.6.1 For Section 2.1

Example 2.6.1. Distance to the nearest point of a Poisson point process. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ. For a given $x \in \mathbb{R}^d$, let $R = \inf_{X\in\Phi} |X - x|$. Note that*

$$\mathbf{P}(R \geq r) = \mathbf{P}(\Phi(B(x,r)) = 0) = e^{-\lambda\kappa_d\ r^d},$$

where $B(x,r) = \left\{y \in \mathbb{R}^d : |y - x| < r\right\}$ and κ_d is the volume of the unit ball in $\mathbb{R}^d$. Note first that R is almost surely finite since

$$\mathbf{P}(R = \infty) = \mathbf{P}\left(\bigcap_{n\in\mathbb{N}} \{R \geq n\}\right) = \lim_{n\to\infty} e^{-\lambda\kappa_d\ n^d} = 0,$$

where the first equality follows from the continuity from above of measures [44, Theorem E p.38]. Note moreover that for all $\omega \in \Omega$, the measure $\Phi(\omega)$ is locally finite, then the number of points within $B(x, 2R(\omega))$ is finite and therefore the infimum $\inf_{X\in\Phi} |X - x|$ is attained; that is $R = \min_{X\in\Phi} |X - x|$. Moreover,

$$\mathbf{E}[R] = \int_0^\infty \mathbf{P}(R \geq r)\,\mathrm{d}r = \int_0^\infty e^{-\lambda\kappa_d\ r^d}\mathrm{d}r = \frac{\Gamma(1/d)}{d\ (\lambda\kappa_d)^{1/d}}.$$

In the particular case $d = 2$, we get $\mathbf{E}[R] = \frac{1}{2\sqrt{\lambda}}$.

Example 2.6.2. Inter-events of Poisson point processes. *Consider a homogeneous Poisson point process Φ on $\mathbb{R}$ with intensity λ. By Proposition 2.1.9, the points T_k of Φ are disjoint. We may number them in the increasing order and in such a way that $T_0 \leq 0 < T_1$. Consider the following* inter-event *random variables*

$$R_k = \begin{cases} T_1, & \text{if } k = 1 \\ -T_0, & \text{if } k = 0 \\ T_k - T_{k-1}, & \text{if } k \in \mathbb{Z} \setminus \{0, 1\} \end{cases}$$

illustrated in Figure 2.1. We aim to show that $\{R_k\}_{k\in\mathbb{Z}}$ are i.i.d. random variables exponentially distributed with parameter λ. First, observe that

$$\mathbf{P}\left(R_1 > t\right) = \mathbf{P}\left(T_1 > t\right) = \mathbf{P}\left(\Phi\left((0, t]\right) = 0\right) = e^{-\lambda t},$$

then R_1 has an exponential distribution with parameter λ. Moreover, using the strong Markov property of Poisson point processes on $\mathbb{R}_+$; see, e.g., [19, Theorem 1.1 p.370] (which will be generalized in Section 12.1 below), we get

$$\begin{aligned} \mathbf{P}\left(R_k > t \mid R_1, \ldots, R_{k-1}\right) &= \mathbf{P}\left(T_k - T_{k-1} > t \mid T_1, \ldots, T_{k-1}\right) \\ &= \mathbf{P}\left(T_k - T_{k-1} > t \mid T_{k-1}\right) \\ &= \mathbf{P}\left(\Phi\left((T_{k-1}, T_{k-1} + t]\right) = 0 \mid T_{k-1}\right) = e^{-\lambda t}, \end{aligned}$$

which shows that $\{R_k\}_{k\geq 1}$ are independent exponential random variables with parameter λ. Following the same lines as above, we may prove that $\{R_k\}_{k\leq 0}$ are independent exponential random variables with parameter λ. By the independence of $\Phi\left(\mathbb{R}_+^*\right)$ and $\Phi\left(\mathbb{R}_-\right)$ we get the announced result.

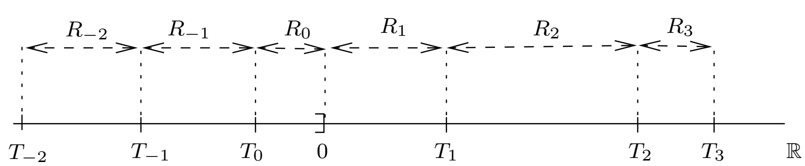

Figure 2.1: Homogeneous Poisson point process on $\mathbb{R}$

2.6.2 For Section 2.2

Example 2.6.3. Choosing uniformly an atom of a point process. *Let $\mathbb{G}$ be a l.c.s.h. space and $\tilde{\Phi} = \sum_{k\in\mathbb{N}} \delta_{(X_k, Z_k)}$ be an i.i.d. marked point process on $\mathbb{G} \times \mathbb{R}$ such that the mark distribution F has a density. Let $\Phi = \sum_{k\in\mathbb{N}} \delta_{X_k}$ be the associated ground process assumed simple and finite. Let $K = K(\omega)$ be the index of the minimal mark $Z_K = \min_{k\in\mathbb{N}} \{Z_k\}$. Then, given Φ, X_K is uniformly distributed among the set of atoms of Φ.*

Indeed, for any $j \in \mathbb{N}$,

$$\begin{aligned}
\mathbf{P}\left(X_K = X_j \mid \Phi, \Phi(\mathbb{G}) \geq j\right) &= \mathbf{P}\left(Z_K = Z_j \mid \Phi, \Phi(\mathbb{G}) \geq j\right) \\
&= \mathbf{P}\left(Z_j \leq Z_k, \forall k \in \mathbb{N}\setminus\{j\} \mid \Phi, \Phi(\mathbb{G}) \geq j\right) \\
&= \int_{\mathbb{R}} \mathbf{P}\left(z \leq Z_k, \forall k \in \mathbb{N}\setminus\{j\} \mid \Phi, \Phi(\mathbb{G}) \geq j\right) F(\mathrm{d}z) \\
&= \int_{\mathbb{R}} (1 - F(z))^{\Phi(\mathbb{G})-1} F(\mathrm{d}z) \\
&= \left[\frac{(1 - F(z))^{\Phi(\mathbb{G})}}{\Phi(\mathbb{G})}\right]_{-\infty}^{+\infty} = \frac{1}{\Phi(\mathbb{G})}.
\end{aligned}$$

Example 2.6.4. I.i.d. marks from sequence of nested partitions. *Let $\mathbb{G}$ be a l.c.s.h. space. We will give a particular way of constructing the marks $\{Z_k\}_{k\in\mathbb{N}}$ of an i.i.d. marked point process $\tilde{\Phi} = \sum_{k\in\mathbb{N}} \delta_{(X_k, Z_k)}$ on $\mathbb{G}\times\mathbb{R}$ with a given mark distribution F. We assume that the associated ground process $\Phi = \sum_{k\in\mathbb{N}} \delta_{X_k}$ is simple. Let $\mathcal{K}_n = \{K_{n,j}\}_{j\in\mathbb{N}}$ ($n \in \mathbb{N}$) be a sequence of nested partitions of $\mathbb{G}$ as in Lemma 1.6.3. Let $\{Y_{n,j}\}_{n,j\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common distribution F independent of Φ. Given the ground point process $\Phi = \sum_{k\in\mathbb{N}} \delta_{X_k}$, for each $k \in \mathbb{N}$, let $n(X_k, \Phi)$ be the smallest index $n \in \mathbb{N}$ such that X_k is the only atom of Φ in some $K_{n,j}$. We may then take*

$$Z_k := Y_{n(X_k,\Phi),j(X_k,\Phi)}, \quad k \in \mathbb{N}.$$

Indeed, for all $k \in \mathbb{N}$ and $B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned}
\mathbf{P}\left(Z_k \in B \mid \Phi\right) &= \mathbf{P}(Y_{n(X_k,\Phi),j(X_k,\Phi)} \in B \mid \Phi) \\
&= \sum_{n,j\in\mathbb{N}} \mathbf{P}(Y_{n,j} \in B \mid \Phi, n(X_k,\Phi) = n, j(X_k,\Phi) = j) \\
&\quad \mathbf{P}(n(X_k,\Phi) = n, j(X_k,\Phi) = j \mid \Phi) \\
&= \sum_{n,j\in\mathbb{N}} F(B)\mathbf{P}(n(X_k,\Phi) = n, j(X_k,\Phi) = j \mid \Phi) = F(B).
\end{aligned}$$

We may prove along the same lines as above that, for any $k \neq l \in \mathbb{N}$ and $B, C \in \mathcal{B}(\mathbb{G})$,

$$\mathbf{P}\left(Z_k \in B, Z_l \in C \mid \Phi\right) = F(B)F(C),$$

which shows that, given Φ, $\{Z_k\}_{k\in\mathbb{N}}$ is an i.i.d. sequence of random variables with common distribution F.

Example 2.6.5. Poisson-Monte Carlo integration. *Continuing Exercise 1.7.4, we may write (1.7.1) as follows*

$$\lim_{n\to\infty} \frac{1}{n} \int_B f \sum_{j=1}^{n} \mathrm{d}\Phi_n = \int_B f \mathrm{d}x.$$

If we take Φ *as a homogeneous Poisson point process on* $\mathbb{R}^d$ *with intensity* 1*, then, by Corollary 2.2.3,* $\sum_{j=1}^{n} \Phi_n$ *is a homogeneous Poisson point process with intensity* n*. Thus one may simulate a Poisson point process* $\tilde{\Phi}_n$ *with sufficiently high intensity* n*, and estimate the integral* $\int_B f \mathrm{d}x$ *by* $\frac{1}{n} \int_B f \mathrm{d}\tilde{\Phi}_n$*.*

Example 2.6.6. Jittering. *Let* $\Phi = \sum_{k\in\mathbb{Z}} \delta_{(T_k,Z_k)}$ *be a point process on* $\mathbb{R}^d \times \mathbb{R}^d$ *and let* g *be the function defined on* $\mathbb{R}^d \times \mathbb{R}^d$ *by* $g(t,z) = t+z$*. Then the image of* Φ *by* g *is* $\tilde{\Phi} = \Phi \circ g^{-1} = \sum_{k\in\mathbb{Z}} \delta_{T_k+Z_k}$ *whose atom* $\tilde{T}_k = T_k + Z_k$ *may be seen as a random displacement (*jittering*) of* T_k *as illustrated in Figure 2.2.*

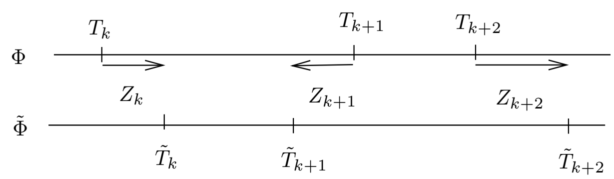

Figure 2.2: Jittered point process

Example 2.6.7. Thinning from marking. *Let* $\mathbb{G}$ *be a l.c.s.h. space,* $p : \mathbb{G} \to [0,1]$ *be some measurable function, and* $\hat{\Phi} = \sum_{k\in\mathbb{N}} \delta_{(T_k,Z_k)}$ *be an independently marked point process on* $\mathbb{G} \times \{0,1\}$*, such that*

$$\mathbf{P}(Z_k = 1 \mid \Phi) = p(T_k), \quad \mathbf{P}(Z_k = 0 \mid \Phi) = 1 - p(T_k),$$

then

$$\tilde{\Phi} = \sum_{k\in\mathbb{N}} Z_k \delta_{T_k}$$

is the thinning of the point process $\Phi = \sum_{k\in\mathbb{N}} \delta_{T_k}$*. Observe that* $\tilde{\Phi}(\cdot)$ *can be seen as* $\hat{\Phi}(\cdot \times \{1\})$*.*

Example 2.6.8. *Cf. [15, Lemma 1].* Propagation loss in wireless networks. *Consider a wireless network composed of base stations whose locations are modelled by a homogeneous Poisson point process on* $\mathbb{R}^2$*, say* $\Phi = \sum_{n\in\mathbb{N}} \delta_{X_n}$*, of intensity* λ*. Consider a user located at* 0*. The propagation loss* Y_n *between the base station located at* X_n *and the user comprises firstly a deterministic term due to the* distance *denoted by* $l(|X_n|)$ *where*

$$l(r) = (Kr)^{\beta}, \quad r \in \mathbb{R}_+$$

for two given constants $K \in \mathbb{R}_+^*$ *and* $\beta > 2$*. Moreover, the propagation loss* Y_n *comprises a random term called* fading *and denoted by* Z_n*; that is*

$$Y_n = \frac{l(|X_n|)}{Z_n}.$$

The fading random variables $Z_0, Z_1, \ldots$ are i.i.d. marks for Φ; i.e., the point process $\sum_{n\in\mathbb{N}} \delta_{(X_n,Z_n)}$ is an i.i.d. marked point process associated to Φ in the sense of Definition 2.2.18.

The point process $\tilde{\Phi} = \sum_{n\in\mathbb{N}} \delta_{Y_n}$ may be viewed as an independent displacement of the point process $\Phi = \sum_{n\in\mathbb{N}} \delta_{X_n}$ by the probability kernel

$$p(x, B) = \mathbf{P}\left(\frac{l(|x|)}{Z} \in B\right), \quad x \in \mathbb{R}^2, B \in \mathcal{B}(\mathbb{R}_+),$$

where $Z = Z_0$. Assume that $\mathbf{E}\left[Z^{\frac{2}{\beta}}\right] < \infty$. By the displacement theorem 2.2.17, the point process $\tilde{\Phi}$ is Poisson on $\mathbb{R}_+$ with intensity measure

$$\begin{aligned}
M_{\tilde{\Phi}}([0,u)) &= \int_{\mathbb{R}^2} p(x,[0,u))\,\lambda \mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^2} \mathbf{P}\left(\frac{l(|x|)}{Z} \in [0,u)\right) \mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathbf{1}\left\{\frac{l(|x|)}{z} < u\right\} \mathrm{d}x \mathbf{P}_Z(\mathrm{d}z) \\
&= \lambda \int_{\mathbb{R}_+} \frac{\pi (zu)^{\frac{2}{\beta}}}{K^2} \mathbf{P}_Z(\mathrm{d}z) = a u^{\frac{2}{\beta}},
\end{aligned} \tag{2.6.1}$$

where

$$a = \pi \lambda K^{-2} \mathbf{E}\left[Z^{\frac{2}{\beta}}\right]. \tag{2.6.2}$$

We may then deduce the probability distribution of the lowest propagation loss $Y = \inf_{n\in\mathbb{N}} Y_n$,

$$\begin{aligned}
\mathbf{P}(Y \geq u) &= \mathbf{P}(Y_n \geq u, \forall n \in \mathbb{N}) \\
&= \mathbf{P}\left(\tilde{\Phi}([0,u)) = 0\right) \\
&= e^{-M_{\tilde{\Phi}}([0,u))} = e^{-a u^{\frac{2}{\beta}}}.
\end{aligned} \tag{2.6.3}$$

On the other hand, it follows from (2.6.1) that the number of points of the point process $\{Y_n\}_{n\in\mathbb{N}}$ is almost surely finite in any finite interval. Thus the infimum $Y = \inf_{n\in\mathbb{N}} Y_n$ is almost surely achieved for some base station; i.e., $Y = \min_{n\in\mathbb{N}} Y_n$. Thus it may be interpreted as the propagation loss with the serving *base station; i.e., the one with strongest received power.*

Example 2.6.9. Arrival process in time and space. *We consider users arriving to a network where each user arrives at a some time instant to some location in $\mathbb{R}^d$. We will describe the arrival process as a point process on $\mathbb{R} \times \mathbb{R}^d$. Given a locally finite measure Λ on $\mathbb{R}^d$, assume that for each $B \in \mathcal{B}_c\left(\mathbb{R}^d\right)$,*

(i) *users arrival times to B is a homogeneous Poisson process $\sum_{n\in\mathbb{Z}} \delta_{T_n}$ on $\mathbb{R}$ with intensity $\Lambda(B)$;*

(ii) the position X_n of the arrival at time T_n is picked at random in B independently of any thing else according to $\Lambda(\mathrm{d}x)/\Lambda(B)$.

We define, for each $B \in \mathcal{B}_c(\mathbb{R}^d)$, the arrival process *to B as the marked point process $\sum_{n\in\mathbb{Z}} \delta_{(T_n,X_n)}$ which is, by Theorem 2.2.21, a Poisson point process on $\mathbb{R}\times B$ with intensity measure*

$$\Lambda(B)\,\mathrm{d}t\frac{\Lambda(\mathrm{d}x)}{\Lambda(B)} = \mathrm{d}t \times \Lambda(\mathrm{d}x).$$

Then, we define the arrival process *to the whole network as the Poisson point process on $\mathbb{R}\times\mathbb{R}^d$ with intensity measure $\mathrm{d}t\times\Lambda(\mathrm{d}x)$. Proposition 2.2.4 shows that such a process exists.*

In the particular case when the measure Λ has a density; i.e., $\Lambda(\mathrm{d}x) = \lambda\mathrm{d}x$, then the arrival process is a homogeneous Poisson point process on $\mathbb{R}\times\mathbb{R}^d$ with intensity λ.

Example 2.6.10. Space-time arrival process. *The arrival process described in Example 2.6.9 may be described alternatively as follows. Given a locally finite measure Λ on $\mathbb{R}^d$, assume that within each time interval $I \in \mathcal{B}_c(\mathbb{R})$,*

(i) users arrival locations is a homogeneous Poisson process $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ on $\mathbb{R}^d$ with intensity measure $\Lambda(\mathrm{d}x)\times\ell(I)$ where ℓ is the Lebesgue measure;

(ii) the arrival times $\{T_n\}_{n\in\mathbb{Z}}$ are i.i.d. marks of Φ with the uniform distribution on I.

Following the same lines as in Example 2.6.9, we may construct a marked point process $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(X_n,T_n)}$ which is a Poisson point process on $\mathbb{R}^d\times\mathbb{R}$ with intensity measure $\Lambda(\mathrm{d}x)\times\mathrm{d}t$. Such a process is called a space-time arrival process.

2.6.3 For Section 2.4

Example 2.6.11. Interference in wireless networks. *Consider the model of wireless networks described in Example 2.6.8. Assuming that each base station transmits a power equal to 1, the interference, defined as the total power received by the user from all the base stations in the network, equals*

$$I = \sum_{n\in\mathbb{N}} \frac{1}{Y_n} = \int_{\mathbb{R}_+} \frac{1}{u}\tilde{\Phi}(\mathrm{d}u) = \tilde{\Phi}(f),$$

where $f(u) = \frac{1}{u}$. This is a cumulative shot-noise whose Laplace transform is deduced from Proposition 2.4.3

$$\begin{aligned}\mathcal{L}_I(t) &= \exp\left[-\int_{\mathbb{R}_+}\left(1-e^{-tf(u)}\right)M_{\tilde{\Phi}}(\mathrm{d}u)\right]\\ &= \exp\left[-\frac{2}{\beta}a\int_0^\infty\left(1-e^{-\frac{t}{u}}\right)u^{\frac{2}{\beta}-1}\mathrm{d}u\right]\\ &= \exp\left[-\frac{2}{\beta}at^{\frac{2}{\beta}}\int_0^\infty\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v\right],\end{aligned} \tag{2.6.4}$$

where the second equality is due to (2.6.1). For the third one, we make the change of variable $v = \frac{t}{u}$ and a is given by (2.6.2). Integrating by parts, we get

$$\begin{aligned}\int_0^\infty\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v &= \left[\left(1-e^{-v}\right)\left(\frac{-\beta}{2}v^{-\frac{2}{\beta}}\right)\right]_0^\infty - \int_0^\infty e^{-v}\left(\frac{-\beta}{2}v^{-\frac{2}{\beta}}\right)\mathrm{d}v\\ &= \frac{\beta}{2}\int_0^\infty e^{-v}v^{-\frac{2}{\beta}}\mathrm{d}v = \frac{\beta}{2}\Gamma\left(1-\frac{2}{\beta}\right).\end{aligned}$$

Thus

$$\mathcal{L}_I(t) = \exp\left[-a\Gamma\left(1-\frac{2}{\beta}\right)t^{\frac{2}{\beta}}\right]. \tag{2.6.5}$$

In particular, it follows from Lemma 13.C.9 that

$$\mathbf{P}(I=0) = 0. \tag{2.6.6}$$

Joint probability distribution of propagation loss and interference in wireless networks. *Letting $Y = \inf_{n\in\mathbb{N}} Y_n$, we deduce from Proposition 2.4.5 that*

$$\begin{aligned}\mathbf{E}\left[\mathbf{1}\{Y\geq s\}e^{-tI}\right] &= \mathbf{E}\left[1\left\{\frac{1}{Y}\leq\frac{1}{s}\right\}e^{-t\Phi(f)}\right]\\ &= \exp\left[-\int_{\mathbb{R}_+}\left(1-e^{-\frac{t}{u}}1\left\{\frac{1}{u}\leq\frac{1}{s}\right\}\right)M_{\tilde{\Phi}}(\mathrm{d}u)\right]\\ &= \exp\left[-\frac{2}{\beta}a\int_s^\infty\left(1-e^{-\frac{t}{u}}\right)u^{\frac{2}{\beta}-1}\mathrm{d}u - \frac{2}{\beta}a\int_0^s u^{\frac{2}{\beta}-1}\mathrm{d}u\right]\\ &= \exp\left[-\frac{2}{\beta}at^{\frac{2}{\beta}}\int_0^{\frac{t}{s}}\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v - as^{\frac{2}{\beta}}\right],\end{aligned}$$

where the last equality is obtained by the change of variable $v = \frac{t}{u}$. Note that

$$\begin{aligned}\int_0^{\frac{t}{s}}\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v &= \int_0^\infty\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v - \int_{\frac{t}{s}}^\infty\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v\\ &= \frac{\beta}{2}\Gamma\left(1-\frac{2}{\beta}\right) - \int_{\frac{t}{s}}^\infty\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v\end{aligned}$$

and

$$\int_{\frac{t}{s}}^{\infty}\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v = -\int_{\frac{t}{s}}^{\infty} v^{-1-\frac{2}{\beta}}\mathrm{d}v - \int_{\frac{t}{s}}^{\infty} e^{-v}v^{-1-\frac{2}{\beta}}\mathrm{d}v$$
$$= -\frac{\beta}{2}\left(\frac{t}{s}\right)^{-\frac{2}{\beta}} - \Gamma\left(-\frac{2}{\beta},\frac{t}{s}\right),$$

where $\Gamma(\alpha,x)=\int_x^\infty e^{-v}v^{a-1}\mathrm{d}v$, $\alpha\in\mathbb{R}_-^*$, $x\in\mathbb{R}_+^*$ *is the upper incomplete gamma function. Then*

$$\mathbf{E}\left[\mathbf{1}\{Y\geq s\}e^{-tI}\right]=\exp\left[-\frac{2}{\beta}at^{\frac{2}{\beta}}\left\{\frac{\beta}{2}\Gamma\left(1-\frac{2}{\beta}\right)+\Gamma\left(-\frac{2}{\beta},\frac{t}{s}\right)\right\}\right].$$

Example 2.6.12. Illustration of Theorem 2.2.21 for wireless networks. *We shall retrieve the results already obtained in Examples 2.6.8 and 2.6.11 using Theorem 2.2.21. Recall that the fading random variables* $\{Z_n\}_{n\in\mathbb{N}}$ *are i.i.d. marks of the process of base station locations* $\Phi=\sum_{n\in\mathbb{N}}\delta_{X_n}$. *By Theorem 2.2.21, the marked point process*

$$\tilde{\Phi}=\sum_{n\in\mathbb{N}}\delta_{(X_n,Z_n)}$$

is Poisson with intensity

$$\tilde{\Lambda}(\mathrm{d}x\times\mathrm{d}z)=F_Z(\mathrm{d}z)\,\lambda\mathrm{d}x.$$

Letting $f(x,z)=\frac{z}{l(|x|)}$ *and* $V:=\sup_{n\in\mathbb{N}}f(X_n,Z_n)$, *we deduce from Proposition 2.4.4 that*

$$\mathbf{P}(V\leq u)=\exp\left[-\int_{\mathbb{R}^2\times\mathbb{R}_+}\mathbf{1}\{f(x,z)>u\}\tilde{\Lambda}(\mathrm{d}x\times\mathrm{d}z)\right]$$
$$=\exp\left[-\int_{\mathbb{R}^2\times\mathbb{R}_+}\mathbf{1}\left\{\frac{z}{l(|x|)}>u\right\}F_Z(\mathrm{d}z)\,\lambda\mathrm{d}x\right]$$
$$=\exp\left[-\lambda\int_{\mathbb{R}_+}\left[\int_{\mathbb{R}^2}\mathbf{1}\left\{\frac{z}{(K|x|)^\beta}>u\right\}\mathrm{d}x\right]F_Z(\mathrm{d}z)\right]$$
$$=\exp\left[-\lambda\int_{\mathbb{R}_+}\left[\int_{\mathbb{R}^2}\mathbf{1}\left\{|x|<\frac{1}{K}\left(\frac{z}{u}\right)^{\frac{1}{\beta}}\right\}\mathrm{d}x\right]F_Z(\mathrm{d}z)\right]$$
$$=\exp\left[-\lambda\int_{\mathbb{R}_+}\frac{\pi}{K^2u^{\frac{2}{\beta}}}z^{\frac{2}{\beta}}F_Z(\mathrm{d}z)\right]=e^{-au^{-\frac{2}{\beta}}},$$

where $a=\lambda\pi K^{-2}\mathbf{E}\left[Z^{\frac{2}{\beta}}\right]$. *Noting that* $V=1/Y$ *where* $Y=\inf_{n\in\mathbb{N}}Y_n$ *we check that the above result is consistent with (2.6.3). The probability distribution of* V *is called a* Fréchet *probability distribution with shape parameter* $\frac{2}{\beta}$ *and scale parameter* $a^{\frac{\beta}{2}}$.

Assuming that each base station transmits a power equal to 1*, the total power received by the user from all the base stations in the network equals*

$$\begin{aligned}\sum_{n\in\mathbb{N}}\frac{1}{Y_n} &= \sum_{n\in\mathbb{N}}\frac{Z_n}{l\left(|X_n|\right)}\\ &= \sum_{n\in\mathbb{N}} f(X_n, Z_n)\\ &= \int_{\mathbb{R}^2\times\mathbb{R}_+} f(x,z)\tilde{\Phi}\left(\mathrm{d}x\times\mathrm{d}z\right) = \tilde{\Phi}\left(f\right)\end{aligned}$$

which is a cumulative shot-noise. We deduce from Proposition 2.4.3 that the Laplace transform of the total received power is given by, for all $t\in\mathbb{R}_+$*,*

$$\begin{aligned}\mathcal{L}_{\tilde{\Phi}(f)}\left(t\right) &= \exp\left[-\int_{\mathbb{R}^2\times\mathbb{R}_+}\left(1-e^{-tf(x,z)}\right)\tilde{\Lambda}\left(\mathrm{d}x\times\mathrm{d}z\right)\right]\\ &= \exp\left[-\int_{\mathbb{R}^2\times\mathbb{R}_+}\left(1-e^{-\frac{tz}{l(|x|)}}\right)F_Z\left(\mathrm{d}z\right)\lambda\mathrm{d}x\right]\\ &= \exp\left[-2\pi\lambda\int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+}\left(1-e^{-\frac{tz}{l(r)}}\right)F_Z\left(\mathrm{d}z\right)\right]r\mathrm{d}r\right]\\ &= \exp\left[-2\pi\lambda\int_{\mathbb{R}_+}\left[1-\mathcal{L}_Z\left(\frac{t}{l\left(r\right)}\right)\right]r\mathrm{d}r\right],\end{aligned}$$

where $\mathcal{L}_Z\left(u\right)=\mathbf{E}\left[e^{-uZ}\right]$ *is the Laplace transform of the fading random variable* Z*. For* $l\left(r\right)=\left(Kr\right)^\beta$*, we continue the last but one equality by making the change of variable* $v=\frac{tz}{l(r)}$

$$\begin{aligned}\mathcal{L}_{\tilde{\Phi}(f)}\left(t\right) &= \exp\left[-\frac{2}{\beta}\pi\lambda K^{-2}\mathbf{E}\left[Z^{\frac{2}{\beta}}\right]t^{\frac{2}{\beta}}\int_{\mathbb{R}_+}\left(1-e^{-v}\right)v^{-\frac{1}{\beta}-2}\mathrm{d}v\right]\\ &= \exp\left[-\frac{2}{\beta}at^{\frac{2}{\beta}}\int_0^\infty\left(1-e^{-v}\right)v^{-1-\frac{2}{\beta}}\mathrm{d}v\right]\end{aligned}$$

which is the same result as (2.6.4).

Example 2.6.13. Poisson shot-noise with marks. *Let* $\Phi=\sum_{n\in\mathbb{Z}}\delta_{X_n}$ *be a Poisson point process on* $\mathbb{R}^d$ *of intensity* λ*, and let* $\{Z_n\}_{n\in\mathbb{Z}}$ *be a i.i.d. sequence of nonnegative random variables and independent of* Φ*. For any* $B\in\mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$*, let*

$$\bar{\Phi}\left(B\right)=\sum_{n\in\mathbb{Z}}Z_n\mathbf{1}\left\{X_n\in B\right\}.$$

We will show that, for any $\theta\in\mathbb{R}_+$*,*

$$\mathbf{E}\left[e^{-\theta\bar{\Phi}(B)}\right]=\exp\left[-\left(1-\mathcal{L}_{Z_0}\left(\theta\right)\right)M_\Phi(B)\right].$$

Indeed, letting $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(X_n, Z_n)}$ *and* $f(x, z) = z\mathbf{1}\{x \in B\}$*, then* $\bar{\Phi}(B) = \sum_{n\in\mathbb{Z}} f(X_n, Z_n) = \tilde{\Phi}(f)$*. Thus it follows from Example 2.4.7(iii) that*

$$\begin{aligned}
\mathbf{E}\left[e^{-\theta\bar{\Phi}(B)}\right] &= \mathbf{E}\left[e^{-\theta\tilde{\Phi}(f)}\right] \\
&= \exp\left[-\int_{\mathbb{G}}\left(1 - \mathbf{E}\left[e^{-\theta Z_0 1\{x\in B\}}\right]\right) M_\Phi(\mathrm{d}x)\right] \\
&= \exp\left[-\int_B \left(1 - \mathbf{E}\left[e^{-\theta Z_0}\right]\right) M_\Phi(\mathrm{d}x)\right] \\
&= \exp\left[-\left(1 - \mathcal{L}_{Z_0}(\theta)\right) M_\Phi(B)\right].
\end{aligned}$$

Example 2.6.14. Wide-sense stationarity. *Consider the random filtering* $Y(t)$ *defined in Example 2.4.8. Besides the assumptions of Example 2.4.8(ii), assume that* Φ *is homogeneous Poisson point process on* $\mathbb{R}^d$ *with intensity* λ *and that the marks* $\{Z_n\}_{n\in\mathbb{Z}}$ *are independent of the ground process* Φ*. Then*

$$\mathbf{E}[Y(t)] = \lambda\int_{\mathbb{R}^d}\mathbf{E}[h(t-x, Z_0)]\,\mathrm{d}x = \lambda\int_{\mathbb{R}^d}\mathbf{E}[h(x, Z_0)]\,\mathrm{d}x, \quad t \in \mathbb{R}^d$$

and

$$\operatorname{cov}(Y(t), Y(t+\tau)) = \lambda\int_{\mathbb{R}^d}\mathbf{E}[h(x, Z_0)\,h^*(x+\tau, Z_0)]\,\mathrm{d}x, \quad t, \tau \in \mathbb{R}^d.$$

Note that the above two quantities don't depend on the parameter t*. Thus* $\{Y(t)\}_{t\in\mathbb{R}^d}$ *is a wide-sense stationary stochastic process [18, §B1.2.2].*

Example 2.6.15. Exponential shot-noise. *Consider the random filtering* $Y(t)$ *defined in Example 2.4.8. Assume that the ground process* Φ *is Poisson on* $\mathbb{R}_+$ *and that the marks* $\{Z_n\}_{n\in\mathbb{Z}}$ *are independent of* Φ *and are integrable; i.e.,* $\mathbf{E}[|Z_n|] < \infty$*. Let*

$$h(t, z) = ze^{-\alpha t}\mathbf{1}_{\{t\geq 0\}}, \quad t, z \in \mathbb{R},$$

where α *is a given positive number, then the random filtering (2.4.5) equals*

$$Y(t) = \sum_{n\in\mathbb{Z}} h(t - X_n, Z_n) = \sum_{n\in\mathbb{Z}} Z_n e^{-\alpha(t-X_n)}.$$

The left hand side of Condition (2.4.6) writes $\mathbf{E}[|Z_0|]\,e^{-\alpha t}\int_0^t e^{\alpha x} M_\Phi(\mathrm{d}x)$ *which is finite since the intensity measure of a Poisson point process is locally finite by definition. Then the Laplace transform of* $Y(t)$ *follows from (2.4.7), for* $\theta \in \mathbb{R}_+$*,*

$$\begin{aligned}
\mathbf{E}\left[e^{-\theta Y(t)}\right] &= \exp\left(-\int_0^t (1 - \mathbf{E}\left[e^{-\theta Z_0 h(t-x)}\right]) M_\Phi(\mathrm{d}x)\right) \\
&= \exp\left(-\int_0^t (1 - L_{Z_0}\left(\theta e^{-\alpha(t-x)}\right)) M_\Phi(\mathrm{d}x)\right).
\end{aligned}$$

Example 2.6.16. Shot-noise Cox point process. *Let Φ be a point process on $\mathbb{R}^d$ with mean measure $\lambda \mathrm{d}t$ and $h : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function such that*

$$\rho := \int_{\mathbb{R}^d} h(t)\,\mathrm{d}t < \infty.$$

A Cox point process $\bar{\Phi}$ on $\mathbb{R}^d$ directed by the random measure $\Lambda(\mathrm{d}t) = \lambda(t)\,\mathrm{d}t$ where

$$\lambda(t) = \int_{\mathbb{R}^d} h(t-x)\,\Phi(\mathrm{d}x) = \sum_{X\in\Phi} h(t-X) \tag{2.6.7}$$

is called a shot-noise Cox point process. *This may be interpreted by saying that, conditionally to Φ, each point X of Φ generates* descendants *according to a Poisson point process of intensity measure $h(t-X)\,\mathrm{d}t$.*

It follows from (2.3.1) that the mean measure of $\bar{\Phi}$ is

$$\begin{aligned}
M_{\bar{\Phi}}(\mathrm{d}t) &= \mathbf{E}[\Lambda(\mathrm{d}t)] \\
&= \mathbf{E}[\lambda(t)]\,\mathrm{d}t \\
&= \left[\int_{\mathbb{R}^d} h(t-x)\,M_\Phi(\mathrm{d}x)\right]\mathrm{d}t \\
&= \left[\int_{\mathbb{R}^d} h(t-x)\,\lambda \mathrm{d}x\right]\mathrm{d}t = \rho\lambda\mathrm{d}t,
\end{aligned} \tag{2.6.8}$$

where the third equality follows from the Campbell averaging theorem 1.2.5.

It follows from (2.3.2) that the Laplace transform of $\bar{\Phi}$ equals, for $f \in \mathfrak{F}_+(\mathbb{G})$,

$$\begin{aligned}
\mathcal{L}_{\bar{\Phi}}(f) &= \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d}(1-e^{-f(t)})\lambda(t)\,\mathrm{d}t\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d}(1-e^{-f(t)})\int_{\mathbb{R}^d} h(t-x)\,\Phi(\mathrm{d}x)\,\mathrm{d}t\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}(1-e^{-f(t)})h(t-x)\,\mathrm{d}t\right]\Phi(\mathrm{d}x)\right)\right] \\
&= \mathcal{L}_\Phi\left(\bar{f}\right),
\end{aligned}$$

where $\bar{f}(x) = \int_{\mathbb{R}^d}(1-e^{-f(t)})h(t-x)\,\mathrm{d}t$, $x \in \mathbb{R}^d$.

Note that the above Laplace transform equals that obtained in Example 2.3.13 for a Cox cluster point process if the descendant process mean measure equals $h(t)\,\mathrm{d}t$. This may be justified by the interpretation we gave just after Equation (2.6.7). Consequently, a Cox cluster point process is sometimes called a generalized shot-noise Cox point process.

Example 2.6.17. Spatial Hawkes process. *Let Φ_0 be a point process on $\mathbb{R}^d$ with mean measure $\lambda_0\mathrm{d}t$ called the* ancestors process *and $h : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function such that*

$$\rho := \int_{\mathbb{R}^d} h(t)\,\mathrm{d}t < \infty.$$

The spatial Hawkes process *is constructed recursively (in terms of* branching*) as follows*

$$\Phi = \sum_{n \geq 0} \Phi_n,$$

such that Φ_n *is a Cox point process on* $\mathbb{R}^d$ *directed by the random measure* $\Lambda_n(\mathrm{d}t) = \lambda_n(t)\,\mathrm{d}t$ *where*

$$\lambda_n(t) = \int_{\mathbb{R}^d} h(t-x)\,\Phi_{n-1}(\mathrm{d}x) = \sum_{X \in \Phi_{n-1}} h(t-X).$$

As in Example 2.6.16, this may be interpreted by saying that, conditionally to Φ_{n-1}*, each point* X *of* Φ_{n-1} *generates* descendants *according to a Poisson point process of intensity measure* $h(t-X)\,\mathrm{d}t$ *(the function* h *is called the* fertility rate function*).*

Note that Φ_1 *is a shot-noise Cox point process as defined in Example 2.6.16). Then its mean measure is given by Equation (2.6.8); that is* $M_{\Phi_1}(\mathrm{d}t) = \rho\lambda_0\mathrm{d}t$*. Hence we may show by induction that* Φ_n *is a shot-noise Cox point process with mean measure* $M_{\Phi_n}(\mathrm{d}t) = \rho^n\lambda_0\mathrm{d}t$*. If* $\rho < 1$*, then* $\sum_{n\geq 0}\rho^n < \infty$ *and by Corollary 2.2.2,* $\Phi = \sum_{n\geq 0}\Phi_n$ *is a well defined point process with mean measure*

$$M_\Phi(\mathrm{d}t) = \sum_{n\geq 0}\rho^n\lambda_0\mathrm{d}t = \frac{\lambda_0}{1-\rho}\mathrm{d}t.$$

2.7 Exercises

2.7.1 For Section 2.1

Exercise 2.7.1. Conditional distribution of points of Poisson point process. *Let* Φ *be a Poisson point process on a l.c.s.h space* $\mathbb{G}$ *with intensity measure* Λ*. Consider pairwise disjoint sets* $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G})$*, and let* $B = \bigcup_{j=1}^{k} B_j$*. Show that*

$$\mathbf{P}\left(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k \mid \Phi(B) = n\right) = \frac{n!}{n_1! \cdots n_k!}\frac{1}{\Lambda(B)^n}\prod_{j=1}^{k}\Lambda(B_j)^{n_j},$$

for all $n, n_1, \ldots, n_k \in \mathbb{N}$ *such that* $n_1 + \cdots + n_k = n$*. In words, conditionally to* $\Phi(B) = n$*, the random vector* $(\Phi(B_1), \ldots, \Phi(B_k))$ *has a multinomial distribution.*

Solution 2.7.1. *Let* $n_1, \ldots, n_k$ *be nonnegative integers such that* $n_1 + \cdots + n_k =$

n. Then

$$
\begin{aligned}
&\mathbf{P}(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k \mid \Phi(B) = n) \\
&= \frac{\mathbf{P}(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k, \Phi(B) = n)}{\mathbf{P}(\Phi(B) = n)} \\
&= \frac{\mathbf{P}(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k)}{\mathbf{P}(\Phi(B) = n)} \\
&= \frac{\mathbf{P}(\Phi(B_1) = n_1) \cdots \mathbf{P}(\Phi(B_k) = n_k)}{\mathbf{P}(\Phi(B) = n)} \\
&= e^{-\Lambda(B_1)} \frac{\Lambda(B_1)^{n_1}}{n_1!} \cdots e^{-\Lambda(B_k)} \frac{\Lambda(B_k)^{n_k}}{n_k!} \Big/ \left[\Lambda(B)^n \frac{e^{-\Lambda(B)}}{n!}\right] \\
&= \frac{n!}{n_1! \cdots n_k!} \frac{1}{\Lambda(B)^n} \prod_{j=1}^{k} \Lambda(B_j)^{n_j}.
\end{aligned}
$$

Exercise 2.7.2. Construction of an inhomogeneous Poisson point process *on $\mathbb{R}$. Let $\lambda : \mathbb{R} \to \mathbb{R}_+$ some measurable function and Ψ be a homogeneous Poisson point process on $\mathbb{R}^2$ with unit intensity. Consider the point process on $\mathbb{R}$ whose atoms are the projections on the abscissa axis of the atoms of Ψ located in $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \lambda(x)\}$. Show that Φ is a Poisson point process on $\mathbb{R}$ with intensity measure $\Lambda(\mathrm{d}x) = \lambda(x)\,\mathrm{d}x$.*

Solution 2.7.2. *Indeed, for $B \in \mathcal{B}(\mathbb{R})$, we have $\Phi(B) = \Psi(\{(x, y) : x \in B, 0 \leq y \leq \lambda(x)\})$. Observe that for pairwise disjoint $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R})$, $\{(x, y) : x \in B_1, 0 \leq y \leq \lambda(x)\}, \ldots, \{(x, y) : x \in B_k, 0 \leq y \leq \lambda(x)\}$ are pairwise disjoint. Then $\Phi(B_1), \ldots, \Phi(B_k)$ are independent. Moreover,*

$$
\begin{aligned}
\int_{\mathbb{R}^2} \mathbf{1}_{\{(x,y):x\in B, 0\leq y\leq \lambda(x)\}} \mathrm{d}x\mathrm{d}y &= \int_B \left(\int_0^{\lambda(x)} \mathrm{d}y\right) \mathrm{d}x \\
&= \int_B \lambda(x)\,\mathrm{d}x = \Lambda(B).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\Phi(B) = \Psi(\{(x, y) : x \in B, 0 \leq y \leq \lambda(x)\}) \\
&\overset{\text{dist.}}{\sim} \mathbf{P}\left(\int_{\mathbb{R}^2} \mathbf{1}_{\{(x,y):x\in B, 0\leq y\leq \lambda(x)\}} \mathrm{d}x\mathrm{d}y\right) = \mathbf{P}(\Lambda(B)),
\end{aligned}
$$

where $\mathbf{P}(\alpha)$ is the Poisson distribution of mean α.

Exercise 2.7.3. Poisson point process in a random interval. *Let Φ be a homogeneous Poisson point process of intensity λ on $\mathbb{R}$ and Z_1, Z_2 be two real random variables independent of Φ such that $Z_1 \leq Z_2$. Compute the probability distribution of $\Phi((Z_1, Z_2])$.*

Solution 2.7.3. *For $n \in \mathbb{N}$,*

$$\begin{aligned}\mathbf{P}(\Phi((Z_1, Z_2]) = n) &= \mathbf{E}[\mathbf{1}\{\Phi((Z_1, Z_2]) = n\}] \\ &= \mathbf{E}[\mathbf{E}[\mathbf{1}\{\Phi((Z_1, Z_2]) = n\} \mid Z_1, Z_2]] \\ &= \mathbf{E}\left[e^{-\lambda(Z_2 - Z_1)} \frac{[\lambda(Z_2 - Z_1)]^n}{n!}\right] \\ &= \frac{\lambda^n}{n!}\mathbf{E}\left[e^{-\lambda(Z_2 - Z_1)}(Z_2 - Z_1)^n\right].\end{aligned}$$

Exercise 2.7.4. Two Poisson point processes. *Let Φ_1 and Φ_2 be two independent Poisson point processes on $\mathbb{R}^d$ of intensity measures Λ_1 and Λ_2 respectively. Calculate the mean number of points of Φ_1 which are at least at distance r of any point of Φ_2. Show that if Φ_2 is homogeneous of intensity λ_2, then this mean number equals $e^{-\lambda_2 \pi r^2}\Lambda_1(\mathbb{R}^d)$.*

Solution 2.7.4. *The mean number of atoms of Φ_1 which are at least at distance r of any point of Φ_2 equals*

$$\begin{aligned}\mathbf{E}\left[\sum_{X \in \Phi_1} \mathbf{1}\{\Phi_2(B(X,r)) = 0\}\right] &= \mathbf{E}\left[\mathbf{E}\left[\sum_{X \in \Phi_1} \mathbf{1}\{\Phi_2(B(X,r)) = 0\} \middle| \Phi_1\right]\right] \\ &= \mathbf{E}\left[\sum_{X \in \Phi_1} \mathbf{P}(\Phi_2(B(X,r)) = 0 \mid \Phi_1)\right] \\ &= \mathbf{E}\left[\sum_{X \in \Phi_1} e^{-\Lambda_2(B(X,r))}\right] \\ &= \mathbf{E}\left[\int_{\mathbb{R}^d} e^{-\Lambda_2(B(x,r))}\Phi_1(\mathrm{d}x)\right] \\ &= \int_{\mathbb{R}^d} e^{-\Lambda_2(B(x,r))}\Lambda_1(\mathrm{d}x),\end{aligned}$$

where the last equality follows from Theorem 1.2.5. If Φ_2 is homogeneous of intensity λ_2, then

$$\mathbf{E}\left[\sum_{X \in \Phi_1} \mathbf{1}\{\Phi_2(B(X,r)) = 0\}\right] = e^{-\lambda_2 \pi r^2}\Lambda_1(\mathbb{R}^d).$$

2.7.2 For Section 2.2

Exercise 2.7.5. Counting points falling in a shape. *Points fall at some times constituting a Poisson point process $\Phi = \sum_{n \in \mathbb{Z}} \delta_{T_n}$ on $\mathbb{R}$ with intensity measure Λ. Their locations are i.i.d marks $\{Z_n\}_{n \in \mathbb{Z}}$ in $\mathbb{R}^2$. For each $t \in \mathbb{R}$, let $S(t)$ be a measurable subset of $\mathbb{R}^2$. Show that*

$$\bar{\Phi} = \sum_{n \in \mathbb{Z}} \delta_{T_n}\mathbf{1}\{Z_n \in S(T_n)\}$$

is a Poisson point process with intensity measure. (We may see $S(t)$ as a moving shape collecting the points falling in it.)

Solution 2.7.5. *The point process $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(T_n, Z_n)}$ is Poisson with intensity measure $M_{\tilde{\Phi}}(\mathrm{d}t \times \mathrm{d}z) = \Lambda(\mathrm{d}t)\,\mathbf{P}_Z(\mathrm{d}z)$. Then $\bar{\Phi}$ is an independent displacement of $\tilde{\Phi}$ through the kernel*

$$p((t,z), B) = \mathbf{1}\{t \in B\}\, 1\{z \in S(t)\}, \quad t \in \mathbb{R}, z \in \mathbb{R}^2, B \in \mathcal{B}(\mathbb{G}).$$

If follows from the displacement theorem 2.2.17 that $\bar{\Phi}$ is a Poisson point process with intensity measure

$$M_{\bar{\Phi}}(\mathrm{d}t) = \int \mathbf{1}\{z \in S(t)\}\, \mathbf{P}_Z(\mathrm{d}z)\, \Lambda(\mathrm{d}t) = \mathbf{P}(Z \in S(t))\, \Lambda(\mathrm{d}t).$$

Exercise 2.7.6. Lightning strikes. *Lightning strikes at some times constituting a homogeneous Poisson point process $\Phi = \sum_{n\geq 1} \delta_{T_n}$ on $\mathbb{R}_+$ with intensity λ. Lightning strokes are modelled as discs with centers $\{Z_n\}_{n\geq 1}$ and radii $\{R_n\}_{n\geq 1}$ which are i.i.d. marks of Φ. A tree is modelled by a disc centered at the origin of radius a. Show that the first time τ the tree is hit by lightning is an exponential random variable of parameter $\lambda\mathbf{P}(|Z_1| - R_1 \leq a)$.*

Solution 2.7.6. *The point process counting the lightning hitting the tree*

$$\bar{\Phi} = \sum_{n\geq 1} \delta_{T_n} \mathbf{1}\{|Z_n| - R_n \leq a\}$$

is a thinning of Φ. Thus $\bar{\Phi}$ is a homogeneous Poisson point process of intensity

$$\hat{\lambda} = \lambda\mathbf{P}(|Z_1| - R_1 \leq a).$$

Then τ is an exponential random variable of parameter $\hat{\lambda} = \lambda\mathbf{P}(|Z_1| - R_1 \leq a)$.

Exercise 2.7.7. Antipersonnel mines. *Antipersonnel mines are placed as a homogeneous Poisson point process centered at the origin. Someone walking straightly away from the origin is injured if he is at a distance lower than a from a mine. What is the probability that he is injured before reaching a distance x. Conditionally to this event, what is the mean distance he remained safe.*

Solution 2.7.7. *Until distance x from the origin, the number of harmful mines is a Poisson random variable of mean $\lambda\left(\pi a^2 + 2ax\right)$. Then the harmful mines constitute a Poisson point process Φ on $\mathbb{R}_+$ with intensity measure $\Lambda([0,x]) = \lambda\left(\pi a^2 + 2ax\right)$. The probability that the person is injured before reaching a distance R equals*

$$\mathbf{P}(\Phi([0,x]) \geq 1) = 1 - e^{-\Lambda([0,x])} = 1 - e^{-\lambda\pi a^2} e^{-2\lambda a x}.$$

The distance along which the person remains safe is the first point X_1 of Φ. For any $u \in [0, x]$, we have

$$\begin{aligned}\mathbf{P}\left(X_1 \le u \mid \Phi([0,x]) \ge 1\right) &= \mathbf{P}\left(\Phi([0,u]) \ge 1 \mid \Phi([0,x]) \ge 1\right)\\ &= \frac{\mathbf{P}\left(\Phi([0,u]) \ge 1\right)}{\mathbf{P}\left(\Phi([0,x]) \ge 1\right)}\\ &= \frac{1 - e^{-\lambda\pi a^2} e^{-2\lambda a u}}{1 - e^{-\lambda\pi a^2} e^{-2\lambda a x}}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}\left[X_1 \mid \Phi([0,x]) \ge 1\right] &= \int_0^x \mathbf{P}\left(X_1 > u \mid \Phi([0,x]) \ge 1\right) \mathrm{d}u\\ &= x - \int_0^x \mathbf{P}\left(X_1 \le u \mid \Phi([0,x]) \ge 1\right) \mathrm{d}u\\ &= x - \frac{x - e^{-\lambda\pi a^2} \int_0^x e^{-2\lambda a u} \mathrm{d}u}{1 - e^{-\lambda\pi a^2} e^{-2\lambda a x}}\\ &= x - \frac{x - e^{-\lambda\pi a^2} \frac{1 - e^{-2\lambda a x}}{2\lambda a}}{1 - e^{-\lambda\pi a^2} e^{-2\lambda a x}}.\end{aligned}$$

Exercise 2.7.8. Obstacles obstructing communication. *Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a point process on $\mathbb{R}^2$ and let $\{Z_n\}_{n\in\mathbb{Z}}$ be i.i.d. marks of Φ where each Z_n is a random geometric shape. Two fixed points A and B in $\mathbb{R}^2$ communicate if the line segment $[A, B]$ does not intersect any of the obstacles $X_n + Z_n$, $n \in \mathbb{Z}$. Calculate the probability that A and B communicate. Show that this probability equals*

$$\exp\left(-\lambda\left(2\left|AB\right| \mathbf{E}\left[R_1\right] + \pi\mathbf{E}\left[R_1^2\right]\right)\right)$$

when Φ is homogeneous of intensity λ and Z_1 is a disc of random radius R_1 centered at the origin.

Solution 2.7.8. *Let $I = [A, B]$ and Λ be the mean measure of Φ. Consider the point process*

$$\Phi' = \sum_{n\in\mathbb{Z}} \delta_{X_n} \mathbf{1}\left\{(X_n + Z_n) \cap I \ne \emptyset\right\}$$

counting the shapes intersecting the line segment I. It is a thinning of Φ, the probability of keeping the point at x being

$$p(x) = \mathbf{P}\left((x + Z_1) \cap I \ne \emptyset\right) = \mathbf{P}\left(x \in I + \check{Z}_1\right),$$

where $\check{Z}_1 = \{-t : t \in Z_1\}$. By Corollary 2.2.7, Φ' is a Poisson point process *with intensity measure* $\Lambda'(\mathrm{d}x) = p(x)\Lambda(\mathrm{d}x)$, *then*

$$\begin{aligned}\mathbf{P}(A \text{ and } B \text{ communicate}) &= \mathbf{P}\left(\Phi'\left(\mathbb{R}^2\right) = 0\right) \\ &= e^{-\Lambda'\left(\mathbb{R}^2\right)} \\ &= \exp\left(-\int_{\mathbb{R}^2} p(x)\Lambda(\mathrm{d}x)\right) \\ &= \exp\left(-\mathbf{E}\left[\int_{\mathbb{R}^2} \mathbf{1}\left\{x \in I + \check{Z}_1\right\}\Lambda(\mathrm{d}x)\right]\right) \\ &= \exp\left(-\mathbf{E}\left[\Lambda\left(I + \check{Z}_1\right)\right]\right).\end{aligned}$$

If Φ *is homogeneous of intensity* λ *and the shape is a disc of center* 0 *and random radius* R_1, *then*

$$\begin{aligned}\mathbf{E}\left[\Lambda\left(I + \check{Z}_1\right)\right] &= \lambda\mathbf{E}\left[\left|I + \check{Z}_1\right|\right] \\ &= \lambda\mathbf{E}\left[\left|I + Z_1\right|\right] \\ &= \lambda\mathbf{E}\left[2\left|I\right|R_1 + \pi R_1^2\right] \\ &= \lambda\left(2\left|I\right|\mathbf{E}\left[R_1\right] + \pi\mathbf{E}\left[R_1^2\right]\right),\end{aligned}$$

where the third equality follows from the fact that $I + Z_1$ *may be obtained by translating* Z_1 *along the segment* I *which gives a rectangle of length* $|I|$ *and width* $2R_1$ *and two half discs of radius* R_1.

Exercise 2.7.9. Policemen catching thieves. *We consider policemen and thieves arriving to some location and remaining there for some duration. The arrival times of thieves (resp. policemen) is a homogeneous Poisson point process on* $\mathbb{R}$ *of intensities* λ *(resp.* λ'*). The sojourn durations of thieves (resp. policemen) are i.i.d. marks of their arrival process of probability distribution* Q *(resp.* Q'*). The arrivals and sojourns of policemen are independent from those of thieves. Sow that the mean number of thieves caught (i.e., meeting a policeman) per unit time equals*

$$\lambda\left(1 - \mathbf{E}\left[e^{-\lambda' Z}\right]e^{-\lambda'\mathbf{E}\left[Z'\right]}\right),$$

where Z *and* Z' *are random variables with respective distributions* Q *and* Q'.

Solution 2.7.9. *Let* $\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$ *be the thieves arrival process and consider the process counting the thieves meeting a policeman*

$$\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{T_n}\mathbf{1}\left\{\textit{thief } n \textit{ is caught}\right\}$$

which is a thinning of Φ*; the probability of keeping the* n*-th point being*

$$p = \mathbf{P}\left(\textit{thief } n \textit{ is caught}\right)$$

which does not depend on n *since, by stationarity, all the thieves have the same probability to be caught. The mean number of thieves caught per unit time is* *precisely the intensity of the process* $\tilde{\Phi}$

$$\frac{\mathbf{E}\left[\tilde{\Phi}\left([0,t]\right)\right]}{t} = \mathbf{E}\left[\tilde{\Phi}\left([0,1]\right)\right] = \lambda p,$$

where the last equality follows from Proposition 2.2.6.

It remains to compute p*. By stationarity, we may consider without loss of generality that a thief arrives at time* 0 *and remains for a random duration* Z*. Let* $\Phi' = \sum_{k\in\mathbb{Z}} \delta_{T'_k}$ *be the policemen arrival process and* Z'_k *the sojourn duration of the policeman arriving at* T'_k*. The thief may be caught either by a policeman arriving at time* $T'_k \in [0, Z]$*, or by a policeman arriving at time* $T'_k < 0$ *such that* $T'_k + Z'_k > 0$*. Let*

$$\Phi'_1 = \sum_{k\in\mathbb{Z}} \delta_{T'_k} 1\left\{T'_k \in [0, Z]\right\}$$

be the process counting the policemen arriving at time $T'_k \in [0, Z]$*. Given* Z*,* Φ'_1 *is a Poisson point process with intensity measure*

$$\Lambda'_1\left(\mathrm{d}t\right) = \lambda' \mathbf{1}\left\{t \in [0, Z]\right\}.$$

Then

$$\mathbf{P}\left(\Phi'_1\left(\mathbb{R}\right) = 0 \mid Z\right) = e^{-\lambda' Z}.$$

On the other hand, let

$$\Phi'_2 = \sum_{k\in\mathbb{Z}} \delta_{T'_k} 1\left\{-Z'_k < T'_k < 0\right\}$$

be the process counting the policemen arriving at time $T'_k < 0$ *such that* $T'_k + Z'_k > 0$*. It is a thinning of* Φ'*, thus its a Poisson point process with intensity*

$$\begin{aligned}\Lambda'_2\left(\mathrm{d}t\right) &= \lambda' 1\left\{t < 0\right\} \mathbf{P}\left(-Z' < t\right)\\ &= \lambda' \mathbf{1}\left\{t < 0\right\} \mathbf{P}\left(Z' > -t\right),\end{aligned}$$

where Z' *is a random variable with probability distribution* Q'*. Then*

$$\begin{aligned}\mathbf{P}\left(\Phi'_2\left(\mathbb{R}\right) = 0\right) &= \exp\left[-\lambda' \int_{-\infty}^{0} \mathbf{P}\left(Z' > -t\right) \mathrm{d}t\right]\\ &= \exp\left[-\lambda' \int_{0}^{\infty} \mathbf{P}\left(Z' > s\right) \mathrm{d}s\right] = e^{-\lambda' \mathbf{E}[Z']}.\end{aligned}$$

Thus

$$\begin{aligned}p &= 1 - \mathbf{E}\left[\mathbf{P}\left(\Phi'_1\left(\mathbb{R}\right) = 0, \Phi'_2\left(\mathbb{R}\right) = 0 \mid Z\right)\right]\\ &= 1 - \mathbf{E}\left[e^{-\lambda' Z} e^{-\lambda' \mathbf{E}[Z']}\right] = 1 - \mathbf{E}\left[e^{-\lambda' Z}\right] e^{-\lambda' \mathbf{E}[Z']}.\end{aligned}$$

Therefore the mean number of thieves caught per unit time is

$$\lambda p = \lambda \left(1 - \mathbf{E}\left[e^{-\lambda' Z}\right] e^{-\lambda' \mathbf{E}[Z']}\right).$$

Exercise 2.7.10. Road traffic. *Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a homogeneous point process on $\mathbb{R}$ with intensity λ representing the initial positions of vehicles and let $\{Z_n\}_{n\in\mathbb{Z}}$ be i.i.d. marks of Φ where Z_n is real-valued random variable representing the n-th vehicle's speed. Therefore the position of the n-th vehicle at time $t \geq 0$ is $X_n(t) = X_n + tZ_n$.*

1. *Show that, for given $t \geq 0$, $\Phi' = \sum_{n\in\mathbb{Z}} \delta_{X_n(t)}$ is a homogeneous Poisson point process on $\mathbb{R}$ with intensity λ.*

2. *Let τ_n be the time at which the n-th vehicle passes through the origin. Show that $\bar{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{\tau_n}$ is a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity $\lambda\mathbf{E}[|Z_1|]$.*

Solution 2.7.10. *By Theorem 2.2.21, $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(X_n,Z_n)}$ is a Poisson point process on $\mathbb{R}^2$ with intensity measure $\tilde{\Lambda}(\mathrm{d}x, \mathrm{d}z) = \lambda \mathrm{d}x \mathbf{P}_{Z_1}(\mathrm{d}z)$.*

1. *Let $Y_n := X_n + tZ_n$ then $\Phi' = \sum_{n\in\mathbb{Z}} \delta_{Y_n}$ may be seen as an independent displacement of $\tilde{\Phi}$ with respect to the kernel*

$$\mathbf{P}\left(Y_n \in B \mid \tilde{\Phi}\right) = \mathbf{1}\{X_n + tZ_n \in B\}, \quad B \in \mathcal{B}(\mathbb{G}).$$

By the displacement theorem 2.2.17, Φ' is a Poisson point process on $\mathbb{R}$ with intensity measure

$$\begin{aligned}
\Lambda'(B) &= \int_{\mathbb{R}^2} \mathbf{1}\{x + tz \in B\} \tilde{\Lambda}(\mathrm{d}x, \mathrm{d}z) \\
&= \lambda \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}\{x \in B - tz\} \mathrm{d}x \right) \mathbf{P}_{Z_1}(\mathrm{d}z) \\
&= \lambda \int_{\mathbb{R}} |B - tz| \mathbf{P}_{Z_1}(\mathrm{d}z) \\
&= \lambda \int_{\mathbb{R}} |B| \mathbf{P}_{Z_1}(\mathrm{d}z) = \lambda |B|,
\end{aligned}$$

where $|B|$ denotes the Lebesgue measure of B. Then Φ' is homogeneous with intensity λ.

2. *Note that $\tau_n = -\frac{X_n}{Z_n}$. Then $\bar{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{\tau_n}$ may be seen as an independent displacement of $\tilde{\Phi}$ with respect to the kernel*

$$\mathbf{P}\left(\tau_n \in B \mid \tilde{\Phi}\right) = \mathbf{1}\left\{-\frac{X_n}{Z_n} \in B\right\}, \quad B \in \mathcal{B}(\mathbb{G}).$$

By the displacement theorem 2.2.17, $\bar{\Phi}$ is a Poisson point process on $\mathbb{R}$

with intensity measure

$$\begin{aligned}
\Lambda'(B) &= \int_{\mathbb{R}^2} 1\left\{-\frac{x}{z} \in B\right\} \tilde{\Lambda}(\mathrm{d}x, \mathrm{d}z) \\
&= \lambda \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}\{x \in -zB\} \mathrm{d}x \right) \mathbf{P}_{Z_1}(\mathrm{d}z) \\
&= \lambda \int_{\mathbb{R}} |-zB| \, \mathbf{P}_{Z_1}(\mathrm{d}z) \\
&= \lambda |B| \int_{\mathbb{R}} |z| \, \mathbf{P}_{Z_1}(\mathrm{d}z) = \lambda |B| \, \mathbf{E}[|Z_1|].
\end{aligned}$$

Then Φ' is homogeneous with intensity $\lambda \mathbf{E}[|Z_1|]$.

Exercise 2.7.11. *[99, Lemma 1] Show that we may extend the results of Example 2.6.8 to $\mathbb{R}^d$ and also replace the constant intensity of base stations by an isotropic power-law function r^α with $-d < \alpha < \beta - d$. Show that this generalization can be done by simply replacing $2/\beta$ by $\alpha/\beta + d/\beta$ in (2.6.1) and λK^{-2} by $\kappa_d/((1+\alpha/d)K^{d+\alpha})$ in (2.6.2), where κ_d is the volume of the unit-radius d-dimensional ball.*

Solution 2.7.11. *The point process of propagation losses $\tilde{\Phi} = \sum_{n\in\mathbb{N}} \delta_{Y_n}$ may be viewed as an independent displacement of the point process of base station locations $\Phi = \sum_{n\in\mathbb{N}} \delta_{X_n}$ by the probability kernel*

$$p(x, B) = \mathbf{P}\left(\frac{l(|x|)}{Z} \in B\right), \quad x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}_+),$$

where $Z = Z_0$. By the displacement theorem 2.2.17, the point process $\tilde{\Phi}$ is Poisson on $\mathbb{R}_+$ with intensity measure

$$\begin{aligned}
M_{\tilde{\Phi}}([0,u)) &= \int_{\mathbb{R}^d} p(x, [0,u)) \, \lambda |x|^\alpha \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathbf{P}\left(\frac{l(|x|)}{Z} \in [0,u)\right) \lambda |x|^\alpha \, \mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbf{1}\left\{\frac{l(|x|)}{s} < u\right\} |x|^\alpha \, \mathrm{d}x \mathbf{P}_Z(\mathrm{d}s).
\end{aligned}$$

Note that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{1}\left\{\frac{l(|x|)}{s} < u\right\} |x|^\alpha \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathbf{1}\left\{|x| < \frac{(su)^{\frac{1}{\beta}}}{K}\right\} |x|^\alpha \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathbf{1}\{|x| < R\} |x|^\alpha \,\mathrm{d}x \\
&= \int_{r=0}^{R}\int_{\phi_1=0}^{\pi}\dots\int_{\phi_{d-2}=0}^{\pi}\int_{\phi_{d-1}=0}^{2\pi} r^{\alpha+d-1}\sin^{d-2}\phi_1\sin^{d-3}\phi_2\dots\sin\phi_{d-2}\mathrm{d}r\mathrm{d}\phi_1\dots\mathrm{d}\phi_{d-1} \\
&= \frac{R^{d+\alpha}}{d+\alpha}\int_{\phi_1=0}^{\pi}\dots\int_{\phi_{d-2}=0}^{\pi}\int_{\phi_{d-1}=0}^{2\pi} \sin^{d-2}\phi_1\sin^{d-3}\phi_2\dots\sin\phi_{d-2}\mathrm{d}\phi_1\dots\mathrm{d}\phi_{d-1} \\
&= \frac{R^{d+\alpha}}{d+\alpha}d\int_{r=0}^{1}\int_{\phi_1=0}^{\pi}\dots\int_{\phi_{d-2}=0}^{\pi}\int_{\phi_{d-1}=0}^{2\pi} r^{d-1}\sin^{d-2}\phi_1\sin^{d-3}\phi_2\dots\sin\phi_{d-2}\mathrm{d}r\mathrm{d}\phi_1\dots\mathrm{d}\phi_{d-1} \\
&= \frac{R^{d+\alpha}}{d+\alpha}d\times\kappa_d,
\end{aligned}
$$

where $R = \frac{(su)^{\frac{1}{\beta}}}{K}$ *and the third equality is due to a change of coordinates from Cartesian to spherical; that is*

$$
\begin{cases}
x_1 = r\cos\phi_1 \\
x_2 = r\sin\phi_1\cos\phi_2 \\
x_3 = r\sin\phi_1\sin\phi_2\cos\phi_3 \\
\vdots \\
x_d = r\sin\phi_1\sin\phi_2\dots\sin\phi_{d-1},
\end{cases}
\tag{2.7.1}
$$

where $r \in \mathbb{R}_+, \phi_1, \dots, \phi_{d-2} \in [0,\pi], \phi_{d-1} \in [0, 2\pi)$ *whose Jacobian equals*

$$
\mathrm{d}x = r^{d-1}\sin^{d-2}\phi_1\sin^{d-3}\phi_2\dots\sin\phi_{d-2}\mathrm{d}r\mathrm{d}\phi_1\dots\mathrm{d}\phi_{d-1}. \tag{2.7.2}
$$

Then

$$
\begin{aligned}
M_{\tilde{\Phi}}([0,u)) &= \lambda\int_{\mathbb{R}_+}\left(\frac{R^{d+\alpha}}{d+\alpha}d\times\kappa_d\right)\mathbf{P}_Z(\mathrm{d}s) \\
&= \lambda\frac{\kappa_d}{\left(1+\frac{\alpha}{d}\right)K^{d+\alpha}}\int_{\mathbb{R}_+}(su)^{\frac{\alpha}{\beta}+\frac{d}{\beta}}\mathbf{P}_Z(\mathrm{d}s) = au^{\frac{\alpha}{\beta}+\frac{d}{\beta}},
\end{aligned}
$$

where

$$
a = \lambda\frac{\kappa_d}{\left(1+\frac{\alpha}{d}\right)K^{d+\alpha}}\mathbf{E}\left[Z^{\frac{\alpha}{\beta}+\frac{d}{\beta}}\right].
$$

2.7.3 For Section 2.3

Exercise 2.7.12. *Let* $(\Omega, \mathcal{A}, \mathbf{P})$ *be a probability space,* Φ *be a homogeneous point process on* $\mathbb{R}$ *of unit intensity,* Z *be a nonnegative random variable,* $T \in \mathbb{R}_+$

and

$$L = Z^{\Phi([0,T])} e^{-(Z-1)T}.$$

Show that the measure Q defined by

$$Q(A) = \mathbf{E}[L \times 1_A], \quad A \in \mathcal{A}$$

is a probability and that under Q, Φ restricted to $[0,T]$ is a Cox point process with directing measure $\Lambda(\mathrm{d}t) = Z \times \mathrm{d}t$.

Solution 2.7.12. *Note that*

$$\begin{aligned} Q(\Omega) &= \mathbf{E}[L] \\ &= \mathbf{E}\left[Z^{\Phi([0,T])} e^{-(Z-1)T}\right] \\ &= \mathbf{E}\left[e^{-(Z-1)T} \mathbf{E}\left[Z^{\Phi([0,T])} \mid Z\right]\right] \\ &= \mathbf{E}\left[e^{-(Z-1)T} e^{(Z-1)T}\right] = 1, \end{aligned}$$

then Q is a probability. Under Q, the Laplace transform of Φ restricted to $[0,T]$ equals

$$\begin{aligned} \mathbf{E}\left[L \times \exp\left(-\int_0^T f \mathrm{d}\Phi\right)\right] &= \mathbf{E}\left[Z^{\Phi([0,T])} e^{-(Z-1)T} \times \exp\left(-\int_0^T f \mathrm{d}\Phi\right)\right] \\ &= \mathbf{E}\left[e^{-(Z-1)T} \times \mathbf{E}\left[Z^{\Phi([0,T])} \exp\left(-\int_0^T f \mathrm{d}\Phi\right) \middle| Z\right]\right] \\ &= \mathbf{E}\left[e^{-(Z-1)T} \times \mathbf{E}\left[\exp\left(-\int_0^T (f - \log Z) \mathrm{d}\Phi\right) \middle| Z\right]\right] \\ &= \mathbf{E}\left[e^{-(Z-1)T} \times \exp\left(-\int_0^T 1 - e^{-[f(t) - \log Z]} \mathrm{d}t\right)\right] \\ &= \mathbf{E}\left[\exp\left(-\int_0^T \left(1 - e^{-f(t)}\right) Z \mathrm{d}t\right)\right], \end{aligned}$$

which is the Laplace transform of a Cox point process with directing measure $\Lambda(\mathrm{d}t) = Z \times \mathrm{d}t$.

Exercise 2.7.13. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$. Show that if Φ is simple, then $\mathbf{P}(\Phi(B) \geq 2) \leq \mathbf{E}\left[\Phi^{(2)}\left(B^2\right)\right]$ for any $B \in \mathcal{B}(\mathbb{G})$.*

Solution 2.7.13. *If Φ is simple, then for any $B \in \mathcal{B}(\mathbb{G})$,*

$$\begin{aligned} \mathbf{P}(\Phi(B) \geq 2) &= \mathbf{P}\left(\exists (x,y) \in B^2 : x \neq y, \Phi(\{x\}) \geq 1, \Phi(\{y\}) \geq 1\right) \\ &= \mathbf{P}\left(\exists (x,y) \in B^2 : \Phi^{(2)}(\{(x,y)\}) \geq 1\right) \\ &= \mathbf{E}\left[\mathbf{1}\left\{\exists (x,y) \in B^2 : \Phi^{(2)}(\{(x,y)\}) \geq 1\right\}\right] \\ &\leq \mathbf{E}\left[\int_{B^2} (\Phi - \delta_x)(\{\mathrm{d}y\})\ \Phi(\mathrm{d}x)\right] = \mathbf{E}\left[\Phi^{(2)}\left(B^2\right)\right]. \end{aligned}$$

2.7.4 For Chapter 2

Exercise 2.7.14. Average of total received power. *Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$, be a point process on $\mathbb{R}^2$ with mean measure $\lambda \mathrm{d}x$ representing the locations of transmitters and let*

$$I_\epsilon = \sum_{n\in\mathbb{Z}} P_{\mathrm{tr}} \times (K\,|X_n - y|)^{-\beta} \times \mathbf{1}\{|X_n - y| > \epsilon\}$$

be the total received power at location $y \in \mathbb{R}^2$ excluding transmitters which are closer than distance $\epsilon > 0$ to the receiver.

1. *Show that*
$$\mathbf{E}[I_\epsilon] = \frac{2\pi\lambda P_{\mathrm{tr}}}{(\beta - 2)\,\epsilon^{\beta-2} K^\beta}. \tag{2.7.3}$$
Discuss the motivation of the assumption $\epsilon > 0$ and $\beta > 2$.

2. *Assuming $\epsilon = 1/K$, show that*
$$\mathbf{E}[I_{1/K}] = \frac{2\pi\lambda P_{\mathrm{tr}}}{(\beta - 2) K^2}.$$
Interpret the choice $\epsilon = 1/K$ observing the value of the received power at distances $r < 1/K$.

Solution 2.7.14. *1. Consider a location $y \in \mathbb{R}^2$,*

$$I_\epsilon = \sum_{n\in\mathbb{Z}} P_{\mathrm{tr}}(K\,|X_n - y|)^{-\beta} \mathbf{1}\{|X_n - y| > \epsilon\} = \int_{\mathbb{R}^2} f(x)\,\Phi(\mathrm{d}x),$$

where $f(x) := P_{\mathrm{tr}}(K\,|x - y|)^{-\beta} \mathbf{1}\{|x - y| > \epsilon\}$. By Campbell averaging formula

$$\begin{aligned}
\mathbf{E}[I_\epsilon] &= \int_{\mathbb{R}^2} f(x)\, M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{R}^2} f(x)\, \lambda \mathrm{d}x \\
&= \lambda P_{\mathrm{tr}} K^{-\beta} \int_{\mathbb{R}^2} |x - y|^{-\beta}\, \mathbf{1}\{|x - y| > \epsilon\}\, \mathrm{d}x \\
&= \lambda P_{\mathrm{tr}} K^{-\beta} \int_{\mathbb{R}^2} |x|^{-\beta}\, \mathbf{1}\{|x| > \epsilon\}\, \mathrm{d}x \\
&= \lambda P_{\mathrm{tr}} K^{-\beta} 2\pi \int_0^\infty r^{-\beta} \mathbf{1}\{r > \epsilon\}\, r \mathrm{d}r \\
&= 2\pi\lambda P_{\mathrm{tr}} K^{-\beta} \int_\epsilon^\infty r^{1-\beta} \mathrm{d}r = \frac{2\pi\lambda P_{\mathrm{tr}}}{(\beta - 2)\,\epsilon^{\beta-2} K^\beta}.
\end{aligned}$$

Then the assumption $\epsilon > 0$ and $\beta > 2$ ensures that $\mathbf{E}[I_\epsilon] < \infty$.

2. For $\epsilon = 1/K$, we get

$$\mathbf{E}[I_{1/K}] = \frac{2\pi\lambda P_{\mathrm{tr}}}{(\beta-2)K^2}.$$

For a distance $r < 1/K$, the received power $P_{\mathrm{rec}} > P_{\mathrm{tr}}$ which is physically meaningless. Taking $\epsilon = 1/K$, avoids this drawback.

Exercise 2.7.15. Laplace transform of total received power. *Consider the setting of Exercise 2.7.14 and assume moreover that Φ is a Poisson point process on $\mathbb{R}^2$. Show that the Laplace transform of $I = I_0$ is equal to*

$$\mathcal{L}_I(\xi) := \mathbf{E}\left[e^{-\xi I}\right] = \exp\left[-\frac{\lambda\pi P_{\mathrm{tr}}^{2/\beta}}{K^2}\Gamma\left(1-\frac{2}{\beta}\right)\xi^{2/\beta}\right],$$

where $\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}\mathrm{d}s$.

Solution 2.7.15. *At a given location $y \in \mathbb{R}^2$,*

$$I = \sum_{n\in\mathbb{Z}} P_{\mathrm{tr}}(K\,|X_n - y|)^{-\beta} = \int_{\mathbb{R}^2} f(x)\,\Phi(\mathrm{d}x),$$

where $f(x) := P_{\mathrm{tr}}(K\,|x-y|)^{-\beta}$. By the Laplace transform expression of Poisson point processes

$$\begin{aligned}
\mathbf{E}\left[e^{-\xi I}\right] &= \mathbf{E}\left[e^{-\int_{\mathbb{R}^2}\xi f(x)\Phi(\mathrm{d}x)}\right] \\
&= \exp\left[-\int_{\mathbb{R}^2}\left(1-e^{-\xi f(x)}\right)\lambda\mathrm{d}x\right] \\
&= \exp\left[-\lambda\int_{\mathbb{R}^2}\left(1-e^{-\xi P_{\mathrm{tr}}(K|x-y|)^{-\beta}}\right)\mathrm{d}x\right] \\
&= \exp\left[-\lambda\int_{\mathbb{R}^2}\left(1-e^{-\xi P_{\mathrm{tr}}(K|x|)^{-\beta}}\right)\mathrm{d}x\right] \\
&= \exp\left[-\lambda 2\pi\int_0^\infty\left(1-e^{-\xi P_{\mathrm{tr}}K^{-\beta}r^{-\beta}}\right)r\mathrm{d}r\right].
\end{aligned}$$

Making the change of variable $v = \left(\xi P_{\mathrm{tr}}K^{-\beta}\right)r^{-\beta}$; that is

$$\begin{aligned}
r &= \left(\xi P_{\mathrm{tr}}K^{-\beta}\right)^{1/\beta}v^{-1/\beta}, \\
\mathrm{d}r &= \left(\xi P_{\mathrm{tr}}K^{-\beta}\right)^{1/\beta}\left(-\frac{1}{\beta}v^{-1-1/\beta}\right)\mathrm{d}v,
\end{aligned}$$

we get

$$\int_0^\infty\left(1-e^{-\xi P_{\mathrm{tr}}K^{-\beta}r^{-\beta}}\right)r\mathrm{d}r = \left(\xi P_{\mathrm{tr}}K^{-\beta}\right)^{2/\beta}\frac{1}{\beta}\int_0^\infty\left(1-e^{-v}\right)v^{-1-2/\beta}\mathrm{d}v.$$

Integrating by parts, we get

$$\int_0^\infty \left(1-e^{-v}\right) v^{-1-\frac{2}{\beta}} \mathrm{d}v = \left[\left(1-e^{-v}\right)\left(\frac{-\beta}{2}v^{-\frac{2}{\beta}}\right)\right]_0^\infty - \int_0^\infty e^{-v}\left(\frac{-\beta}{2}v^{-\frac{2}{\beta}}\right)\mathrm{d}v$$
$$= \frac{\beta}{2}\int_0^\infty e^{-v}v^{-\frac{2}{\beta}}\mathrm{d}v = \frac{\beta}{2}\Gamma\left(1-\frac{2}{\beta}\right).$$

Thus

$$\begin{aligned}\mathbf{E}\left[e^{-\xi I}\right] &= \exp\left[-\lambda 2\pi\left(\xi P_{\mathrm{tr}}K^{-\beta}\right)^{2/\beta}\frac{1}{\beta}\frac{\beta}{2}\Gamma\left(1-\frac{2}{\beta}\right)\right]\\ &= \exp\left[-\frac{\lambda\pi P_{\mathrm{tr}}^{2/\beta}}{K^2}\Gamma\left(1-\frac{2}{\beta}\right)\xi^{2/\beta}\right],\end{aligned}$$

which is consistent with (2.6.5).

Exercise 2.7.16. Fading versus fading-less.

1. *Let $\Phi = \sum_{n\in\mathbb{Z}}\delta_{X_n}$, be a Poisson point process on $\mathbb{R}^2$ with intensity measure Λ. Assume that Λ has a density λ in polar coordinates; i.e., if $x \in \mathbb{R}^2$ has polar coordinates (r,θ), then $\Lambda(\mathrm{d}x) = \lambda(r,\theta)r\mathrm{d}r\mathrm{d}\theta/(2\pi)$. We assume that λ is bounded. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be an independent sequence of marks of Φ. These marks are i.i.d with probability distribution H on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+}\frac{1}{y^2}H(\mathrm{d}y) < \infty$. For all $n \in \mathbb{N}$, let $X'_n := Z_nX_n$ (with the usual convention for the multiplication of a vector by a scalar). Show that $\Phi' = \sum_{n\in\mathbb{Z}}\delta_{X'_n}$ a Poisson point process on $\mathbb{R}^2$. Give its intensity measure.*

2. *Let $\beta > 2$. Let $\Phi = \sum_{n\in\mathbb{Z}}\delta_{X_n}$, be a Poisson point process on $\mathbb{R}^2$ with intensity measure Λ, with the same properties as in 1. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of marks of Φ, representing fading variables, i.i.d. with probability distribution G on $\mathbb{R}_+$. Let $I(0)$ be the shot noise of Φ for these fading variables: $I(0) = \sum_{n\in\mathbb{Z}}\frac{Z_n}{|X_n|^\beta}$. Use 1. to show that, under appropriate conditions on G, there exists a Poisson point process $\Psi = \sum_{n\in\mathbb{Z}}\delta_{Y_n}$ on $\mathbb{R}^2$ such that $I(0)$ has the same distribution as the fading-less shot-noise $J(0) = \sum_{n\in\mathbb{Z}}\frac{1}{|Y_n|^\beta}$. Give the intensity measure of Ψ.*

Solution 2.7.16. *1. The point process Φ' may be seen as an independent displacement of Φ by the kernel*

$$p((r,\theta), A\times B) = \mathbf{P}(Z_0 r \in A)\mathbf{1}_{\{\theta\in B\}}, \quad A \in \mathcal{B}(\mathbb{R}_+), B \in \mathcal{B}([0,2\pi)).$$

Its mean measure is given in polar coordinates by

$$\begin{aligned}
M_{\Phi'}(A \times B) &= \frac{1}{2\pi} \int_{\theta \in [0,2\pi)} \int_{r \in \mathbb{R}_+} p\left((r,\theta), A \times B\right) \lambda(r,\theta) r \mathrm{d}r \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{\theta \in B} \int_{r \in \mathbb{R}_+} \mathbf{P}\left(Z_0 r \in A\right) \lambda(r,\theta) r \mathrm{d}r \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{\theta \in B} \int_{r \in \mathbb{R}_+} \int_{y \in \mathbb{R}_+} \mathbf{1}\{yr \in A\} H(\mathrm{d}y) \lambda(r,\theta) r \mathrm{d}r \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{\theta \in B} \int_{v \in A} \int_{y \in \mathbb{R}_+} \frac{1}{y^2} H(\mathrm{d}y) \lambda\left(\frac{v}{y}, \theta\right) v \mathrm{d}v \mathrm{d}\theta
\end{aligned}$$

where the last equality follows by the change of variable $r \to v = yr$*. Hence* $M_{\Phi'}$ *admits the density*

$$\lambda'(v,\theta) = \int_{y \in \mathbb{R}_+} \lambda\left(\frac{v}{y}, \theta\right) \frac{1}{y^2} H(\mathrm{d}y) \tag{2.7.4}$$

The above integral is finite since λ *is bounded and* $\int_{\mathbb{R}_+} \frac{1}{y^2} H(\mathrm{d}y) < \infty$*. Then* $M_{\Phi'}$ *is locally finite. By the displacement theorem 2.2.17,* Φ' *is a Poisson point process with intensity measure* $M_{\Phi'}$*.*

2. *Note that*

$$I(0) = \sum_{n \in \mathbb{Z}} \frac{Z_n}{|X_n|^\beta} = \sum_{n \in \mathbb{Z}} \frac{1}{|Y_n|^\beta}$$

where $Y_n = Z_n^{-1/\beta} X_n = T_n X_n$ *with* $T_n = Z_n^{-1/\beta}$*. By Question 1, if*

$$\mathbf{E}\left[T_0^{-2}\right] = \mathbf{E}\left[Z_0^{2/\beta}\right] < \infty,$$

then $\Psi = \sum_{n \in \mathbb{Z}} \delta_{Y_n}$ *a Poisson point process on* $\mathbb{R}^2$ *with intensity measure having density (2.7.4) with* H *the distribution of* $Z_0^{-1/\beta}$*.*

In the special case where $\lambda(r,\theta)$ *is a constant, denoted by* λ*, we get that* Ψ *is a homogeneous Poisson point process with intensity* $\lambda \mathbf{E}\left[Z_0^{2/\beta}\right]$*, which is consistent with (2.6.2).*

Exercise 2.7.17. *Let* $\Phi = \sum_{n \in \mathbb{Z}} \delta_{X_n}$ *be a homogeneous Poisson point process on* $\mathbb{R}^2$ *of intensity* $\lambda \in \mathbb{R}_+^*$ *representing the location of transmitters. Consider an attenuation function of the type* $l(r) = r^\beta$ *and a fixed transmission power* $P \in \mathbb{R}_+^*$*. Let* X^* *be the point of* Φ *which is the closest to the origin.*

1. *Let* $M := \frac{P}{|X^*|^\beta}$ *be the power received at 0 from* X^**. Show that the distribution of* M *is Pareto-like; namely, has a tail distribution function equivalent to* C/t^γ*, with* $C \in \mathbb{R}_+^*$ *and* $\gamma \in \mathbb{R}_+^*$*, when* t *tends to infinity.*

2. *Let* $M := \frac{PZ}{|X^*|^\beta}$ *where* Z *is an exponential random variable with unit mean. Discuss the tail of* M*.*

3. *Let R be a positive real number and let $X^*(R)$ be the point the closest to the origin and outside the ball $B(0,R)$ of center 0 and radius R. Let $M_R := \frac{PZ}{|X^*(R)|^\beta}$ where Z is a nonnegative random variable. Show that if Z is light-tailed, then so is M_R. (A random variable is said to be* light-tailed *if its cumulative distribution function decreases exponentially fast.)*

Solution 2.7.17. *1. Consider,*

$$
\begin{aligned}
\mathbf{P}\left(M \geq t\right) &= \mathbf{P}\left(\frac{P}{|X^*|^\beta} \geq t\right) \\
&= \mathbf{P}\left(|X^*| \leq \left(\frac{P}{t}\right)^{1/\beta}\right) \\
&= 1 - \mathbf{P}\left(\Phi(B\left(0, \left(\frac{P}{t}\right)^{1/\beta}\right)\right) = 0) = 1 - \exp\left[-\lambda\pi\left(\frac{P}{t}\right)^{2/\beta}\right].
\end{aligned}
$$

(Note that the above result is consistent with (2.6.3).) Now, we use the approximation $\exp\left[-\lambda\pi\left(\frac{P}{t}\right)^{2/\beta}\right] = 1 - \lambda\pi\left(\frac{P}{t}\right)^{2/\beta} + o\left(\left(\frac{P}{t}\right)^{2/\beta}\right)$, *for large enough t, and obtain*

$$\mathbf{P}(M \geq t) = Ct^{-2/\beta}.$$

This implies that the law of M is Pareto-like, with $\gamma = 2/\beta$. (Note that $\lambda\pi P^{2/\beta}$ gets absorbed in the constant term C.)

2. We have

$$
\begin{aligned}
\mathbf{P}\left(M \geq t\right) &= \mathbf{P}\left(\frac{PZ}{|X^*|^\beta} \geq t\right) \\
&= \int_{\mathbb{R}_+} \mathbf{P}\left(\frac{PZ}{r^\beta} \geq t\right) 2\pi\lambda r e^{-\lambda\pi r^2} \mathrm{d}r \\
&= 2\pi\lambda \int_{\mathbb{R}_+} \mathbf{P}\left(Z \geq \frac{tr^\beta}{P}\right) r e^{-\lambda\pi r^2} \mathrm{d}r \\
&= 2\pi\lambda \int_{\mathbb{R}_+} e^{-\frac{tr^\beta}{P}} e^{-\lambda\pi r^2} r \mathrm{d}r,
\end{aligned}
$$

where for the second equality we have used the distribution of $R_1 = |X^*|$ (from part 1 of Exercise 2.7.23). Now using the change of variable $y = \frac{tr^\beta}{P}$, we obtain

$$\mathbf{P}(M \geq t) = \frac{2\pi\lambda}{\beta}\left(\frac{P}{t}\right)^{2/\beta} \int_{\mathbb{R}_+} e^{-y} y^{2/\beta - 1} e^{-\lambda\pi\left(\frac{yP}{t}\right)^{2/\beta}} \mathrm{d}y.$$

From the monotone convergence theorem, we have that for large enough t,

$$
\begin{aligned}
\mathbf{P}\left(M \geq t\right) &\approx \frac{2\pi\lambda}{\beta}\left(\frac{P}{t}\right)^{2/\beta} \int_{\mathbb{R}_+} e^{-y} y^{2/\beta - 1} \mathrm{d}y \\
&= \frac{2\pi\lambda}{\beta}\left(\frac{P}{t}\right)^{2/\beta} \Gamma\left(\frac{2}{\beta}\right),
\end{aligned}
$$

which shows that the distribution of M is again Pareto-like with $\gamma = \frac{2}{\beta}$.

3. First we find the distribution of $|X^*(R)|$, for $r > R$

$$\begin{aligned}\mathbf{P}\left(|X^*(R)| \le r\right) &= 1 - \mathbf{P}(|X^*(R)| > r)\\ &= 1 - \mathbf{P}(\Phi(B(0,r)\backslash B(0,R)) = 0)\\ &= 1 - e^{-\lambda\pi(r^2-R^2)}.\end{aligned}$$

For $r \le R$, we have $\mathbf{P}(|X^*(R)| \le r) = 0$. Now, consider

$$\begin{aligned}\mathbf{P}\left(M_R \ge t\right) &= \mathbf{P}\left(\frac{PZ}{|X^*(R)|^\beta} \ge t\right)\\ &= \int_{\mathbb{R}_+} \mathbf{P}\left(\frac{Ps}{|X^*(R)|^\beta} \ge t\right) 2\pi\lambda \mathbf{P}_Z(\mathrm{d}s)\\ &= \int_{\mathbb{R}_+} \mathbf{P}\left(|X^*(R)| \le \left(\frac{Ps}{t}\right)^{1/\beta}\right) \mathbf{P}_Z(\mathrm{d}s)\\ &= \int_{\frac{tR^\beta}{P}}^{\infty} \left[1 - e^{-\lambda\pi\left(\left(\frac{Ps}{t}\right)^{2/\beta} - R^2\right)}\right] \mathbf{P}_Z(\mathrm{d}s)\\ &\le \int_{\frac{tR^\beta}{P}}^{\infty} \lambda\pi \left[\left(\frac{Ps}{t}\right)^{2/\beta} - R^2\right] \mathbf{P}_Z(\mathrm{d}s)\\ &\le \int_{\frac{tR^\beta}{P}}^{\infty} \lambda\pi \left(\frac{Ps}{t}\right)^{2/\beta} \mathbf{P}_Z(\mathrm{d}s)\\ &= \lambda\pi\left(\frac{P}{t}\right)^{2/\beta} \int_{\frac{tR^\beta}{P}}^{\infty} s^{2/\beta} \mathbf{P}_Z(\mathrm{d}s).\end{aligned}$$

In the above analysis, we used $1 - x \le e^{-x}$ to obtain first inequality. Now having Z light-tailed, is a sufficient condition for M_R to be light-tailed.

Exercise 2.7.18. *Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a homogeneous Poisson point process on $\mathbb{R}^2$ of intensity $\lambda \in \mathbb{R}_+^*$ independently marked with i.i.d. nonnegative random variables $\{Z_n\}_{n\in\mathbb{Z}}$. Let R be a positive real number. Give the Laplace transform of*

$$I_R = \sum_{n\in\mathbb{Z}} \mathbf{1}\left\{X_n \in B(0,R)\right\} \frac{Z_n}{l(|X_n|)},$$

for a general measurable function $l : \mathbb{R}_+ \to \mathbb{R}_+$.

Solution 2.7.18. *By definition of the Laplace transform*

$$\mathcal{L}_{I_R}(s) = \mathbf{E}\left[e^{-sI_R}\right] = \mathbf{E}\left[e^{-s\sum_{n\in\mathbb{Z}} \mathbf{1}\{X_n\in B(0,R)\}\frac{Z_n}{l(|X_n|)}}\right].$$

Now using the Laplace functional for Poisson point process, $\tilde{\Phi} := \sum_{n\in\mathbb{Z}} \delta_{X_n,Z_n}$, *with* $f(x,p) = s\mathbf{1}\{x \in B(0,R)\}\frac{p}{l(|x|)}$*, we get*

$$\begin{aligned}
\mathcal{L}_{I_R}(s) &= \exp\left(-\int_{\mathbb{R}^2\times\mathbb{R}_+}\left[1-e^{-s\mathbf{1}\{x\in B(0,R)\}\frac{p}{l(|x|)}}\right]M_{\tilde{\Phi}}(\mathrm{d}x\times\mathrm{d}p)\right)\\
&= \exp\left(-\int_{B(0,R)}\int_{\mathbb{R}_+}\left[1-e^{-s\frac{p}{l(|x|)}}\right]\mathbf{P}_Z(\mathrm{d}p)\lambda\mathrm{d}x\right)\\
&= \exp\left(-\int_{B(0,R)}\left[1-\mathcal{L}_Z\left(\frac{s}{l(|x|)}\right)\right]\lambda\mathrm{d}x\right)\\
&= \exp\left(-\int_0^R\int_0^{2\pi}\left[1-\mathcal{L}_Z\left(\frac{s}{l(r)}\right)\right]\lambda\mathrm{d}\theta r\mathrm{d}r\right)\\
&= \exp\left(-2\pi\lambda\int_0^R\left[1-\mathcal{L}_Z\left(\frac{s}{l(r)}\right)\right]r\mathrm{d}r\right),
\end{aligned}$$

where $\mathcal{L}_Z$ *denotes Laplace transform of* $Z = Z_1$*.*

Exercise 2.7.19. Time evolution of users' locations. *Consider the context of Example 2.6.9. Assume moreover that the duration of the call arriving at location* X_n *is an exponential random variable* Z_n *independent of any thing else with mean* $1/\mu$ *for some* $\mu \in \mathbb{R}_+^*$*. At time* 0*, the users' locations is assumed to be a Poisson point process* $\Phi_0 = \sum_{n\in\mathbb{Z}}\delta_{X_n^0}$ *with intensity measure* Λ_0 *on* $\mathbb{R}^2$*. Show that at time* $t \in \mathbb{R}_+$*, the point process of users' locations*

$$\Phi_t = \sum_{n\in\mathbb{Z}}\mathbf{1}\{T_n\in(0,t], Z_n > t-T_n\}\delta_{X_n} + \sum_{n\in\mathbb{Z}}\mathbf{1}\{Z_n^0 > t\}\delta_{X_n^0} \tag{2.7.5}$$

is a Poisson point processon $\mathbb{R}^2$ *with intensity measure*

$$M_{\Phi_t}(\mathrm{d}x) = \frac{\Lambda(\mathrm{d}x)}{\mu}\left(1-e^{-\mu t}\right) + \Lambda_0(\mathrm{d}x)e^{-\mu t}.$$

Deduce that when $t\to\infty$*,* $M_{\Phi_t}(\mathrm{d}x) \to \frac{\Lambda(\mathrm{d}x)}{\mu}$*.*

Solution 2.7.19. *By Theorem 2.2.21,*

$$\tilde{\Phi} = \sum_{n\in\mathbb{Z}}\delta_{(T_n,X_n,Z_n)}$$

is a Poisson point process on $\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}_+$ *with intensity measure*

$$M_{\tilde{\Phi}}(\mathrm{d}t\times\mathrm{d}x\times\mathrm{d}z) = \mathrm{d}t\times\Lambda(\mathrm{d}x)\times\mathbf{P}_{Z_0}(\mathrm{d}z),$$

where $\mathbf{P}_{Z_0}$ *is the probability distribution of* Z_0*. By Corollary 2.2.7, the thinning*

$$\sum_{n\in\mathbb{Z}}\mathbf{1}\{T_n\in(0,t], Z_n > t-T_n\}\delta_{(T_n,X_n,Z_n)}$$

is a Poisson point process on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_+$ *with intensity measure*

$$\mathbf{1}\{s \in (0,t], z > t-s\} \, \mathrm{d}s \times \Lambda(\mathrm{d}x) \times \mathbf{P}_{Z_0}(\mathrm{d}z).$$

Then, the projection process

$$\Phi_t^1 := \sum_{n \in \mathbb{Z}} \mathbf{1}\{T_n \in (0,t], Z_n > t - T_n\} \, \delta_{X_n}$$

is a Poisson point process on $\mathbb{R}^2$ *with intensity measure*

$$\begin{aligned} M_{\Phi_t^1}(\mathrm{d}x) &= \Lambda(\mathrm{d}x) \int_0^t \mathbf{P}(Z_0 > t-s) \, \mathrm{d}s \\ &= \Lambda(\mathrm{d}x) \int_0^t e^{-\mu(t-s)} \mathrm{d}s = \frac{\Lambda(\mathrm{d}x)}{\mu}\left(1 - e^{-\mu t}\right). \end{aligned}$$

Similarly,

$$\Phi_t^0 := \sum_{n \in \mathbb{Z}} \mathbf{1}\left\{Z_n^0 > t\right\} \delta_{X_n^0}$$

is a Poisson point process on $\mathbb{R}^2$ *with intensity measure*

$$M_{\Phi_t^0}(\mathrm{d}x) = \Lambda_0(\mathrm{d}x) \, \mathbf{P}(Z_0 > t) = \Lambda_0(\mathrm{d}x) \, e^{-\mu t}.$$

Since $M_{\Phi_t} = M_{\Phi_t^1} + M_{\Phi_t^0}$, *we get the announced expression (2.7.5).*

Exercise 2.7.20. Thinned Poisson point process. *Let* Φ *be a homogeneous Poisson point process on* $\mathbb{R}^d$ *with intensity* $\lambda \in \mathbb{R}_+^*$. *The points of* Φ *are independently thinned with the fixed thinning probability* $p \in (0,1)$. *Let* Ψ *be the point process on* $\mathbb{R}^d$ *obtained by the thinning of* Φ *with retention probability* p. *Show that* Ψ *and* $\Psi' = \Phi \backslash \Psi$ *are independent Poisson point processes. Give their intensity measures.*

Solution 2.7.20. *Let* $\Phi = \sum_{k \in \mathbb{N}} \delta_{X_k}$ *and let* $U_0, U_1, \ldots$ *be a sequence of i.i.d. random variables independent of* Φ *uniformly distributed in* $[0,1]$, *then*

$$\Psi = \sum_{k \in \mathbb{N}} \mathbf{1}\{U_k \leq p\} \, \delta_{X_k}, \quad \Psi' = \sum_{k \in \mathbb{N}} \mathbf{1}\{U_k > p\} \, \delta_{X_k}.$$

Let $B, B' \in \mathcal{B}\left(\mathbb{R}^d\right)$ *and* $\alpha, \alpha' \in \mathbb{R}_+$, *and note that*

$$\begin{aligned} \alpha\Psi(B) + \alpha'\Psi'(B') &= \sum_{k \in \mathbb{N}} \left(\alpha \mathbf{1}\{U_k \leq p, X_k \in B\} + \alpha' \mathbf{1}\{U_k > p, X_k \in B'\}\right) \\ &= \sum_{k \in \mathbb{N}} f(X_k, U_k), \end{aligned}$$

where

$$f(x,u) := \alpha \mathbf{1}\{u \leq p, x \in B\} + \alpha' \mathbf{1}\{u > p, x \in B'\}$$

is a cumulative shot-noise with marks. Then it follows from Example 2.4.7(iii) that

$$\begin{aligned}
\mathbf{E}\left[e^{\alpha\Psi(B)+\alpha'\Psi'\left(B'\right)}\right] &= \exp\left[-\int_{\mathbb{G}}\left(1-\mathbf{E}\left[e^{-f(x,U_1)}\right]\right)\lambda \mathrm{d}x\right] \\
&= \exp\left[-\int_{B}\left(1-\mathbf{E}\left[e^{-\alpha\mathbf{1}\{U_1\leq p\}}\right]\right)\lambda \mathrm{d}x\right. \\
&\qquad \left.-\int_{B'}\left(1-\mathbf{E}\left[e^{-\alpha'\mathbf{1}\{U_1> p\}}\right]\right)\lambda \mathrm{d}x\right] \\
&= \exp\left[-\int_{B}\left(1-\mathbf{E}\left[e^{-\alpha\mathbf{1}\{U_1\leq p\}}\right]\right)\lambda \mathrm{d}x\right] \\
&\qquad \exp\left[-\int_{B'}\left(1-\mathbf{E}\left[e^{-\alpha'\mathbf{1}\{U_1> p\}}\right]\right)\lambda \mathrm{d}x\right] \\
&= \mathbf{E}\left[e^{\alpha\Psi(B)}\right]\mathbf{E}\left[e^{\alpha'\Psi'\left(B'\right)}\right],
\end{aligned}$$

which shows that $\Psi(B)$ and $\Psi'(B')$ are independent. This being true for all $B, B' \in \mathcal{B}\left(\mathbb{R}^d\right)$, it follows that Ψ and Ψ' are independent. By Corollary 2.2.7 they are homogeneous Poisson point processes with respective intensities $p\lambda$ and $(1-p)\lambda$.

Alternative solution. *Here we provide a* combinatorial argument. *Consider the point process Ψ. Since Ψ is obtained from Φ by thinning, $\forall k \in \mathbb{N}^*$, $\forall A_1, A_2, \ldots, A_k \in \mathcal{B}(\mathbb{R}^d)$, the random variables $\Psi(A_1), \Psi(A_2), \ldots, \Psi(A_k)$ are mutually independent. In order to show that Ψ is a homogeneous Poisson point process with intensity $p\lambda$, it remains to show that, for $A \in \mathcal{B}(\mathbb{R}^d)$*

$$\mathbf{P}(\Psi(A)=k) = e^{-p\lambda|A|}\frac{(p\lambda|A|)^k}{k!},$$

where $|A|$ is the Lebesgue measure of A. Consider,

$$\begin{aligned}
\mathbf{P}\left(\Psi(A)=k\right) &= \sum_{n=k}^{\infty}\mathbf{P}(\Psi(A)=k, \Phi(A)=n) \\
&= \sum_{n=k}^{\infty}\mathbf{P}(\Psi(A)=k \mid \Phi(A)=n)\mathbf{P}(\Phi(A)=n) \\
&= \sum_{n=k}^{\infty}\frac{n!}{k!\,(n-k)!}p^k(1-p)^{(n-k)}e^{-\lambda|A|}\frac{(\lambda|A|)^n}{n!} \\
&= e^{-\lambda|A|}\frac{(p\lambda|A|)^k}{k!}\sum_{n=k}^{\infty}\frac{((1-p)\lambda|A|)^{n-k}}{(n-k)!} \\
&= e^{-\lambda|A|}\frac{(p\lambda|A|)^k}{k!}e^{(1-p)\lambda|A|} = e^{-p\lambda|A|}\frac{(p\lambda|A|)^k}{k!}.
\end{aligned}$$

This establishes that Ψ is a homogeneous Poisson point process with intensity

$p\lambda$. Similarly, we can show that Ψ' is a homogeneous Poisson point process with intensity $(1-p)\lambda$. Next, we address the issue of independence of Ψ and Ψ'. Consider,

$$\begin{aligned}
\mathbf{P}\left(\Psi(A)=k_1, \Psi'(A)=k_2\right) &= \mathbf{P}(\Psi(A)=k_1, \Phi(A)=k_1+k_2) \\
&= \mathbf{P}(\Psi(A)=k_1 \mid \Phi(A)=k_1+k_2)\mathbf{P}(\Phi(A)=k_1+k_2) \\
&= \frac{(k_1+k_2)!}{k_1!k_2!}p^{k_1}(1-p)^{k_2}e^{-\lambda|A|}\frac{(\lambda|A|)^{k_1+k_2}}{(k_1+k_2)!} \\
&= e^{-p\lambda|A|}\frac{(p\lambda|A|)^{k_1}}{k_1!}e^{-(1-p)\lambda|A|}\frac{((1-p)\lambda|A|)^{k_2}}{k_2!} \\
&= \mathbf{P}(\Psi(A)=k_1)\mathbf{P}(\Psi'(A)=k_2).
\end{aligned}$$

This proves that Ψ and Ψ' are independent.

Exercise 2.7.21. Independent thinning of a Poisson point process. *This exercise generalizes the result of Exercise 2.7.20. Let Φ be a Poisson point process of intensity measure Λ on $\mathbb{R}^d$. Consider its independent thinning $\tilde{\Phi}$, where $p:\mathbb{R}^d \to [0,1]$ is some measurable function representing the* retention function *(i.e., given Φ, each point $X \in \Phi$ is erased with probability $1-p(X)$ independently from the other points).*

1. *Using Laplace transforms, show that $\tilde{\Phi}$ is a Poisson point process of intensity $\tilde{\Lambda}(\mathrm{d}x) := p(x)\Lambda(\mathrm{d}x)$.*

2. *Show that the removed points $\Phi - \tilde{\Phi}$ form also a Poisson point process of intensity $(1-p(x))\Lambda(\mathrm{d}x)$. Moreover $\tilde{\Phi}$ and $\Phi-\tilde{\Phi}$ are independent. Use the Laplace transforms.*

Solution 2.7.21. *Let $\mathbb{G} = \mathbb{R}^d$.*

1. Let $U_0, U_1, \ldots$ be a sequence of i.i.d. random variables independent of Φ uniformly distributed in $[0,1]$, the thinning of Φ is defined as

$$\tilde{\Phi} = \sum_{k\in\mathbb{N}} \mathbf{1}\left\{U_k \le p\left(X_k\right)\right\}\delta_{X_k}.$$

The Laplace transform of $\tilde{\Phi}$ is given for all measurable $f:\mathbb{G}\to\mathbb{R}_+$ by

$$\begin{aligned}
\mathcal{L}_{\tilde{\Phi}}(f) &= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f\mathrm{d}\tilde{\Phi}\right)\right]\\
&= \mathbf{E}\left[\exp\left(-\sum_{k\in\mathbb{N}} \mathbf{1}\{U_k \le p(X_k)\} f(X_k)\right)\right]\\
&= \mathbf{E}\left[\prod_{k\in\mathbb{N}} e^{-\mathbf{1}\{U_k\le p(X_k)\}f(X_k)}\right]\\
&= \mathbf{E}\left[\mathbf{E}\left[\prod_{k\in\mathbb{N}} e^{-\mathbf{1}\{U_k\le p(X_k)\}f(X_k)} \middle| \Phi\right]\right]\\
&= \mathbf{E}\left[\prod_{k\in\mathbb{N}}\left(1-p(X_k)+p(X_k)e^{-f(X_k)}\right)\right]\\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} \tilde{f}\mathrm{d}\Phi\right)\right] = \mathcal{L}_{\Phi}\left(\tilde{f}\right),
\end{aligned}$$

where $\tilde{f}(x) = -\log\left[1-p(x)\left(1-e^{-f(x)}\right)\right]$. Since Φ is Poisson

$$\begin{aligned}
\mathcal{L}_{\Phi}\left(\tilde{f}\right) &= \exp\left(-\int_{\mathbb{G}}(1-e^{-\tilde{f}})\mathrm{d}\Lambda\right)\\
&= \exp\left(-\int_{\mathbb{G}}\left(1-e^{-f(x)}\right))p(x)\Lambda(\mathrm{d}x)\right)\\
&= \exp\left(-\int_{\mathbb{G}}\left(1-e^{-f(x)}\right))\tilde{\Lambda}(\mathrm{d}x)\right),
\end{aligned}$$

which is the Laplace transform of a Poisson point process of intensity measure $\tilde{\Lambda}$. The characterization of a Poisson point process by its Laplace transform completes the proof.

2. Clearly $\Phi-\tilde{\Phi}$ has the same distribution as a thinning of Φ with retention function $1-p$. Then by Point 1, $\Phi-\tilde{\Phi}$ is a Poisson point process of intensity $(1-p(x))\Lambda(\mathrm{d}x)$. In particular, its Laplace transform is given for all measurable $g:\mathbb{G}\to\mathbb{R}_+$ by

$$\mathcal{L}_{\Phi-\tilde{\Phi}}(g) = \exp\left(-\int_{\mathbb{G}}\left(1-e^{-g(x)}\right))(1-p(x))\Lambda(\mathrm{d}x)\right).$$

In order to show that $\tilde{\Phi}$ and $\Phi-\tilde{\Phi}$ are independent, it is enough to show that

$$\mathbf{E}\left[e^{-\int_{\mathbb{G}} f\mathrm{d}\tilde{\Phi}-\int_{\mathbb{G}} g\mathrm{d}\left(\Phi-\tilde{\Phi}\right)}\right] = \mathcal{L}_{\tilde{\Phi}}(f)\,\mathcal{L}_{\Phi-\tilde{\Phi}}(g),$$

for all measurable functions $f,g:\mathbb{G}\to\mathbb{R}_+$. Observe that

$$\begin{aligned}
\int_{\mathbb{G}} f\mathrm{d}\tilde{\Phi}+\int_{\mathbb{G}} g\mathrm{d}\left(\Phi-\tilde{\Phi}\right) &= \sum_{k\in\mathbb{N}}\left(\mathbf{1}\{U_k\le p(X_k)\}f(X_k)+\mathbf{1}\{U_k>p(X_k)\}g(X_k)\right)\\
&= \sum_{k\in\mathbb{N}}\varphi(X_k,U_k),
\end{aligned}$$

where

$$\varphi(x,u) := \mathbf{1}\{u \leq p(x)\} f(x) + \mathbf{1}\{u > p(x)\} g(x).$$

Note that $\hat{\Phi} = \sum_{k\in\mathbb{N}} \delta_{(X_k,U_k)}$ is obtained from Φ by independent (and even i.i.d) marking. Then $\hat{\Phi}$ is a Poisson point process with intensity measure

$$M_{\hat{\Phi}}(\mathrm{d}x \times \mathrm{d}u) := \Lambda(\mathrm{d}x)\mathrm{d}u.$$

Then it follows from the Laplace transform of Poisson point processes that

$$\begin{aligned}
&\mathbf{E}\left[e^{-\int_{\mathbb{G}} f\mathrm{d}\tilde{\Phi} - \int_{\mathbb{G}} g\mathrm{d}\left(\Phi-\tilde{\Phi}\right)}\right] \\
&= \mathbf{E}\left[e^{-\int_{\mathbb{G}} \varphi \mathrm{d}\hat{\Phi}}\right] \\
&= \exp\left(-\int_{\mathbb{G}\times[0,1]} (1-e^{-\varphi})\mathrm{d}M_{\hat{\Phi}}\right) \\
&= \exp\left(-\int_{\mathbb{G}}\left[\int_{[0,1]} (1-e^{-\mathbf{1}\{u\leq p(x)\}f(x)}e^{-\mathbf{1}\{u>p(x)\}g(x)})\mathrm{d}u\right]\Lambda(\mathrm{d}x)\right) \\
&= \exp\left(-\int_{\mathbb{G}}\left[\int_{[0,p(x)]} (1-e^{-f(x)})\mathrm{d}u + \int_{(p(x),1]} (1-e^{-g(x)})\mathrm{d}u\right]\Lambda(\mathrm{d}x)\right) \\
&= \exp\left(-\int_{\mathbb{G}}\left[(1-e^{-f(x)})p(x) + (1-p(x))(1-e^{-g(x)})\right]\Lambda(\mathrm{d}x)\right) \\
&= \mathcal{L}_{\tilde{\Phi}}(f)\,\mathcal{L}_{\Phi-\tilde{\Phi}}(g).
\end{aligned}$$

Exercise 2.7.22. I.i.d. shifts of points. *Let $\Phi = \sum_{n\in\mathbb{N}^*} \delta_{X_n}$ be a homogeneous Poisson point process on $\mathbb{R}^d$ of intensity $\lambda \in \mathbb{R}_+^*$. Let $\{Z_n\}_{n\in\mathbb{N}^*}$ be a sequence of i.i.d. $\mathbb{R}^d$-valued random variables which are independent of Φ. Show that $\Psi = \sum_n \delta_{X_n+Z_n}$ is also a homogeneous Poisson point process of intensity λ on $\mathbb{R}^d$.*

Solution 2.7.22. *The point process $\Psi = \sum_{n\in\mathbb{N}^*} \delta_{X_n+Z_n}$ may be viewed as an independent displacement of the point process $\Phi = \sum_{n\in\mathbb{N}^*} \delta_{X_n}$ by the probability kernel given by, for $x \in \mathbb{R}^2, B \in \mathcal{B}\left(\mathbb{R}^d\right)$,*

$$p(x,B) = \mathbf{P}(x + Z_1 \in B) = \mathbf{P}_{Z_1}(B - x).$$

By the displacement theorem 2.2.17, the point process Ψ is Poisson on $\mathbb{R}^d$ with *intensity measure given by, for $A \in \mathcal{B}\left(\mathbb{R}^d\right)$,*

$$\begin{aligned}
M_\Psi(B) &= \int_{\mathbb{R}^d} p(x, B)\,\lambda \mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^d} \mathbf{P}_{Z_1}(B - x)\,\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} 1\{y \in B - x\}\,\mathbf{P}_{Z_1}(\mathrm{d}y)\right] \mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} 1\{y \in B - x\}\,\mathrm{d}x\right] \mathbf{P}_{Z_1}(\mathrm{d}y) \\
&= \lambda \int_{\mathbb{R}^d} |B|\,\mathbf{P}_{Z_1}(\mathrm{d}y) = \lambda\,|B|\,,
\end{aligned}$$

where $|B|$ is the Lebesgue measure of B. This implies that Ψ is a homogeneous Poisson point process of intensity λ.

Exercise 2.7.23. Points closest to origin. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}^2$ of intensity $\lambda \in \mathbb{R}_+^*$.*

1. *Let R_1 be the distance between the origin and the point of Φ which is the closest to the origin. Compute the probability density function of R_1.*

2. *Let R_2 be the distance between the origin and the point of Φ which is the second closest to the origin. Compute joint probability density function of (R_1, R_2).*

Solution 2.7.23. *1. We have that,*

$$\begin{aligned}
\mathbf{P}(R_1 \le r_1) &= 1 - \mathbf{P}(R_1 > r_1) \\
&= 1 - \mathbf{P}(\Phi(\bar{B}(0, r_1)) = 0) \\
&= 1 - e^{-\lambda\pi r_1^2}.
\end{aligned}$$

Differentiating the above expression with respect to r_1 gives the probability density function of R_1

$$f_{R_1}(r_1) = 2\pi\lambda r_1 e^{-\lambda\pi r_1^2}, \quad r_1 \in \mathbb{R}_+.$$

2. We consider two cases:

- $0 \le r_2 \le r_1$,

$$\begin{aligned}
\mathbf{P}(R_1 > r_1, R_2 > r_2) &= \mathbf{P}(R_1 > r_1)\mathbf{P}(R_2 > r_2 \mid R_1 > r_1) \\
&= \mathbf{P}(R_1 > r_1) = e^{-\lambda\pi r_1^2}.
\end{aligned}$$

- $0 \leq r_1 < r_2$,

$$\begin{aligned}
&\mathbf{P}(R_1 > r_1, R_2 > r_2) \\
&= \mathbf{P}(R_1 > r_1)\mathbf{P}(R_2 > r_2 \mid R_1 > r_1) \\
&= \mathbf{P}(\Phi(\bar{B}(0, r_1)) = 0)\mathbf{P}(\Phi(\bar{B}(0, r_2) \backslash \bar{B}(0, r_1)) \leq 1 \mid \Phi(\bar{B}(0, r_1)) = 0) \\
&= e^{-\lambda\pi r_1^2}\left[e^{-\lambda\pi(r_2^2 - r_1^2)} + \lambda\pi(r_2^2 - r_1^2)e^{-\lambda\pi(r_2^2 - r_1^2)}\right] \\
&= e^{-\lambda\pi r_2^2} + \lambda\pi(r_2^2 - r_1^2)e^{-\lambda\pi r_2^2}.
\end{aligned}$$

The joint probability of other partitions, e.g., $\mathbf{P}(R_1 \leq r_1, R_2 > r_2)$*, can be obtained in a similar manner. Differentiating the above expression with respect to* r_1 *and* r_2 *respectively, we get the joint probability density function of* R_1 *and* R_2

$$f_{R_1,R_2}(r_1, r_2) = 4\lambda^2\pi^2 r_1 r_2 e^{-\lambda\pi r_2^2}\mathbf{1}\{r_1 < r_2\}.$$

Exercise 2.7.24. n-th closest point. *Let* Φ *be a Poisson point process on* $\mathbb{R}^2$ *with intensity measure* Λ*. Compute the distribution of the distance* R_n *from* 0 *to the* n*-th closest point of* Φ*.*

Solution 2.7.24. *Recall that* $\Phi(\bar{B}(0, r))$ *is a Poisson random variable with mean* $\Lambda(\bar{B}(0, r))$*, then*

$$\mathbf{P}(R_n > r) = \mathbf{P}(\Phi(\bar{B}(0, r)) \leq n - 1) = e^{-\Lambda(\bar{B}(0,r))}\sum_{k=0}^{n-1}\frac{\Lambda(\bar{B}(0, r))^k}{k!}.$$

Exercise 2.7.25. Vanishing shot-noise. *Let* $\Phi = \sum_{k\in\mathbb{N}^*}\delta_{X_k}$ *be a homogeneous Poisson point process on* $\mathbb{R}^2$ *with intensity* $\lambda \in \mathbb{R}_+^*$ *and let* $Z_1, Z_2, \ldots$ *be a sequence of i.i.d. nonnegative random variables independent of* Φ*. Assume that* $\mathbf{P}(Z_1 > 0) > 0$*. Let*

$$I = \sum_{k\in\mathbb{N}} Z_k g(|X_k|),$$

for some measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$*. Show that* $\mathbf{P}(I = 0) > 0$ *if and only if* g *has bounded support (i.e.,* $g(r) = 0$ *for all* $r > R$ *for some* $R > 0$*).*

Solution 2.7.25. *Note that* I *is a cumulative shot-noise with marks. Let* $\operatorname{supp}(g)$ *be the support of* g*. Then by Example 2.4.7(iii)*

$$\begin{aligned}
\mathbf{P}(I = 0) &= \exp\left[-\lambda\int_{\mathbb{R}^2}\mathbf{P}(Z_1 g(|x|) > 0)\,\mathrm{d}x\right] \\
&= \exp\left[-2\pi\lambda\mathbf{P}(Z_1 > 0)\int_0^\infty \mathbf{1}\{g(r) > 0\}\,\mathrm{d}r\right] \\
&= \exp\left[-2\pi\lambda\mathbf{P}(Z_1 > 0)\left|\operatorname{supp}(g)\right|\right].
\end{aligned}$$

Then $\mathbf{P}(I = 0) > 0$ *iff* $\operatorname{supp}(g)$ *is bounded.*

Exercise 2.7.26. *Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a Poisson point process of intensity measure Λ on $\mathbb{R}^2$ representing the locations of transmitters at time 1. At time 2, the points of Φ move independently according to some Markov kernel $K : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\int_{\mathbb{R}^2} K(x, A)\Lambda(dx) < \infty,$$

for all bounded $A \in \mathcal{B}\left(\mathbb{R}^2\right)$. That is the point at X_n at time 1 moves to a point Y_n sampled according to the kernel $K(X_n, .)$ independently of everything else. Let $\Psi = \sum_{n\in\mathbb{Z}} \delta_{Y_n}$ be the point process of the locations of transmitters at time 2.

1. *Let I_j be the interference created at the origin of the plane by these transmitters at time $j \in \{1, 2\}$; that is*

$$I_1 = \sum_{n\in\mathbb{Z}} \frac{1}{l(|X_n|)}, \quad I_2 = \sum_{n\in\mathbb{Z}} \frac{1}{l(|Y_n|)},$$

where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is a given measurable function. Give the joint Laplace transform of (I_1, I_2) and the covariance of (I_1, I_2).

2. *Same question when*

$$I_1 = \sum_{n\in\mathbb{Z}} \frac{Z_n^1}{l(|X_n|)}, \quad I_2 = \sum_{n\in\mathbb{Z}} \frac{Z_n^2}{l(|Y_n|)},$$

where $\left\{Z_n^j\right\}_{j\in\{1,2\},n\in\mathbb{Z}}$ form an i.i.d. collection of random variables with common probability distribution $\mathbf{P}_Z$ on $\mathbb{R}_+$ independent from Φ and Ψ.

Solution 2.7.26. *1. For any $t_1, t_2 \in \mathbb{R}_+$, consider,*

$$\begin{aligned}
\mathcal{L}_{I_1,I_2}(t_1, t_2) &= \mathbf{E}[e^{-t_1 I_1 - t_2 I_2}] \\
&= \mathbf{E}\left[e^{-t_1 \sum_{n\in\mathbb{Z}} \frac{1}{l(|X_n|)} - t_2 \sum_{n\in\mathbb{Z}} \frac{1}{l(|Y_n|)}}\right] \\
&= \mathbf{E}\left[e^{-t_1 \sum_{n\in\mathbb{Z}} \frac{1}{l(|X_n|)}} \prod_{n\in\mathbb{Z}} \int_{\mathbb{R}^2} e^{-\frac{t_2}{l(|y|)}} K(X_n, \mathrm{d}y)\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{n\in\mathbb{Z}} \frac{t_1}{l(|X_n|)} - \log\left(\int_{\mathbb{R}^2} e^{-\frac{t_2}{l(|y|)}} K(X_n, \mathrm{d}y)\right)\right)\right].
\end{aligned}$$

Now using the expression (2.1.1) of Laplace functional for Poisson point processes with $f(x) = \frac{t_1}{l(|x|)} - \log\left(\int_{\mathbb{R}^2} e^{-\frac{t_2}{l(|y|)}} K(x, \mathrm{d}y)\right)$, we get

$$\begin{aligned}
\mathcal{L}_{I_1,I_2}(t_1, t_2) &= \exp\left(-\int_{\mathbb{R}^2} \left[1 - e^{-\frac{t_1}{l(|x|)} + \log\left(\int_{\mathbb{R}^2} e^{-\frac{t_2}{l(|y|)}} K(x,\mathrm{d}y)\right)}\right] \Lambda(\mathrm{d}x)\right) \\
&= \exp\left(-\int_{\mathbb{R}^2} \left[1 - e^{-\frac{t_1}{l(|x|)}} \int_{\mathbb{R}^2} e^{-\frac{t_2}{l(|y|)}} K(x, \mathrm{d}y)\right] \Lambda(\mathrm{d}x)\right) \\
&= \exp\left(-\int_{\mathbb{R}^2} \left[1 - \int_{\mathbb{R}^2} e^{-\frac{t_1}{l(|x|)} - \frac{t_2}{l(|y|)}} K(x, \mathrm{d}y)\right] \Lambda(\mathrm{d}x)\right).
\end{aligned}$$

Next, consider

$$\frac{\partial}{\partial t_1}\mathcal{L}_{I_1,I_2}(t_1,t_2) = \mathcal{L}_{I_1,I_2}(t_1,t_2)\left[-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|x|)}e^{-\frac{t_1}{l(|x|)}-\frac{t_2}{l(|y|)}}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x)\right].$$

Therefore,

$$\mathbf{E}[I_1] = -\frac{\partial}{\partial t_1}\mathcal{L}_{I_1,I_2}(t_1,t_2)|_{(t_1,t_2)=(0,0)} = \int_{\mathbb{R}^2}\frac{1}{l(|x|)}\Lambda(\mathrm{d}x).$$

Similarly,

$$\frac{\partial}{\partial t_2}\mathcal{L}_{I_1,I_2}(t_1,t_2) = \mathcal{L}_{I_1,I_2}(t_1,t_2)\left[-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|y|)}e^{-\frac{t_1}{l(|x|)}-\frac{t_2}{l(|y|)}}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x)\right].$$

This implies that

$$\mathbf{E}[I_2] = -\frac{\partial}{\partial t_2}\mathcal{L}_{I_1,I_2}(t_1,t_2)|_{(t_1,t_2)=(0,0)} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|y|)}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x).$$

Observe that

$$\begin{aligned}\frac{\partial^2}{\partial t_2\partial t_1}\mathcal{L}_{I_1,I_2}(t_1,t_2) = \mathcal{L}_{I_1,I_2}(t_1,t_2)&\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|y|)}e^{-\frac{t_1}{l(|x|)}-\frac{t_2}{l(|y|)}}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x)\right]\\ &\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|x|)}e^{-\frac{t_1}{l(|x|)}-\frac{t_2}{l(|y|)}}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x)\right]\\ &+\mathcal{L}_{I_1,I_2}(t_1,t_2)\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|x|)l(|y|)}e^{-\frac{t_1}{l(|x|)}-\frac{t_2}{l(|y|)}}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x)\right].\end{aligned}$$

Following similar analysis, we get

$$\begin{aligned}\mathbf{E}[I_1I_2] &= \frac{\partial^2}{\partial t_2\partial t_1}\mathcal{L}_{I_1,I_2}(t_1,t_2)|_{(t_1,t_2)=(0,0)}\\ &= \int_{\mathbb{R}^2}\frac{1}{l(|x|)}\Lambda(\mathrm{d}x)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|y|)}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x) + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|x|)}\frac{1}{l(|y|)}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x).\end{aligned}$$

Now we can obtain covariance of (I_1, I_2) *as follows*

$$\mathrm{cov}(I_1,I_2) = \mathbf{E}[I_1I_2] - \mathbf{E}[I_1]\mathbf{E}[I_2] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|x|)}\frac{1}{l(|y|)}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x).$$

2: Following steps similar to analysis part 1 and invoking Laplace functional for Poisson point processes, $\sum_{n\in\mathbb{Z}}\delta_{X_n,Z_n^j}$, *we obtain*

$$\mathcal{L}_{I_1,I_2}(t_1,t_2) = \exp\left[-\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\left(1-\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}e^{-\frac{t_1s_1}{l(|x|)}-\frac{t_2s_2}{l(|y|)}}\mathbf{P}_Z(\mathrm{d}s_2)K(x,\mathrm{d}y)\right)\mathbf{P}_Z(\mathrm{d}s_1)\Lambda(\mathrm{d}x)\right].$$

In this case, we obtain

$$\mathrm{cov}(I_1,I_2) = \mathbf{E}[Z]^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{l(|x|)}\frac{1}{l(|y|)}K(x,\mathrm{d}y)\Lambda(\mathrm{d}x).$$

Exercise 2.7.27. M/GI/∞ system. *We consider a system where users arrive, remain for some time, then leave. The user's* arrival *instants to the system are modelled by a homogeneous Poisson point process on* $\mathbb{R}$*, say* $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$*, of intensity* $\lambda \in \mathbb{R}_+^*$ *with the usual convention*

$$\ldots < X_{-1} < X_0 \leq 0 < X_1 < X_2 < \ldots$$

The sojourn duration *of the user arriving at* X_n *is modelled by a random variable denoted by* $Z_n \in \mathbb{R}_+$*. The point process*

$$\tilde{\Phi} = \sum_{n\in\mathbb{N}} \delta_{(X_n, Z_n)}$$

is assumed to be an i.i.d. marked point process in the sense of Definition 2.2.18. The number of users *in the system at time* t *is*

$$Y(t) = \sum_{n\in\mathbb{Z}} \mathbf{1}\{X_n \leq t < X_n + Z_n\}.$$

Show that:

1. *For all* $t \in \mathbb{R}$*,* $Y(t)$ *is a Poisson random variable of mean* $\lambda \mathbf{E}[Z_1]$*.*
2. *For all* $t \in \mathbb{R}, \tau \in \mathbb{R}_+$*,*

$$\operatorname{cov}(Y(t), Y(t+\tau)) = \lambda \mathbf{E}\left[\max(Z_1 - \tau)^+\right] = \lambda \int_\tau^\infty \mathbf{P}(Z_1 > z)\,\mathrm{d}z,$$

   where $x^+ := \max(x, 0)$*.*

3. *The point process of* departure times

$$D = \sum_{n\in\mathbb{Z}} \delta_{X_n + Z_n}$$

   is a homogeneous Poisson point process of intensity λ*.*

Solution 2.7.27. *The mean measure of* $\tilde{\Phi}$ *is given by (2.2.8)*

$$M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z) = \mathbf{P}_{Z_1}(\mathrm{d}z) M_\Phi(\mathrm{d}x)$$

which is clearly locally finite. Then it follows from Theorem 2.2.21 that $\tilde{\Phi}$ *is a Poisson point process.*

1. *Letting*

$$A(t) = \{(x, z) \in \mathbb{R} \times \mathbb{R}_+ : x \leq t < x + z\},$$

   then

$$Y(t) = \tilde{\Phi}(A(t))$$

is a Poisson random variable of mean

$$
\begin{aligned}
M_{\tilde{\Phi}}(A(t)) &= \int_{A(t)} \mathbf{P}_{Z_1}(\mathrm{d}z) M_\Phi(\mathrm{d}x) \\
&= \int_{-\infty}^{t} \mathbf{P}(Z_1 > t-x) M_\Phi(\mathrm{d}x).
\end{aligned}
$$

Assume now that Φ is homogeneous with intensity $\lambda \in \mathbb{R}_+^*$. Then

$$
\begin{aligned}
M_{\tilde{\Phi}}(A(t)) &= \lambda \int_{-\infty}^{t} \mathbf{P}(Z_1 > t-x)\mathrm{d}x \\
&= \lambda \int_0^\infty \mathbf{P}(Z_1 > u)\mathrm{d}u = \lambda \mathbf{E}[Z_1].
\end{aligned}
$$

2. *We deduce from Proposition 2.4.6, that*

$$
\begin{aligned}
\mathrm{cov}(Y(t), Y(t+\tau)) &= \mathrm{cov}(\tilde{\Phi}(A(t)), \tilde{\Phi}(A(t+\tau))) \\
&= M_{\tilde{\Phi}}(A(t) \cap A(t+\tau)) \\
&= \int_{A(t)\cap A(t+\tau)} \lambda \mathbf{P}_{Z_1}(\mathrm{d}z)\mathrm{d}x \\
&= \lambda \int_\tau^\infty \left(\int_{t+\tau-z}^{t} \mathrm{d}x \right) \mathbf{P}_{Z_1}(\mathrm{d}z) \\
&= \lambda \int_\tau^\infty (z-\tau) \mathbf{P}_{Z_1}(\mathrm{d}z) \\
&= \lambda \int_0^\infty (z-\tau)^+ \mathbf{P}_{Z_1}(\mathrm{d}z) \\
&= \lambda \mathbf{E}\left[(Z_1-\tau)^+ \right].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathbf{E}\left[(Z_1-\tau)^+ \right] &= \mathbf{E}\left[(Z_1-\tau) 1_{\{Z_1>\tau\}} \right] \\
&= \mathbf{E}\left[\int_\tau^{Z_1} \mathrm{d}z 1_{\{Z_1>\tau\}} \right] \\
&= \mathbf{E}\left[\int_\tau^\infty 1_{\{z<Z_1\}} \mathrm{d}z \right] = \int_\tau^\infty \mathbf{P}(Z_1 > z)\,\mathrm{d}z.
\end{aligned}
$$

3. *For any measurable $A \subset \mathbb{R}$, $D(A) = \sum_{n\in\mathbb{Z}} \mathbf{1}\{X_n + Z_n \in A\} = \tilde{\Phi}\left(\tilde{A}\right)$ where*

$$
\tilde{A} = \{(x,z) \in \mathbb{R} \times \mathbb{R}_+ : x+z \in A\}.
$$

Note that

$$\begin{aligned} M_{\tilde{\Phi}}\left(\tilde{A}\right) &= \int_{\tilde{A}} \lambda \mathbf{P}_{Z_1}(\mathrm{d}z)\mathrm{d}x \\ &= \lambda \int_{\mathbb{R}_+} \left(\int_{A-z} \mathrm{d}x\right) \mathbf{P}_{Z_1}(\mathrm{d}z) \\ &= \lambda \int_{\mathbb{R}_+} |A-z|\, \mathbf{P}_{Z_1}(\mathrm{d}z) \\ &= \lambda \int_{\mathbb{R}_+} |A|\, \mathbf{P}_{Z_1}(\mathrm{d}z) = \lambda\, |A|\,, \end{aligned}$$

where $|A|$ is the Lebesgue measure of A. Moreover, if $A_1, \ldots, A_n$ are disjoint measurable subsets of $\mathbb{R}$, then $\tilde{A}_1, \ldots, \tilde{A}_n$ are disjoint, then $D(A_1) = \tilde{\Phi}\left(\tilde{A}_1\right), \ldots, D(A_n) = \tilde{\Phi}\left(\tilde{A}_n\right)$ are independent random variables of respective means $M_{\tilde{\Phi}}\left(\tilde{A}_1\right) = \lambda\, |A_1|\,, \ldots, M_{\tilde{\Phi}}\left(\tilde{A}_n\right) = \lambda\, |A_n|$. Therefore the departure process D is a Poisson point process on $\mathbb{R}$ with intensity measure $\lambda \mathrm{d}x$; i.e., a homogeneous Poisson point process of intensity λ.

Exercise 2.7.28. Spatial M/GI/∞ system. *Points are dropped in $\mathbb{R}^d$ according to a homogeneous Poisson point process $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ of intensity $\lambda \in \mathbb{R}_+^*$. We associate to each point X_n of Φ a subset Z_n of $\mathbb{R}^d$. The point process*

$$\tilde{\Phi} = \sum_{n\in\mathbb{N}} \delta_{(X_n, Z_n)}$$

is assumed to be an i.i.d. marked point process in the sense of Definition 2.2.18. We consider $X_n + Z_n$ as the geographic zone covered *by X_n. The* number of points *of Φ covering a given location $t \in \mathbb{R}^d$ is*

$$Y(t) = \sum_{n\in\mathbb{Z}} \mathbf{1}\{t \in X_n + Z_n\}$$

as depicted in Figure 2.3. Show that for all $t \in \mathbb{R}^d$, $Y(t)$ is a Poisson random variable of mean $\lambda \mathbf{E}[|Z_1|]$. (This is a spatial extension of Exercise 2.7.27.)

Solution 2.7.28. *The mean measure of $\tilde{\Phi}$ is given by (2.2.8)*

$$M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z) = \mathbf{P}_{Z_1}(\mathrm{d}z) M_{\Phi}(\mathrm{d}x)\,,$$

which is clearly locally finite. Then it follows from Theorem 2.2.21 that $\tilde{\Phi}$ is a Poisson point process. Letting

$$A(t) = \left\{(x,z) : x \in \mathbb{R}^d, z \subset \mathbb{R}^d : t \in x + z\right\},$$

then

$$Y(t) = \tilde{\Phi}(A(t))$$

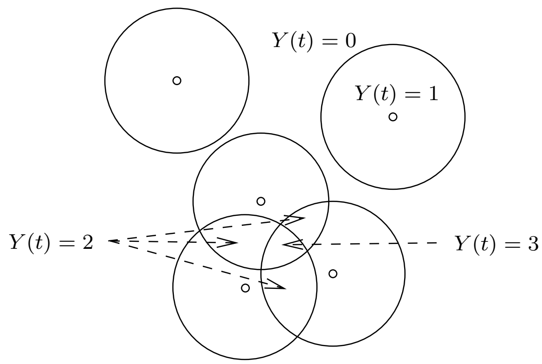

Figure 2.3: Spatial $M/GI/\infty$ process.

is a Poisson random variable of mean

$$M_{\tilde{\Phi}}\left(A\left(t\right)\right) = \int_{A(t)} \mathbf{P}_{Z_1}(\mathrm{d}z) M_{\Phi}\left(\mathrm{d}x\right)$$
$$= \int_{\mathbb{R}^d} \mathbf{P}(t - x \in Z_1) M_{\Phi}\left(\mathrm{d}x\right).$$

Assume now that Φ *is homogeneous with intensity* $\lambda \in \mathbb{R}_+^*$*. Then*

$$M_{\tilde{\Phi}}\left(A\left(t\right)\right) = \lambda \int_{\mathbb{R}^d} \mathbf{P}(t - x \in Z_1)\mathrm{d}x$$
$$= \lambda \int_{\mathbb{R}^d} \mathbf{P}(u \in Z_1)\mathrm{d}u = \lambda \mathbf{E}\left[|Z_1|\right].$$

Chapter 3

Palm theory

3.1 Palm distributions

Lemma 3.1.1. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$. There exists a unique σ-finite measure $\mathcal{C}_\Phi$ on $\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})$ characterized by*

$$\mathcal{C}_\Phi(B \times L) = \mathbf{E}[\Phi(B)\mathbf{1}\{\Phi \in L\}], \quad B \in \mathcal{B}(\mathbb{G}), L \in \bar{\mathcal{M}}(\mathbb{G}). \tag{3.1.1}$$

This measure is called the Campbell measure *of Φ.*

Proof. Let $\mathcal{R}$ be the class of all finite disjoint unions of sets of the form $B \times L$ where $B \in \mathcal{B}(\mathbb{G}), L \in \bar{\mathcal{M}}(\mathbb{G})$. Note first that $\mathcal{R}$ is an algebra of sets. Moreover $\mathcal{C}_\Phi$ is σ-finite on $\mathcal{R}$ [52, §10.1]. Indeed, let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ covering $\mathbb{G}$, and let $L_{mn} = \{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B_m) \leq n\}$. Since $C_\Phi(L_{mn} \times B_n) = \int_{L_{mn}} \mu(B_n)\mathbf{P}_\Phi(\mathrm{d}\mu) \leq n\mathbf{P}_\Phi(L_{mn}) \leq n$, then C_Φ is finite on each set $L_{mn} \times B_n$. These sets cover $\bar{\mathbb{M}}(\mathbb{G}) \times \mathbb{G}$, since for any given $(\mu, x) \in \bar{\mathbb{M}}(\mathbb{G}) \times \mathbb{G}$, there exists some m such that $B_m \ni x$, and since μ is locally finite, we can find some n such that $\mu(B_m) \leq n$.) Then it follows from the Carathéodory's extension theorem [44, §13 Theorem A] that $\mathcal{C}_\Phi$ admits a unique extension on $\sigma(\mathcal{R})$, which is precisely the product σ-algebra on $\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})$. □

Note that, for $B \in \mathcal{B}(\mathbb{G})$,

$$\mathcal{C}_\Phi\left(B \times \bar{\mathbb{M}}(\mathbb{G})\right) = \mathbf{E}[\Phi(B)] = M_\Phi(B).$$

That is the mean measure M_Φ is the projection of $\mathcal{C}_\Phi$ onto $\mathbb{G}$.

The measure $\mathcal{C}_\Phi$ is σ-finite by Lemma 3.1.1. Since $\left(\bar{\mathbb{M}}(\mathbb{G}), \bar{\mathcal{M}}(\mathbb{G})\right)$ is Polish [52, §15.7.7], if M_Φ is σ-finite, then it follows from the measure disintegration theorem 14.D.10 that $\mathcal{C}_\Phi$ admits a disintegration

$$\mathcal{C}_\Phi(B \times L) = \int_B \mathbf{P}_\Phi^x(L)\, M_\Phi(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{G}), L \in \bar{\mathcal{M}}(\mathbb{G}), \tag{3.1.2}$$

where $\mathbf{P}_\Phi^{\cdot}(\cdot)$ is a probability kernel from $\mathbb{G}$ to $\bar{\mathbb{M}}(\mathbb{G})$ called the disintegration probability kernel of $\mathcal{C}_\Phi$ with respect to M_Φ. Moreover, the disintegration probability kernel $\{\mathbf{P}_\Phi^x(\cdot)\}_{x\in\mathbb{G}}$ is unique almost everywhere with respect to M_Φ. This means that we may modify the choice of $\mathbf{P}_\Phi^x(\cdot)$ for x within a set of M_Φ-measure 0 to obtain another disintegration probability kernel.

Definition 3.1.2. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure. Let $\mathbf{P}_\Phi^{\cdot}(\cdot)$ be a disintegration probability kernel of $\mathcal{C}_\Phi$ with respect to M_Φ. Then $\mathbf{P}_\Phi^x(\cdot)$ is called a* Palm distribution *of Φ at $x\in\mathbb{G}$ and $\{\mathbf{P}_\Phi^x\}_{x\in\mathbb{G}}$ is called a* family of Palm distributions *of Φ.*

Remark 3.1.3. *Note that, for given $L\in\bar{\mathcal{M}}(\mathbb{G})$, $\mathcal{C}_\Phi(\cdot\times L)\ll M_\Phi(\cdot)$. Then Equation (3.1.2) shows that $\mathbf{P}_\Phi^{\cdot}(L)$ is the Radon-Nikodym derivative of $\mathcal{C}_\Phi(\cdot\times L)$ with respect to the mean measure; that is*

$$\mathbf{P}_\Phi^{\cdot}(L)=\frac{\mathrm{d}\mathcal{C}_\Phi(\cdot\times L)}{\mathrm{d}M_\Phi(\cdot)}.$$

Definition 3.1.4. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure. Let $\{\mathbf{P}_\Phi^x\}_{x\in\mathbb{G}}$ be a family of Palm distributions of Φ. Since for all $x\in\mathbb{G}$, $\mathbf{P}_\Phi^x$ is a probability measure on $\bar{\mathbb{M}}(\mathbb{G})$, it can be identified with a probability distribution of some random measure, say Φ_x, often called* Palm version *of Φ at x. The collection $\{\Phi_x\}_{x\in\mathbb{G}}$ is the* family of Palm versions *of Φ.*

Note that we may modify the choice of Φ_x for x within a set of M_Φ-measure 0 to obtain another family of Palm versions of Φ.

Remark 3.1.5. *The probability space on which Φ_x is defined is not important. Nevertheless, in order to simplify the notation, we aim to use $\mathbf{P}(\cdot)$ and $\mathbf{E}[\cdot]$ for the probability and the expectation respectively with respect to Φ_x writing for example $\mathbf{P}_\Phi^x(L)=\mathbf{P}(\Phi_x\in L)$, or $\int_{\bar{\mathbb{M}}(\mathbb{G})}f(\mu)\,\mathbf{P}_\Phi^x(\mathrm{d}\mu)=\mathbf{E}[f(\Phi_x)]$. To do so, we suppose that Φ_x is defined on a suitable extension of the probability space $(\Omega,\mathcal{A},\mathbf{P})$ on which Φ is defined. Note that this extension of the probability space has no impact on the joint distribution of Φ and Φ_x.*

Example 3.1.6. Palm version of a deterministic measure. *Let μ be a deterministic σ-finite measure on a l.c.s.h. space $\mathbb{G}$ and let $\Phi=\mu$. Then $M_\Phi=\mu$ and $\mathcal{C}_\Phi(B\times L)=\mu(B)\,\mathbf{1}\{\mu\in L\}=\mu(B)\,\delta_\mu(L)$. Thus, we may take $\mathbf{P}_\Phi^x=\delta_\mu$, $(x\in\mathbb{G})$ as family of* Palm distributions *of Φ; which gives the Palm version $\Phi_x=\mu$ for all $x\in\mathbb{G}$.*

Remark 3.1.7. *Note that if $\mathbf{P}(\Phi\in L)=1$, then $\mathcal{C}_\Phi(B\times L)=\mathbf{E}[\Phi(B)]=M_\Phi(B)$, thus $\mathbf{P}_\Phi^x(L)=\mathbf{P}(\Phi_x\in L)=1$ for M_Φ-almost all x. In particular, if Φ is simple then so is Φ_x for M_Φ-almost all x.*

Example 3.1.8. Palm for stochastic integrals. *Let $\{\lambda(x)\}_{x\in\mathbb{R}^d}$ be a nonnegative measurable stochastic process on $\mathbb{R}^d$ such that, for almost all $\omega\in\Omega$, the function $x\mapsto\lambda(x,\omega)$ is locally integrable. Consider the stochastic integral*

$$\Phi(B)=\int_B\lambda(x)\,\mathrm{d}x,\quad B\in\mathcal{B}\left(\mathbb{R}^d\right).$$

It follows from Proposition 1.4.1 that Φ is a random measure on $\mathbb{R}^d$, and from Proposition 1.4.2 that its mean measure equals

$$\mathbf{E}\left[\Phi\left(B\right)\right]=\int_{B}\mathbf{E}\left[\lambda\left(x\right)\right]\mathrm{d}x.$$

Then, for $L\in\bar{\mathcal{M}}\left(\mathbb{G}\right)$,

$$\mathbf{P}_{\Phi}^{x}\left(L\right)=\frac{\mathbf{E}\left[\Phi\left(\mathrm{d}x\right)\mathbf{1}\left\{\Phi\in L\right\}\right]}{\mathbf{E}\left[\Phi\left(\mathrm{d}x\right)\right]}=\frac{\mathbf{E}\left[\lambda\left(x\right)\mathrm{d}x\mathbf{1}\left\{\Phi\in L\right\}\right]}{\mathbf{E}\left[\lambda\left(x\right)\mathrm{d}x\right]}.$$

Thus we may take

$$\mathbf{P}_{\Phi}^{x}\left(L\right)=\frac{\mathbf{E}\left[\lambda\left(x\right)\mathbf{1}\left\{\Phi\in L\right\}\right]}{\mathbf{E}\left[\lambda\left(x\right)\right]},\quad L\in\bar{\mathcal{M}}\left(\mathbb{G}\right),x\in\mathbb{R}^{d}.$$

In this case it is not obvious how to construct Φ_x with the above distribution.

Theorem 3.1.9. Campbell-Little-Mecke formula (C-L-M). *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure and let $\{\mathbf{P}_{\Phi}^{x}\}_{x\in\mathbb{G}}$ be a family of Palm distributions of Φ. Then for all measurable $f:\mathbb{G}\times\bar{\mathbb{M}}\left(\mathbb{G}\right)\to\bar{\mathbb{R}}_{+}$, we have*

$$\mathbf{E}\left[\int_{\mathbb{G}}f\left(x,\Phi\right)\Phi\left(\mathrm{d}x\right)\right]=\int_{\mathbb{G}\times\bar{\mathbb{M}}\left(\mathbb{G}\right)}f\left(x,\mu\right)\mathbf{P}_{\Phi}^{x}\left(\mathrm{d}\mu\right)M_{\Phi}\left(\mathrm{d}x\right). \tag{3.1.3}$$

Proof. **Step 1**. We first show the announced equality for $f(x,\mu)=\mathbf{1}\{x\in B,\mu\in L\}$ where $B\in\mathcal{B}\left(\mathbb{G}\right),L\in\bar{\mathcal{M}}\left(\mathbb{G}\right)$. To do so, note that

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}}f\left(x,\Phi\right)\Phi\left(\mathrm{d}x\right)\right]&=\mathbf{E}\left[\int_{\mathbb{G}}\mathbf{1}\{x\in B,\Phi\in L\}\Phi\left(\mathrm{d}x\right)\right]\\
&=\mathcal{C}_{\Phi}\left(B\times L\right)\\
&=\int_{B}\mathbf{P}_{\Phi}^{x}\left(L\right)M_{\Phi}\left(\mathrm{d}x\right)\\
&=\int_{\mathbb{G}\times\bar{\mathbb{M}}\left(\mathbb{G}\right)}\mathbf{1}\{x\in B,\mu\in L\}\mathbf{P}_{\Phi}^{x}\left(\mathrm{d}\mu\right)M_{\Phi}\left(\mathrm{d}x\right)\\
&=\int_{\mathbb{G}\times\bar{\mathbb{M}}\left(\mathbb{G}\right)}f\left(x,\mu\right)\mathbf{P}_{\Phi}^{x}\left(\mathrm{d}\mu\right)M_{\Phi}\left(\mathrm{d}x\right),
\end{aligned}$$

where we used (3.1.2) for the third equality. **Step 2**. Since M_{Φ} is assumed σ-finite, there exists a countable measurable partition $\{B_j\}_{j\in\mathbb{N}}$ of $\mathbb{G}$ such that $M_{\Phi}\left(B_j\right)<\infty$ for all $j\in\mathbb{N}$. Decomposing the integrals on $\mathbb{G}$ in Equation (3.1.3) into the sum over $j\in\mathbb{N}$ of integrals on B_j, we see that we may assume, without loss of generality, that M_{Φ} is finite on $\mathbb{G}$. **Step 3**. We will now apply the functional monotone class theorem 14.C.1. Let $\mathcal{H}$ be the collection of functions $f:\mathbb{G}\times\bar{\mathbb{M}}\left(\mathbb{G}\right)\to\bar{\mathbb{R}}_{+}$ such that (3.1.3) holds true (i) Applying the results of Step 1 with $B=\mathbb{G}$ and $L=\bar{\mathbb{M}}\left(\mathbb{G}\right)$ shows that the constant function 1 is in $\mathcal{H}$.

(ii) Observe that for any bounded $f \in \mathcal{H}$, the right-hand side of (3.1.3) is finite. Then so is the left-hand side. Then for any bounded $f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha f + \beta g$ is nonnegative, the linearity of the integral and the expectation shows that $\alpha f + \beta g \in \mathcal{H}$. (iii) The monotone convergence theorem shows that $\mathcal{H}$ is closed under nondecreasing pointwise limits. (iv) Let $\mathcal{I}$ be the set of all finite disjoint unions of sets of the form $B \times L$, where $B \in \mathcal{B}(\mathbb{G}), L \in \bar{\mathcal{M}}(\mathbb{G})$. Observe that $\mathcal{I}$ is closed with respect to finite intersections and that, by Step 1, $\mathbf{1}_C \in \mathcal{H}$ for all $C \in \mathcal{I}$. Theorem 14.C.1 and the fact that $\sigma(\mathcal{I})$ is precisely $\mathcal{B}(\mathbb{G}) \otimes \bar{\mathcal{M}}(\mathbb{G})$ allows one to conclude. □

Corollary 3.1.10. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure and let $\{\mathbf{P}_\Phi^x\}_{x \in \mathbb{G}}$ be a family of Palm distributions of Φ. Let $f : \mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G}) \to \bar{\mathbb{C}}$ be measurable. If either of the following conditions*

$$\mathbf{E}\left[\int_{\mathbb{G}} |f(x, \Phi)| \Phi(\mathrm{d}x)\right] < \infty, \quad \textit{or} \int_{\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})} |f(x, \mu)| \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_\Phi(\mathrm{d}x) < \infty$$

holds, then the other one holds, and Equality (3.1.3) is true.

Proof. For $f : \mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G}) \to \bar{\mathbb{R}}$, consider its decomposition into positive and negative parts; that is $f = f^+ - f^-$. Note that

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}} |f(x, \Phi)| \Phi(\mathrm{d}x)\right] &= \mathbf{E}\left[\int_{\mathbb{G}} f^+(x, \Phi) \Phi(\mathrm{d}x)\right] + \mathbf{E}\left[\int_{\mathbb{G}} f^-(x, \Phi) \Phi(\mathrm{d}x)\right] \\
&= \int_{\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})} f^+(x, \mu) \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_\Phi(\mathrm{d}x) \\
&\quad + \int_{\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})} f^-(x, \mu) \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})} |f(x, \mu)| \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_\Phi(\mathrm{d}x).
\end{aligned}$$

Then, either of the two sides of the above equality is finite iff the other one is so; in which case

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}} f(x, \Phi) \Phi(\mathrm{d}x)\right] &= \mathbf{E}\left[\int_{\mathbb{G}} f^+(x, \Phi) \Phi(\mathrm{d}x)\right] - \mathbf{E}\left[\int_{\mathbb{G}} f^-(x, \Phi) \Phi(\mathrm{d}x)\right] \\
&= \int_{\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})} f^+(x, \mu) \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_\Phi(\mathrm{d}x) \\
&\quad - \int_{\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})} f^-(x, \mu) \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G})} f(x, \mu) \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_\Phi(\mathrm{d}x).
\end{aligned}$$

For $f : \mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G}) \to \bar{\mathbb{C}}$, applying the previous result to its real and imaginary parts gives the announced result. □

3.1.1 Reduced Palm distribution

Lemma 3.1.11. *Let $\mathbb{G}$ be a l.c.s.h. space. The mapping $\mathbb{G}\times\bar{\mathbb{M}}(\mathbb{G}) \to \mathbb{R}_+; (x,\mu) \mapsto \mu(\{x\})$ is measurable.*

Proof. It is enough to prove that for all $a \in \mathbb{R}_+$,

$$\left\{(x,\mu) \in \mathbb{G} \times \bar{\mathbb{M}}(\mathbb{G}) : \mu(\{x\}) \geq a\right\}$$

is measurable. By Lemma 1.1.4, $\mathbb{G}$ may be covered by a countable union of compact sets. Then it is enough to prove that for any $a \in \mathbb{R}_+, B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, the set

$$A := \left\{(x,\mu) \in B \times \bar{\mathbb{M}}(\mathbb{G}) : \mu(\{x\}) \geq a\right\}$$

is measurable. Let $\{B_{n,j}\}_{j=1}^{k_n}$ be a sequence of nested partitions of B as in Lemma 1.6.3. Let

$$C := \bigcap_{n\in\mathbb{N}} \bigcup_{j} B_{n,j} \times \left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B_{n,j}) \geq a\right\}.$$

It is clear that $A \subset C$. Now let $(x,\mu) \in C$. For any $n \in \mathbb{N}$, there exists some $j(n)$ such that $x \in B_{n,j(n)}$ and $\mu\left(B_{n,j(n)}\right) \geq a$. Choosing arbitrary $x_n \in B_{n,j(n)}, n \in \mathbb{N}$, it follows from the compactness of $\bar{B}$ that there exists some subsequence x_{σ_n} converging to some $y \in \bar{B}$. By the triangular inequality $|x-y| \leq |x - x_{\sigma_n}| + |x_{\sigma_n} - y| \to 0$, thus $y = x$. Every open set G containing x contains eventually the set $B_{\sigma_n, j(\sigma_n)}$ and therefore $\mu(G) \geq a$. This being true for all open sets G containing x, it follows that $\mu(\{x\}) \geq a$. Then $(x,\mu) \in A$, and therefore $C \subset A$. Therefore $A = C$ and the announced measurability follows. □

Proposition 3.1.12. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure, then*

$$\mathbf{P}(\Phi_x(\{x\}) \geq 1) = 1,$$

for M_Φ-almost all $x \in \mathbb{G}$.

Proof. Note first that by Lemma 3.1.11, the mapping $\mathbb{G}\times\bar{\mathbb{M}}(\mathbb{G}) \to \mathbb{R}_+; (x,\mu) \mapsto \mathbf{1}\{\mu(\{x\}) = 0\}$ is measurable. Then

$$\begin{aligned}
\int_{\mathbb{G}} \mathbf{P}(\Phi_x(\{x\}) = 0)\, M_\Phi(\mathrm{d}x) &= \int_{\mathbb{G}} \mathbf{E}[\mathbf{1}\{\Phi_x(\{x\}) = 0\}]\, M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\mathbb{M}} 1\{\mu(\{x\}) = 0\}\, \mathbf{P}_\Phi^x(\mathrm{d}\mu)\, M_\Phi(\mathrm{d}x) \\
&= \mathbf{E}\left[\int_{\mathbb{G}} \mathbf{1}\{\Phi(\{x\}) = 0\}\, \Phi(\mathrm{d}x)\right] = 0,
\end{aligned}$$

where the third equality follows from the C-L-M theorem 3.1.9. Therefore, $\mathbf{P}(\Phi_x(\{x\}) = 0) = 0$ for M_Φ-almost all $x \in \mathbb{G}$ since the integrated function in the left-hand side of the above equalities is nonnegative. □

The above proposition shows that the point process Φ_x has almost surely an atom at x. This justifies the interpretation of the Palm distribution at x as the probability distribution of the point process given that (conditionally to) it has an atom at x. It follows that $\Phi_x - \delta_x$ is a point process which justifies the following definition.

Definition 3.1.13. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure. Then $\mathbf{P}_{\Phi_x - \delta_x}$ is the* reduced Palm distribution *associated to Φ at x. We will denote*

$$\Phi_x^! := \Phi_x - \delta_x,$$

called reduced Palm version *of Φ at x.*

We may state the Campbell-Little-Mecke theorem in terms of the reduced Palm distributions as follows.

Corollary 3.1.14. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure. For all measurable $f : \mathbb{G} \times \mathbb{M}(\mathbb{G}) \to \mathbb{R}_+$, we have*

$$\mathbf{E}\left[\int_{\mathbb{G}} f(x, \Phi - \delta_x)\, \Phi(\mathrm{d}x)\right] = \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} f(x,\mu)\, \mathbf{P}_{\Phi_x^!}(\mathrm{d}\mu)\, M_\Phi(\mathrm{d}x). \tag{3.1.4}$$

Proof. Let $g(x,\mu) = f(x, \mu - \delta_x)$ then

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}} f(x, \Phi - \delta_x)\, \Phi(\mathrm{d}x)\right] &= \mathbf{E}\left[\int_{\mathbb{G}} g(x,\Phi)\, \Phi(\mathrm{d}x)\right] \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} g(x,\mu)\, \mathbf{P}_{\Phi_x}(\mathrm{d}\mu)\, M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} g(x,\mu+\delta_x)\, \mathbf{P}_{\Phi_x-\delta_x}(\mathrm{d}\mu)\, M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} f(x,\mu)\, \mathbf{P}_{\Phi_x^!}(\mathrm{d}\mu)\, M_\Phi(\mathrm{d}x),
\end{aligned}$$

where the second equality follows from Theorem 3.1.9. □

3.1.2 Mixed Palm version

In the present section, we will show that the distribution of a random measure Φ is characterized by any version of its family of Palm distributions $\{\mathbf{P}_\Phi^x\}_{x\in\mathbb{G}}$.

Definition 3.1.15. *[52, §10.1] Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure M_Φ, and let $f : \mathbb{G} \to \mathbb{R}_+$ be a measurable function such that $0 < M_\Phi(f) < \infty$. The* f-mixed Palm distribution *of Φ, denoted by $\mathbf{P}^f$, is a distribution defined on $\bar{\mathbb{M}}(\mathbb{G})$ by*

$$\mathbf{P}^f(L) = \frac{1}{M_\Phi(f)} \int_{\mathbb{G}} \mathbf{P}_\Phi^x(L)\, f(x)\, M_\Phi(\mathrm{d}x), \quad L \in \bar{\mathcal{M}}(\mathbb{G}). \tag{3.1.5}$$

Note that $\mathbf{P}^f$ does not depend on the particular choice of the family of Palm distributions $\{\mathbf{P}_\Phi^x\}_{x\in\mathbb{G}}$.

Lemma 3.1.16. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure M_Φ, and let $f:\mathbb{G}\to\mathbb{R}_+$ be measurable such that $0<M_\Phi(f)<\infty$. Then*

$$\mathbf{P}^f(L)=\frac{1}{M_\Phi(f)}\mathbf{E}\left[\Phi(f)\mathbf{1}\left\{\Phi\in L\right\}\right],\quad L\in\bar{\mathcal{M}}(\mathbb{G}),$$

or, equivalently,

$$\mathbf{P}^f(\mathrm{d}\mu)=\frac{1}{M_\Phi(f)}\mu(f)\mathbf{P}_\Phi(\mathrm{d}\mu). \tag{3.1.6}$$

Proof. The first equality follows from Theorem 3.1.9. The second equality is the usual shortened writing for measures; it may be checked as follows, for all $L\in\bar{\mathcal{M}}(\mathbb{G})$,

$$\begin{aligned}\frac{1}{M_\Phi(f)}\int_L\mu(f)\mathbf{P}_\Phi(\mathrm{d}\mu)&=\frac{1}{M_\Phi(f)}\int_{\bar{\mathbb{M}}(\mathbb{G})}\mu(f)\mathbf{1}\left\{\mu\in L\right\}\mathbf{P}_\Phi(\mathrm{d}\mu)\\&=\frac{1}{M_\Phi(f)}\mathbf{E}\left[\Phi(f)\mathbf{1}\left\{\Phi\in L\right\}\right]=\mathbf{P}^f(L).\end{aligned}$$

□

Proposition 3.1.17. Characterization of the distribution of a random measure by the mixed Palm distributions. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with given mean measure M_Φ assumed σ-finite. The distribution of Φ is characterized by the family of $\left\{\mathbf{P}^f\right\}$ for measurable functions $f:\mathbb{G}\to\mathbb{R}_+$ which are bounded with support in $\mathcal{B}_c(\mathbb{G})$ such that $M_\Phi(f)>0$.*

Proof. For a function f as in the proposition, let g be a function defined on $\bar{\mathbb{M}}(\mathbb{G})$ by

$$g(\mu)=\begin{cases}\frac{1-e^{-\mu(f)}}{\mu(f)} & \text{if } \mu(f)>0,\\ 1 & \text{if } \mu(f)=0.\end{cases}$$

Observe that

$$\begin{aligned}1-\mathcal{L}_\Phi(f)&=\mathbf{E}\left[1-e^{-\Phi(f)}\right]\\&=\int_{\bar{\mathbb{M}}(\mathbb{G})}\left[1-e^{-\mu(f)}\right]\mathbf{P}_\Phi(\mathrm{d}\mu)\\&=\int_{\bar{\mathbb{M}}(\mathbb{G})}g(\mu)\,\mu(f)\,\mathbf{P}_\Phi(\mathrm{d}\mu)\\&=M_\Phi(f)\int_{\bar{\mathbb{M}}(\mathbb{G})}g(\mu)\,\mathbf{P}^f(\mathrm{d}\mu),\end{aligned}$$

where the last equality follows from (3.1.6). By Corollary 1.2.2, the probability distribution of a random measure Φ is characterized by its Laplace transform. Moreover, by Proposition 1.3.11, the latter is characterized by $\mathcal{L}_\Phi(f)$

for measurable functions $f : \mathbb{G} \to \mathbb{R}_+$ which are bounded with support in $\mathcal{B}_c(\mathbb{G})$. It remains to show that we may restrict this class to functions f satisfying $M_\Phi(f) > 0$. To do so consider a measurable function $f : \mathbb{G} \to \mathbb{R}_+$ such that $M_\Phi(f) = 0$. Then, by the Campbell averaging formula (1.2.2), $\mathbf{E}\left[\int f \mathrm{d}\Phi\right] = \int f \mathrm{d}M_\Phi = 0$ which implies that $\int f \mathrm{d}\Phi = 0$ almost surely. Thus $\mathcal{L}_\Phi(f) = \mathbf{E}\left[\exp\left(-\int f \mathrm{d}\Phi\right)\right] = 1$. Thus the Laplace transform is characterized by its values on measurable functions $f : \mathbb{G} \to \mathbb{R}_+$ which are bounded with support in $\mathcal{B}_c(\mathbb{G})$ such that $M_\Phi(f) > 0$. This concludes the proof. □

Remark 3.1.18. *Since $\mathbf{P}^f$ is a probability on $\bar{\mathbb{M}}(\mathbb{G})$, it is the probability distribution of some random measure, say Φ_f, called the f-mixed* Palm version *of Φ. As in Remark 3.1.5, it can be assumed that the random measures Φ_f are defined on a suitable extension of the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ on which Φ is defined. In this case, we have*

$$\mathbf{P}(\Phi_f \in L) = \mathbf{P}^f(L), \quad L \in \bar{\mathcal{M}}(\mathbb{G}).$$

Example 3.1.19. *Let μ be a deterministic measure on a l.c.s.h. space $\mathbb{G}$ and let $\Phi = \mu$. We have shown in Example 3.1.6 that $\mathbf{P}_\Phi^x = \delta_\mu$, $x \in \mathbb{G}$ is a family of Palm distributions of Φ. Then, for $L \in \bar{\mathcal{M}}(\mathbb{G})$,*

$$\mathbf{P}^f(L) = \frac{1}{M_\Phi(f)} \int_{\mathbb{G}} \mathbf{1}\{\mu \in L\} f(x) M_\Phi(\mathrm{d}x) = \mathbf{1}\{\mu \in L\}.$$

Thus $\mathbf{P}^f = \delta_\mu$. Therefore $\Phi_f = \mu$; that is, the mixed Palm version of a deterministic measure is equal to the original measure.

Example 3.1.20. *Let Φ be a Poisson point process with intensity measure Λ. Take $f(x) = \mathbf{1}\{x \in A\}$ for some relatively compact set A. Then*

$$\begin{aligned}
\mathbf{E}[\Phi_f(A)] &= \int_{\bar{\mathbb{M}}(\mathbb{G})} \mu(A) \mathbf{P}^f(\mathrm{d}\mu) \\
&= \int_{\bar{\mathbb{M}}(\mathbb{G})} \mu(A) \frac{1}{\Lambda(f)} \mu(f) \mathbf{P}_\Phi(\mathrm{d}\mu) \\
&= \frac{1}{\Lambda(A)} \int_{\bar{\mathbb{M}}(\mathbb{G})} \mu(A)^2 \mathbf{P}_\Phi(\mathrm{d}\mu) \\
&= \frac{\mathbf{E}\left[(\Phi(A))^2\right]}{\Lambda(A)} = \Lambda(A) + 1,
\end{aligned}$$

where the second equality follows from Lemma 3.1.16 and the last one from the fact that $\Phi(A)$ is a Poisson random variable.

3.1.3 Local Palm probabilities

Sometimes, it is useful to consider the *local Palm probabilities* denoted by $\mathbf{P}^x$ and defined on $(\Omega, \mathcal{A})$ as follows

$$\mathbf{P}^x(A) = \frac{\mathbf{E}[\Phi(\mathrm{d}x) 1_A]}{M_\Phi(\mathrm{d}x)}, \quad A \in \mathcal{A}, \text{ for } M_\Phi\text{-almost all } x \in \mathbb{G}. \tag{3.1.7}$$

If $(\Omega, \mathcal{A})$ is a Polish space, we may again refer to the measure disintegration theorem 14.D.10 to conclude that $\mathbf{P}^{\cdot}$ has a version which is a kernel from $\mathbb{G}$ to Ω. Note that, in this case,

$$\begin{aligned} \mathbf{P}^x \Phi^{-1}(L) &= \frac{\mathbf{E}\left[\Phi(\mathrm{d}x)\, 1_{\Phi^{-1}(L)}\right]}{M_\Phi(\mathrm{d}x)} \\ &= \frac{\mathbf{E}\left[\Phi(\mathrm{d}x)\, 1_L(\Phi)\right]}{M_\Phi(\mathrm{d}x)} \\ &= \mathbf{P}_\Phi^x(L) = \mathbf{P}\Phi_x^{-1}(L), \quad L \in \bar{\mathcal{M}}(\mathbb{G}), \end{aligned}$$

which proves that

$$\mathbf{P}\Phi_x^{-1} = \mathbf{P}_\Phi^x = \mathbf{P}^x \Phi^{-1}, \quad \text{for } M_\Phi\text{-almost all } x \in \mathbb{G}.$$

Hence the Campbell-Little-Mecke formula (3.1.3) writes

$$\begin{aligned} \mathbf{E}\left[\int_{\mathbb{G}} f(\Phi, x)\, \Phi(\mathrm{d}x)\right] &= \int_{\mathbb{G}} \int_{\bar{\mathbb{M}}(\mathbb{G})} f(\mu, x)\, \mathbf{P}\Phi_x^{-1}(d\mu)\, M_\Phi(\mathrm{d}x) \\ &= \int_{\mathbb{G}} \mathbf{E}\left[f(\Phi_x, x)\right] M_\Phi(\mathrm{d}x) \\ &= \int_{\mathbb{G}} \mathbf{E}^x\left[f(\Phi, x)\right] M_\Phi(\mathrm{d}x), \end{aligned} \tag{3.1.8}$$

where $\mathbf{E}^x$ is expectation with respect to $\mathbf{P}^x$.

3.2 Palm distributions for particular models

3.2.1 Palm for Poisson point processes

Proposition 3.2.1. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ with σ-finite mean measure and $\{\Phi_x\}_{x\in\mathbb{G}}$ be a family of Palm versions of Φ. Then for all measurable $f, g : \mathbb{G} \to \mathbb{R}_+$ such that g is M_Φ-integrable*

$$\left.\frac{\partial}{\partial t}\mathcal{L}_\Phi(f + tg)\right|_{t=0} = -\mathbf{E}\left[\Phi(g)\, e^{-\Phi(f)}\right] = -\int_{\mathbb{G}} g(x)\, \mathcal{L}_{\Phi_x}(f)\, M_\Phi(\mathrm{d}x). \tag{3.2.1}$$

Moreover, the above equality characterizes $\mathcal{L}_{\Phi_x}$; i.e., if there exists a family of Laplace transforms $\mathcal{L}_x : \mathfrak{F}_+(\mathbb{G}) \to \mathbb{R}_+$ such that

$$\left.\frac{\partial}{\partial t}\mathcal{L}_\Phi(f + tg)\right|_{t=0} = -\int_{\mathbb{G}} g(x)\, \mathcal{L}_x(f)\, M_\Phi(\mathrm{d}x), \tag{3.2.2}$$

then $\mathcal{L}_{\Phi_x} = \mathcal{L}_x$ for M_Φ-almost all $x \in \mathbb{G}$.

Proof. By definition of the Laplace transform $\mathcal{L}_\Phi(f + tg) = \mathbf{E}\left[e^{-\Phi(f) - t\Phi(g)}\right]$. Note that $\frac{\partial}{\partial t} e^{-\Phi(f)-t\Phi(g)} = -\Phi(g)\, e^{-\Phi(f)-t\Phi(g)}$ then $\left|\frac{\partial}{\partial t} e^{-\Phi(f)-t\Phi(g)}\right| \le \Phi(g)$ which is integrable since by Theorem 1.2.5

$$\mathbf{E}\left[\Phi(g)\right] = \int_{\mathbb{G}} g \mathrm{d}M_\Phi < \infty.$$

Then we may invert the derivation and the expectation, that is

$$\frac{\partial}{\partial t}\mathbf{E}\left[e^{-\Phi(f)-t\Phi(g)}\right]=\mathbf{E}\left[\frac{\partial}{\partial t}e^{-\Phi(f)-t\Phi(g)}\right],$$

which gives

$$\left.\frac{\partial}{\partial t}\mathcal{L}_{\Phi}\left(f+tg\right)\right|_{t=0}=-\mathbf{E}\left[\Phi\left(g\right)e^{-\Phi(f)}\right],$$

which proves the first announced equality. Let $h\left(x,\mu\right)=g\left(x\right)e^{-\mu(f)}$ and note that

$$\Phi\left(g\right)e^{-\Phi(f)}=\int_{\mathbb{G}}h\left(x,\Phi\right)\Phi\left(\mathrm{d}x\right).$$

Applying the C-L-M theorem 3.1.9 to f gives

$$\begin{aligned}\mathbf{E}\left[\Phi\left(g\right)e^{-\Phi(f)}\right]&=\mathbf{E}\left[\int_{\mathbb{G}}h\left(x,\Phi\right)\Phi\left(\mathrm{d}x\right)\right]\\&=\int_{\mathbb{G}}\int_{\bar{\mathbb{M}}(\mathbb{G})}g\left(x\right)e^{-\mu(f)}\mathbf{P}_{\Phi}^{x}\left(\mathrm{d}\mu\right)M_{\Phi}\left(\mathrm{d}x\right)\\&=\int_{\mathbb{G}}g\left(x\right)\mathcal{L}_{\Phi_x}\left(f\right)M_{\Phi}\left(\mathrm{d}x\right),\end{aligned}$$

which proves the second equality of the proposition. Assume now that there exists a family of Laplace transform $\mathcal{L}_x:\mathfrak{F}_+\left(\mathbb{G}\right)\to\mathbb{R}_+$ such that (3.2.2). Then, using (3.2.1), we deduce that for all $f\in\mathfrak{F}_+\left(\mathbb{G}\right)$,

$$\mathcal{L}_{\Phi_x}\left(f\right)=\mathcal{L}_x\left(f\right),\quad\text{for } M_{\Phi}\text{-almost all } x\in\mathbb{G}.$$

Let $\mathcal{B}_0\left(\mathbb{G}\right)$ be a countable class as in Lemma 1.3.1 (whose existence is ensured by Lemma 1.3.6). Then for all $k\in\mathbb{N}^*,B_1,\ldots,B_k\in\mathcal{B}_0\left(\mathbb{G}\right),a_1,\ldots,a_k\in\mathbb{Q}$,

$$\mathcal{L}_{\Phi_x}\left(\sum_{j=1}^{k}a_j1_{B_j}\right)=\mathcal{L}_x\left(\sum_{j=1}^{k}a_j1_{B_j}\right),\quad\text{for } M_{\Phi}\text{-almost all } x\in\mathbb{G}.$$

By continuity of the Laplace transform of random vectors, it follows that the above equality holds true for all $a_1,\ldots,a_k\in\mathbb{R}$. Then by Corollary 1.3.4, the probability distribution of Φ_x coincides with that of the random measure of Laplace transform $\mathcal{L}_x$. Thus $\mathcal{L}_{\Phi_x}=\mathcal{L}_x$ for M_{Φ}-almost all $x\in\mathbb{G}$. □

Example 3.2.2. *Let $\Phi=\sum_{j=1}^{k}\delta_{X_j}$ be a Binomial point process as in Example 1.1.11; i.e., $X_1,\ldots,X_k$ are i.i.d. random variables with values in $\mathbb{G}$. Then we know from Example 1.2.4 that*

$$\mathcal{L}_{\Phi}\left(f\right)=\left(\mathbf{E}\left[e^{-f(X_1)}\right]\right)^{k}.$$

Then for $f,g\in\mathfrak{F}_+\left(\mathbb{G}\right)$,

$$\mathcal{L}_{\Phi}\left(f+tg\right)=\left(\mathbf{E}\left[e^{-f(X_1)-tg(X_1)}\right]\right)^{k}.$$

Thus

$$\frac{\partial}{\partial t}\mathcal{L}_{\Phi}(f+tg)\Big|_{t=0} = k\left(\mathbf{E}\left[e^{-f(X_1)}\right]\right)^{k-1}\mathbf{E}\left[-g(X_1)e^{-f(X_1)}\right]$$
$$= -\left(\mathbf{E}\left[e^{-f(X_1)}\right]\right)^{k-1}\int g(x)e^{-f(x)}k\mathbf{P}_{X_1}(\mathrm{d}x).$$

It follows from Proposition 3.2.1 that

$$\mathcal{L}_{\Phi_x}(f) = \left(\mathbf{E}\left[e^{-f(X_1)}\right]\right)^{k-1}e^{-f(x)}.$$

Therefore,

$$\Phi_x = \delta_x + \sum_{j=1}^{k-1}\delta_{X_j}$$

is a family of Palm versions of Φ.

Example 3.2.3. *Let* U *be a random variable uniformly distributed in* $[0,1)$ *and consider the point process on* $\mathbb{R}$ *defined by*

$$\Phi = \sum_{k\in\mathbb{Z}}\delta_{k+U}.$$

We aim to compute the Palm distributions $\mathbf{P}_{\Phi}^{x}$. *To do so, we shall use Proposition 3.2.1. The Laplace transform of* Φ *equals*

$$\mathcal{L}_{\Phi}(f) = \mathbf{E}\left[e^{-\sum_{k\in\mathbb{Z}}f(k+U)}\right] = \int_0^1 e^{-\sum_{k\in\mathbb{Z}}f(k+x)}\mathrm{d}x,$$

for all $f\in\mathfrak{F}_+(\mathbb{G})$. *Then, for* $f,g\in\mathfrak{F}_+(\mathbb{G}), t\in\mathbb{R}_+$,

$$\mathcal{L}_{\Phi}(f+tg) = \int_0^1 e^{-\sum_{k\in\mathbb{Z}}f(k+x)-t\sum_{k\in\mathbb{Z}}g(k+x)}\mathrm{d}x,$$

thus

$$\frac{\partial}{\partial t}\mathcal{L}_{\Phi}(f+tg)\Big|_{t=0} = \int_0^1\sum_{k\in\mathbb{Z}}g(k+x)e^{-\sum_{n\in\mathbb{Z}}f(n+x)}\mathrm{d}x$$
$$= \sum_{k\in\mathbb{Z}}\int_0^1 g(k+x)e^{-\sum_{n\in\mathbb{Z}}f(n+x)}\mathrm{d}x$$
$$= \sum_{k\in\mathbb{Z}}\int_k^{k+1} g(y)e^{-\sum_{n\in\mathbb{Z}}f(n+y-k)}\mathrm{d}y$$
$$= \int_{\mathbb{R}} g(x)e^{-\sum_{n\in\mathbb{Z}}f(n+x)}\mathrm{d}x.$$

Then it follows from Proposition 3.2.1 that, for Lebesgue-almost all $x\in\mathbb{R}$,

$$\mathcal{L}_{\Phi_x}(f) = e^{-\sum_{n\in\mathbb{Z}}f(n+x)}.$$

In particular,

$$\Phi_x = \sum_{n\in\mathbb{Z}} \delta_{n+x},$$

(which is deterministic) is a family of Palm versions of Φ*.*

The following theorem gives a characterization of Poisson point processes in terms of Palm distributions.

Theorem 3.2.4. Slivnyak-Mecke. *Let* Φ *be a random measure on a l.c.s.h. space* $\mathbb{G}$ *with locally finite mean measure. Then* Φ *is a Poisson point process if and only if*

$$\Phi_x^! \overset{\text{dist.}}{=} \Phi, \quad \text{for } M_\Phi\text{-almost all } x \in \mathbb{G}.$$

or, equivalently, $\Phi_x \overset{\text{dist.}}{=} \Phi + \delta_x$*, for* M_Φ*-almost all* $x \in \mathbb{G}$*.*

Proof. **Direct part**. Assume that Φ is Poisson and let Λ be its intensity measure. Its Laplace transform is $\mathcal{L}_\Phi(f) = \exp\left[-\Lambda\left(1-e^{-f}\right)\right]$. Then, for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,

$$\mathcal{L}_\Phi(f + t1_B) = \exp\left[-\Lambda\left(1-e^{-f-t1_B}\right)\right],$$

hence

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{L}_\Phi(f+t1_B) &= \frac{\partial}{\partial t}\exp\left[-\Lambda\left(1-e^{-f-t1_B}\right)\right]\\
&= \mathcal{L}_\Phi(f+t1_B)\frac{\partial}{\partial t}\left[-\Lambda\left(1-e^{-f-t1_B}\right)\right]\\
&= -\mathcal{L}_\Phi(f+t1_B)\Lambda\left(1_B e^{-f-t1_B}\right),
\end{aligned}$$

where for the third equality we invert the derivation and the integral, using $\left|1_B(x)e^{-f(x)-t1_B(x)}\right| \le |1_B(x)|$ and $\Lambda(1_B) = \Lambda(B) < \infty$. Hence

$$\left.\frac{\partial}{\partial t}\mathcal{L}_\Phi(f+t1_B)\right|_{t=0} = -\int_B \mathcal{L}_\Phi(f)e^{-f(x)}\Lambda(\mathrm{d}x).$$

On the other hand, by Proposition 3.2.1,

$$\left.\frac{\partial}{\partial t}\mathcal{L}_\Phi(f+t1_B)\right|_{t=0} = -\int_B \mathcal{L}_{\Phi_x}(f)\Lambda(\mathrm{d}x).$$

Thus

$$\mathcal{L}_{\Phi_x}(f) = \mathcal{L}_\Phi(f)e^{-f(x)} = \mathcal{L}_{\Phi+\delta_x}(f), \quad \text{for } \Lambda\text{-almost all } x \in \mathbb{G}.$$

Since the Laplace transform characterizes the probability distribution of the random measure by Corollary 1.2.2, then $\Phi_x \overset{\text{dist.}}{=} \Phi + \delta_x$ for Λ-almost all $x \in \mathbb{G}$.
Converse part. Let Φ be a random measure on $\mathbb{G}$ with locally finite mean measure Λ such that $\Phi_x \overset{\text{dist.}}{=} \Phi + \delta_x$, for Λ-almost all $x \in \mathbb{G}$. We have to prove that Φ is Poisson, which is equivalent to $\mathcal{L}_\Phi(f) = \exp\left[-\Lambda\left(1-e^{-f}\right)\right]$ for all

measurable $f : \mathbb{G} \to \mathbb{R}_+$. It is enough to prove the announced result for f bounded with support in $\mathcal{B}_c(\mathbb{G})$; then for any measurable $f : \mathbb{G} \to \mathbb{R}_+$ find an increasing sequence of bounded with support in $\mathcal{B}_c(\mathbb{G})$ measurable functions converging pointwise to f and invoke the monotone convergence theorem for the general case. Let $f : \mathbb{G} \to \mathbb{R}_+$ be measurable, bounded and with support in $\mathcal{B}_c(\mathbb{G})$ and let $t > 0$.

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{L}_\Phi(tf)\bigg|_t &= \frac{\partial}{\partial t}\mathcal{L}_\Phi(tf+sf)\bigg|_{s=0} \\
&= -\int_{\mathbb{G}} f(x)\,\mathcal{L}_{\Phi_x}(tf)\,\Lambda(\mathrm{d}x) \\
&= -\int_{\mathbb{G}} f(x)\,\mathcal{L}_{\Phi+\delta_x}(tf)\,\Lambda(\mathrm{d}x) \\
&= -\left[\int_{\mathbb{G}} f(x)\,e^{-tf(x)}\Lambda(\mathrm{d}x)\right]\mathcal{L}_\Phi(tf) = -a(t)\,\mathcal{L}_\Phi(tf),
\end{aligned}$$

where

$$a(t) := \int_{\mathbb{G}} f(x)\,e^{-tf(x)}\Lambda(\mathrm{d}x).$$

Then

$$\begin{aligned}
\mathcal{L}_\Phi(tf) &= \exp\left[-\int_0^t a(s)\,\mathrm{d}s\right] \\
&= \exp\left[-\int_0^t\int_{\mathbb{G}} f(x)\,e^{-sf(x)}\Lambda(\mathrm{d}x)\,\mathrm{d}s\right] \\
&= \exp\left[-\int_{\mathbb{G}}\left(\int_0^t f(x)\,e^{-sf(x)}\mathrm{d}s\right)\Lambda(\mathrm{d}x)\right] \\
&= \exp\left[-\int_{\mathbb{G}}\left(1-e^{-tf(x)}\right)\Lambda(\mathrm{d}x)\right].
\end{aligned}$$

□

3.2.2 Palm for Cox point processes

The following proposition shows that the reduced Palm version of a Cox point process (cf. Definition 2.3.1) is itself a Cox point process and gives its directing measure.

Proposition 3.2.5. Palm for Cox point processes. *Let Φ be a Cox point process on a l.c.s.h. space $\mathbb{G}$ directed by the random measure Λ having σ-finite mean measure. Let $\{\Phi_x\}_{x\in\mathbb{G}}$ and $\{\Lambda_x\}_{x\in\mathbb{G}}$ be families of Palm versions of Φ and Λ respectively. Then, for M_Λ-almost all $x \in \mathbb{G}$, the reduced Palm version $\Phi_x^!$ is a Cox point process directed by Λ_x.*

Proof. The Laplace transform of Φ is given by (2.3.2)

$$\mathcal{L}_\Phi(f) = \mathcal{L}_\Lambda\left(1-e^{-f}\right), \quad f \in \mathfrak{F}_+(\mathbb{G}).$$

Then, for $f \in \mathfrak{F}_+(\mathbb{G}), B \in \mathcal{B}(\mathbb{G}), t \in \mathbb{R}_+$,

$$\begin{aligned}
\mathcal{L}_\Phi(f + t1_B) &= \mathcal{L}_\Lambda\left(1 - e^{-f} e^{-t1_B}\right) \\
&= \mathcal{L}_\Lambda\left(1 - e^{-f}\left[1 + 1_B\left(e^{-t} - 1\right)\right]\right) \\
&= \mathcal{L}_\Lambda\left(1 - e^{-f} + e^{-f} 1_B\left(1 - e^{-t}\right)\right).
\end{aligned}$$

Introducing the function $u(t) = 1 - e^{-t}$, we get

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\mathcal{L}_\Phi(f + tg)\right|_{t=0} &= \left.\frac{\partial}{\partial u}\mathcal{L}_\Lambda\left(1 - e^{-f} + e^{-f} 1_B u\right)\right|_{u=0} \\
&= -\int_{\mathbb{G}} e^{-f(x)} 1_B \mathcal{L}_{\Lambda_x}\left(1 - e^{-f}\right) M_\Lambda(\mathrm{d}x),
\end{aligned}$$

where the second equality follows from Proposition 3.2.1. Recall from (2.3.1) that the mean measure of the Cox point process is the same as that of the directing measure. Then again by Proposition 3.2.1, we get

$$\mathcal{L}_{\Phi_x}(f) = e^{-f(x)} \mathcal{L}_{\Lambda_x}\left(1 - e^{-f}\right),$$

for M_Λ-almost all $x \in \mathbb{G}$. Thus

$$\begin{aligned}
\mathcal{L}_{\Phi_x^!}(f) &= \mathcal{L}_{\Phi_x - \delta_x}(f) \\
&= \mathcal{L}_{\Phi_x}(f) e^{f(x)} = \mathcal{L}_{\Lambda_x}\left(1 - e^{-f}\right),
\end{aligned}$$

which by comparison to (2.3.2) shows that $\Phi_x^!$ is a Cox point process directed by Λ_x. □

3.2.3 Palm distribution of Gibbs point processes

In the present section, we give the Palm distribution of Gibbs point processes introduced in Section 2.3.2. We begin with the mean measure of a such process.

Lemma 3.2.6. *Let $\tilde{\Phi}$ be a Gibbs point process on a l.c.s.h. space $\mathbb{G}$ with density f with respect to a Poisson point process Φ. Then its mean measure is*

$$M_{\tilde{\Phi}}(\mathrm{d}x) = \mathbf{E}\left[f(\Phi + \delta_x)\right] M_\Phi(\mathrm{d}x). \tag{3.2.3}$$

That is the mean measure of $\tilde{\Phi}$ has density $x \mapsto \mathbf{E}\left[f(\Phi + \delta_x)\right]$ with respect to the intensity measure of Φ.

Proof. For all $B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned}
M_{\tilde{\Phi}}(B) &= \int_{\mathbb{M}(\mathbb{G})} \mu(B)\, \mathbf{P}_{\tilde{\Phi}}(\mathrm{d}\mu) \\
&= \int_{\mathbb{M}(\mathbb{G})} \mu(B)\, f(\mu)\, \mathbf{P}_{\Phi}(\mathrm{d}\mu) \\
&= \mathbf{E}\left[\Phi(B)\, f(\Phi)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}} \mathbf{1}\{x \in B\}\, f(\Phi)\, \Phi(\mathrm{d}x)\right] \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} \mathbf{1}\{x \in B\}\, f(\mu)\, \mathbf{P}_{\Phi}^{x}(\mathrm{d}\mu)\, M_{\Phi}(\mathrm{d}x),
\end{aligned}$$

where the second equality follows from (2.3.6) and the last equality is due to the C-L-M theorem 3.1.9. Using Slivnyak's theorem 3.2.4, we may continue the above equalities as follows

$$\begin{aligned}
M_{\tilde{\Phi}}(B) &= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} \mathbf{1}\{x \in B\}\, f(\mu)\, \mathbf{P}_{\Phi+\delta_x}(\mathrm{d}\mu)\, M_{\Phi}(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} \mathbf{1}\{x \in B\}\, f(\mu+\delta_x)\, \mathbf{P}_{\Phi}(\mathrm{d}\mu)\, M_{\Phi}(\mathrm{d}x) \\
&= \int_{B} \mathbf{E}\left[f(\Phi+\delta_x)\right] M_{\Phi}(\mathrm{d}x).
\end{aligned}$$

□

Proposition 3.2.7. *Let $\tilde{\Phi}$ be a Gibbs point process on a l.c.s.h. space $\mathbb{G}$ with density f with respect to a Poisson point process Φ. Then, for M_Φ-almost all x, the reduced Palm version $\tilde{\Phi}_x^{!}$ is a Gibbs point process with density $\tilde{f}$ with respect to Φ, where*

$$\tilde{f}(\mu) = \frac{f(\mu+\delta_x)}{\mathbf{E}\left[f(\Phi+\delta_x)\right]}, \quad \mu \in \mathbb{M}(\mathbb{G}).$$

Proof. For all measurable functions $h : \mathbb{G} \times \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{R}}_+$,

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}} h\left(x, \tilde{\Phi}\right) \tilde{\Phi}(\mathrm{d}x)\right] &= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} h(x,\mu)\, \mu(\mathrm{d}x)\, \mathbf{P}_{\tilde{\Phi}}(\mathrm{d}\mu) \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} h(x,\mu)\, \mu(\mathrm{d}x)\, f(\mu)\, \mathbf{P}_{\Phi}(\mathrm{d}\mu) \\
&= \mathbf{E}\left[\int_{\mathbb{G}} h(x,\Phi)\, f(\Phi)\, \Phi(\mathrm{d}x)\right] \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} h(x,\mu)\, f(\mu)\, \mathbf{P}_{\Phi}^{x}(\mathrm{d}\mu)\, M_{\Phi}(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} h(x,\mu)\, f(\mu)\, \mathbf{P}_{\Phi+\delta_x}(\mathrm{d}\mu)\, M_{\Phi}(\mathrm{d}x), \qquad (3.2.4)
\end{aligned}$$

where the fourth equality is due to the C-L-M theorem 3.1.9 and the fifth one follows from Slivnyak's theorem 3.2.4. On the other hand, by the C-L-M theorem 3.1.9,

$$\begin{aligned}\mathbf{E}\left[\int_{\mathbb{G}} h\left(x, \tilde{\Phi}\right) \tilde{\Phi}\left(\mathrm{d}x\right)\right] &= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} h\left(x,\mu\right) \mathbf{P}^{x}_{\tilde{\Phi}}\left(\mathrm{d}\mu\right) M_{\tilde{\Phi}}\left(\mathrm{d}x\right)\\ &= \int_{\mathbb{G}\times\mathbb{M}(\mathbb{G})} h\left(x,\mu\right) \mathbf{E}\left[f\left(\Phi+\delta_x\right)\right] \mathbf{P}^{x}_{\tilde{\Phi}}\left(\mathrm{d}\mu\right) M_{\Phi}\left(\mathrm{d}x\right).\end{aligned}$$

The above two equations show that, for M_{Φ}-almost all x,

$$\mathbf{P}^{x}_{\tilde{\Phi}}\left(\mathrm{d}\mu\right) = \frac{f\left(\mu\right)}{\mathbf{E}\left[f\left(\Phi+\delta_x\right)\right]} \mathbf{P}_{\Phi+\delta_x}\left(\mathrm{d}\mu\right).$$

Thus, for M_{Φ}-almost all x,

$$\begin{aligned}\mathbf{P}_{\tilde{\Phi}^{!}_{x}}\left(\mathrm{d}\mu\right) &= \mathbf{P}_{\tilde{\Phi}_x-\delta_x}\left(\mathrm{d}\mu\right)\\ &= \mathbf{P}_{\tilde{\Phi}_x}\left(\mathrm{d}\mu+\delta_x\right) = \frac{f\left(\mu+\delta_x\right)}{\mathbf{E}\left[f\left(\Phi+\delta_x\right)\right]} \mathbf{P}_{\Phi}\left(\mathrm{d}\mu\right),\end{aligned}$$

which is the announced result. □

Corollary 3.2.8. Papangelou's formula. *Let $\tilde{\Phi}$ be a Gibbs point process on a l.c.s.h. space $\mathbb{G}$ with density f with respect to a Poisson point processΦ. Then for all measurable $h : \mathbb{G} \times \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{R}}_+$,*

$$\mathbf{E}\left[\int_{\mathbb{G}} h\left(x, \tilde{\Phi}\right) \tilde{\Phi}\left(\mathrm{d}x\right)\right] = \int_{\mathbb{G}} \mathbf{E}\left[h\left(x, \Phi+\delta_x\right) f\left(\Phi+\delta_x\right)\right] M_{\Phi}\left(\mathrm{d}x\right). \qquad (3.2.5)$$

Proof. This follows from Equation (3.2.4). □

Using (2.3.7) for the right-hand side of the above equality, we may write Papangelou's formula (3.2.5) as follows

$$\mathbf{E}\left[\int_{\mathbb{G}} h\left(x, \tilde{\Phi}\right) \tilde{\Phi}\left(\mathrm{d}x\right)\right] = \int_{\mathbb{G}} \mathbf{E}\left[h\left(x, \tilde{\Phi}+\delta_x\right) \frac{f\left(\tilde{\Phi}+\delta_x\right)}{f\left(\tilde{\Phi}\right)}\right] M_{\Phi}\left(\mathrm{d}x\right).$$

Example 3.2.9. *Let Φ be a Poisson point processon a l.c.s.h. space $\mathbb{G}$ such that $\lambda := \Lambda(\mathbb{G}) < 1$. Let $g : \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{R}}_+$ be defined by*

$$g\left(\mu\right) = \mu\left(\mathbb{G}\right)!, \quad \mu \in \mathbb{M}\left(\mathbb{G}\right).$$

Recall first two elementary properties of Poisson random variables N with intensity $\lambda < 1$,

$$\mathbf{E}\left[N!\right] = \sum_{n=0}^{\infty} n! e^{-\lambda} \frac{\lambda^n}{n!} = \frac{e^{-\lambda}}{1-\lambda}. \qquad (3.2.6)$$

and

$$\mathbf{E}\left[(N+1)!\right] = \sum_{n=0}^{\infty}(n+1)!e^{-\lambda}\frac{\lambda^n}{n!}$$
$$= e^{-\lambda}\sum_{n=0}^{\infty}(n+1)\,\lambda^n = \frac{e^{-\lambda}}{(1-\lambda)^2}. \qquad (3.2.7)$$

Then by (3.2.6), $\mathbf{E}\left[g\left(\Phi\right)\right] = \frac{e^{-\lambda}}{1-\lambda} \in \mathbb{R}_+^*$. *By Remark 2.3.10, we may consider a Gibbs point process* $\tilde{\Phi}$ *with density*

$$f\left(\mu\right) = \frac{g\left(\mu\right)}{\mathbf{E}\left[g\left(\Phi\right)\right]} = \left(1-\lambda\right)e^{\lambda}\mu\left(\mathbb{G}\right)!, \quad \mu \in \mathbb{M}\left(\mathbb{G}\right), \qquad (3.2.8)$$

with respect to Φ. *The intensity measure of* $\tilde{\Phi}$ *is given by (3.2.3); that is, for any* $B \in \mathcal{B}\left(\mathbb{G}\right)$,

$$\mathbf{E}\left[\tilde{\Phi}\left(B\right)\right] = \int_B \mathbf{E}\left[f\left(\Phi+\delta_x\right)\right]\Lambda\left(\mathrm{d}x\right)$$
$$= \Lambda\left(B\right)\left(1-\lambda\right)e^{\lambda}\mathbf{E}\left[\left(\Phi\left(\mathbb{G}\right)+1\right)!\right]$$
$$= \frac{\Lambda\left(B\right)}{1-\lambda}, \qquad (3.2.9)$$

where the last equality is due to (3.2.7).

Example 3.2.10. Multiclass $M/GI/1/PS$ queue. *We will show that Example 3.2.9 may be seen as a spatial extension of a famous example of queueing theory; namely, the multiclass* M/GI/1/PS *queue. Let* $\mathbb{G}$ *be the set of classes assumed finite and denote by* $j \in \mathbb{G}$ *a particular class. Assume that users of each class* $j \in \mathbb{G}$ *arrive with rate* $\alpha_j \in \mathbb{R}_+^*$ *and require to transmit a volume of data of mean* β_j^{-1} *(where* $\beta_j \in \mathbb{R}_+^*$*). The state of the queue is denoted by* $\mu = \left(\mu_j\right)_{j\in\mathbb{G}}$ *where* μ_j *is the number of users of class* j. *The state space is* $\mathbb{N}^{\mathbb{G}}$. *The stationary distribution of the state process is [25]*

$$\pi\left(\mu\right) = \left(1-\lambda\right)\mu\left(\mathbb{G}\right)!\prod_{j\in\mathbb{G}}\frac{\Lambda_j^{\mu_j}}{\mu_j!}, \quad \mu \in \mathbb{N}^{\mathbb{G}},$$

where $\Lambda_j = \frac{\alpha_j}{\beta_j}$ *and* $\lambda = \sum_{j\in\mathbb{G}}\Lambda_j$. *The above distribution is that of a Gibbs point process with density* f *given by (3.2.8) with respect to a Poisson point process on* $\mathbb{G}$ *with intensity measure* $\Lambda = \left(\Lambda_j\right)_{j\in\mathbb{G}}$. *It follows from (3.2.9) that the mean number of users of class* j *at the stationary state, denoted* N_j, *equals*

$$N_j = \frac{\Lambda_j}{1-\lambda}.$$

From Little's formula [8] the mean delay, denoted T_j, *equals*

$$T_j = \frac{N_j}{\alpha_j} = \frac{1}{\left(1-\lambda\right)\beta_j}.$$

The mean throughput of class j, denoted r_j, is the average required volume β_j^{-1} divided by the expected delay, that is

$$r_j = \frac{\beta_j^{-1}}{T_j} = 1 - \lambda.$$

3.2.4 Palm distribution for marked random measures

The objective of the present section is to build a relation between Palm versions of a marked random measure (cf. Section 2.2.6) and Palm versions of the associated ground random measure.

Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{G})$ and $\mathcal{B}(\mathbb{K})$ respectively. Let $\tilde{\Phi}$ be a marked random measure on $\mathbb{G}\times\mathbb{K}$ and let Φ be the associated ground random measure, i.e., its projection on $\mathbb{G}$. Note that, for $B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned} M_{\tilde{\Phi}}(B\times\mathbb{K}) &= \mathbf{E}\left[\tilde{\Phi}(B\times\mathbb{K})\right] \\ &= \mathbf{E}[\Phi(B)] = M_\Phi(B). \end{aligned}$$

That is M_Φ is the projection of $M_{\tilde{\Phi}}$ onto $\mathbb{G}$. Assume that M_Φ is σ-finite (*which we will assume from now on*), then so is $M_{\tilde{\Phi}}$. Moreover $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ is Polish since it is l.c.s.h [56, Theorem 5.3, p.29,], then it follows from the measure disintegration theorem 14.D.10 that $M_{\tilde{\Phi}}$ admits a disintegration

$$M_{\tilde{\Phi}}(B\times K) = \int \Pi(x,K)\, M_\Phi(\mathrm{d}x), \quad B\in\mathcal{B}(\mathbb{G}), K\in\mathcal{B}(\mathbb{K}), \tag{3.2.10}$$

where $\Pi(\cdot,\cdot)$ is a probability kernel from $\mathbb{G}$ to $\mathbb{K}$. Moreover, for every $\mathcal{B}(\mathbb{G})\otimes\mathcal{B}(\mathbb{K})$-measurable function $\varphi(x,z)$ which is nonnegative or $M_{\tilde{\Phi}}$-integrable

$$\int_{\mathbb{G}\times\mathbb{K}} \varphi(x,z)\, M_{\tilde{\Phi}}(\mathrm{d}x\times\mathrm{d}z) = \int_{\mathbb{G}}\int_{\mathbb{K}} \varphi(x,z)\,\Pi(x,\mathrm{d}z)\, M_\Phi(\mathrm{d}x). \tag{3.2.11}$$

Example 3.2.11. *Let $\tilde{\Phi}$ be an independently marked point process on $\mathbb{G}\times\mathbb{K}$ associated to the ground process Φ through the probability kernel $\tilde{p}$; cf. Definition 2.2.18. Then, by (2.2.8), $M_{\tilde{\Phi}}(\mathrm{d}x\times\mathrm{d}z) = M_\Phi(\mathrm{d}x)\,\tilde{p}(x,\mathrm{d}z)$ which shows that $\Pi(x,\mathrm{d}z) = \tilde{p}(x,\mathrm{d}z)$ for M_Φ-almost all $x\in\mathbb{G}$.*

Remark 3.2.12. *Note that, for given $K\in\mathcal{B}(\mathbb{K})$, $M_{\tilde{\Phi}}(\cdot\times K) \ll M_\Phi(\cdot)$. Then Equation (3.2.10) shows that $\Pi(\cdot,K)$ is the Radon-Nikodym derivative of the measure $M_{\tilde{\Phi}}(\cdot\times K)$ with respect to M_Φ; that is*

$$\Pi(\cdot,K) = \frac{\mathrm{d}M_{\tilde{\Phi}}(\cdot\times K)}{\mathrm{d}M_\Phi(\cdot)}.$$

Proposition 3.2.13. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces, $\tilde{\Phi}$ be a marked random measure on $\mathbb{G}\times\mathbb{K}$, and Φ be its projection on $\mathbb{G}$ having σ-finite mean measure.*

Let $\left\{\tilde{\Phi}_{(x,z)}\right\}_{(x,z)\in\mathbb{G}}$ *be a family of Palm versions of* $\tilde{\Phi}$ *and* $\{\Phi_x\}_{x\in\mathbb{G}}$ *be a family of Palm versions of* Φ*. Then, for* M_Φ*-almost* $x \in \mathbb{G}$*,*

$$\mathbf{P}(\Phi_x \in L) = \int_{\mathbb{K}} \mathbf{P}\left(\tilde{\Phi}_{(x,z)}(\cdot \times \mathbb{K}) \in L\right) \Pi(x, \mathrm{d}z), \quad L \in \bar{\mathcal{M}}(\mathbb{G}),$$

where $\Pi(\cdot,\cdot)$ *is a probability kernel obtained by the mean measure disintegration (3.2.10).*

Proof. By the C-L-M theorem 3.1.9, for all measurable functions $f : \mathbb{G}\times\mathbb{K}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K}) \to \bar{\mathbb{R}}_+$, we have

$$\mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} f\left(x, z, \tilde{\Phi}\right) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] = \int_{\mathbb{G}\times\mathbb{K}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})} f(x, z, \tilde{\mu}) \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\tilde{\mu}) M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z),$$

where $\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})$ be the space of measures $\tilde{\mu}$ on $(\mathbb{G}\times\mathbb{K}, \mathcal{B}(\mathbb{G})\otimes\mathcal{B}(\mathbb{K}))$ such that $\tilde{\mu}(B\times\mathbb{K}) < \infty$ for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$. In particular, for all measurable functions $g : \mathbb{G}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K}) \to \bar{\mathbb{R}}_+$, we have

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} g\left(x, \tilde{\Phi}\right) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] &= \int_{\mathbb{G}\times\mathbb{K}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})} g(x, \tilde{\mu}) \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\tilde{\mu}) M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z) \\
&= \int_{\mathbb{G}\times\mathbb{K}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})} g(x, \tilde{\mu}) \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\tilde{\mu}) \Pi(x, \mathrm{d}z) M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})} g(x, \tilde{\mu}) \left(\int_{\mathbb{K}} \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\tilde{\mu}) \Pi(x, \mathrm{d}z)\right) M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})} g(x, \tilde{\mu}) \bar{\mathbf{P}}_{\tilde{\Phi}}^{x}(\mathrm{d}\tilde{\mu}) M_\Phi(\mathrm{d}x),
\end{aligned}$$

where the second equality follows from (3.2.11) and for the last equality we denote $\bar{\mathbf{P}}_{\tilde{\Phi}}^{x}(\mathrm{d}\tilde{\mu}) := \int_{\mathbb{K}} \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\tilde{\mu}) \Pi(x, \mathrm{d}z)$. For any measurable function $h : \mathbb{G}\times\bar{\mathbb{M}}(\mathbb{G}) \to \bar{\mathbb{R}}_+$, define a function g on $\mathbb{G}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})$ by $g\left(x, \tilde{\Phi}\right) = h(x, \Phi)$. Then the LHS of the above equation equals

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} g\left(x, \tilde{\Phi}\right) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] &= \mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} h(x, \Phi) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}} h(x, \Phi) \tilde{\Phi}(\mathrm{d}x \times \mathbb{K})\right] \\
&= \mathbf{E}\left[\int_{\mathbb{G}} h(x, \Phi) \Phi(\mathrm{d}x)\right]
\end{aligned}$$

and the RHS equals

$$\begin{aligned}
\int_{\mathbb{G}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})} g(x, \tilde{\mu}) \bar{\mathbf{P}}_{\tilde{\Phi}}^{x}(\mathrm{d}\tilde{\mu}) M_\Phi(\mathrm{d}x) &= \int_{\mathbb{G}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})} h(x, \mu) \bar{\mathbf{P}}_{\tilde{\Phi}}^{x}(\mathrm{d}\tilde{\mu}) M_\Phi(\mathrm{d}x) \\
&= \int_{\mathbb{G}\times\bar{\mathbb{M}}(\mathbb{G})} h(x, \mu) \bar{\mathbf{P}}_{\Phi}^{x}(\mathrm{d}\mu) M_\Phi(\mathrm{d}x),
\end{aligned}$$

where $\bar{\mathbf{P}}_\Phi^x$ is the projection of $\bar{\mathbf{P}}_{\tilde{\Phi}}^x$ on $\bar{\mathbb{M}}(\mathbb{G})$, that is

$$\bar{\mathbf{P}}_\Phi^x(L) = \bar{\mathbf{P}}_{\tilde{\Phi}}^x\left\{\tilde{\mu} \in \tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K}) : \tilde{\mu}(\cdot\times\mathbb{K}) \in L\right\},$$

for $L \in \bar{\mathcal{M}}(\mathbb{G})$. Thus

$$\mathbf{E}\left[\int_{\mathbb{G}} h(x,\Phi)\,\Phi(\mathrm{d}x)\right] = \int_{\mathbb{G}\times\bar{\mathbb{M}}(\mathbb{G})} h(x,\mu)\,\bar{\mathbf{P}}_\Phi^x(\mathrm{d}\mu)\,M_\Phi(\mathrm{d}x),$$

Comparing the above equality with the C-L-M theorem 3.1.9 shows that $\bar{\mathbf{P}}_\Phi^x = \mathbf{P}_\Phi^x$ for M_Φ-almost $x \in \mathbb{G}$. Thus $\mathbf{P}_\Phi^x$ is the projection of $\bar{\mathbf{P}}_{\tilde{\Phi}}^x$ on $\bar{\mathbb{M}}(\mathbb{G})$; that is, for $L \in \bar{\mathcal{M}}(\mathbb{G})$,

$$\mathbf{P}_\Phi^x(L) = \int_{\mathbb{K}} \mathbf{P}_{\tilde{\Phi}}^{(x,z)}\left(\tilde{\mu} \in \tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K}) : \tilde{\mu}(\cdot\times\mathbb{K}) \in L\right)\Pi(x,\mathrm{d}z),$$

or, equivalently,

$$\mathbf{P}(\Phi_x \in L) = \int_{\mathbb{K}} \mathbf{P}\left(\tilde{\Phi}_{(x,z)}(\cdot\times\mathbb{K}) \in L\right)\Pi(x,\mathrm{d}z).$$

□

In the particular case of independently marked point processes, we may retrieve $\tilde{\Phi}_{(x,z)}$ from Φ_x as shown in the following proposition.

Proposition 3.2.14. Palm for independently marked point process. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces, $\tilde{\Phi}$ be an independently marked point process on $\mathbb{G}\times\mathbb{K}$ associated to the ground process Φ having σ-finite mean measure through the probability kernel $\tilde{p}$; cf. Definition 2.2.18. Then, for $M_{\tilde{\Phi}}$-almost all (x,z), the reduced Palm version $\tilde{\Phi}^!_{(x,z)}$ is equal in distribution to an independently marked point process on $\mathbb{G}\times\mathbb{K}$ associated to the ground process $\Phi^!_x$ through the probability kernel $\tilde{p}(\cdot,\cdot)$.*

Proof. By Corollary 3.1.14 (Campbell-Little-Mecke theorem in the reduced case), $\tilde{\Phi}^!_{(x,z)}$ satisfies

$$\int_{\mathbb{G}\times\mathbb{K}} \mathbf{E}\left[f\left(x,z,\tilde{\Phi}^!_{(x,z)}\right)\right] M_{\tilde{\Phi}}(\mathrm{d}x\times\mathrm{d}z) = \mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} f\left(x,z,\tilde{\Phi}-\delta_{(x,z)}\right)\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}z)\right]. \tag{3.2.12}$$

for any measurable function $f : \mathbb{G}\times\mathbb{K}\times\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K}) \to \mathbb{R}_+$, where $\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})$ be the space of measures $\tilde{\mu}$ on $(\mathbb{G}\times\mathbb{K}, \mathcal{B}(\mathbb{G})\otimes\mathcal{B}(\mathbb{K}))$ such that $\tilde{\mu}(B\times\mathbb{K}) < \infty$ for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$. Moreover, by (2.2.8),

$$M_{\tilde{\Phi}}(\mathrm{d}x\times\mathrm{d}z) = M_\Phi(\mathrm{d}x)\,\tilde{p}(x,\mathrm{d}z).$$

We write

$$\Phi^!_x = \sum_{n\in\mathbb{Z}:X_n\in\Phi^!_x} \delta_{X_n}.$$

Let $\tilde{\Psi}_x$ be an independently marked point process on $\mathbb{G} \times \mathbb{K}$ associated to the ground process $\Phi_x^!$ through the probability kernel $\tilde{p}(\cdot, \cdot)$. We may write

$$\tilde{\Psi}_x = \sum_{n \in \mathbb{Z}: X_n \in \Phi_x^!} \delta_{(X_n, Y_n)}$$

which may be viewed as the pair $\left(\Phi_x^!, \hat{Y}\right)$, where $\hat{Y} = \{Y_n\}_{n \in \mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}}$. Then, for all $x \in \mathbb{G}, z \in \mathbb{K}$,

$$\begin{aligned}
\mathbf{E}\left[f\left(x, z, \tilde{\Psi}_x\right)\right] &= \mathbf{E}\left[f\left(x, z, \Phi_x^!, \hat{Y}\right)\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[f\left(x, z, \Phi_x^!, \hat{Y}\right) \mid \Phi_x^!\right]\right] \\
&= \mathbf{E}\left[\int_{\mathbb{K}^{\mathbb{Z}}} f\left(x, z, \Phi_x^!, \hat{y}\right) \prod_{n \in \mathbb{Z}: X_n \in \Phi_x^!} \tilde{p}(X_n, \mathrm{d}y_n)\right],
\end{aligned}$$

where $\hat{y} = \{y_n\}_{n \in \mathbb{Z}}$. Thus

$$\begin{aligned}
&\int_{\mathbb{G} \times \mathbb{K}} \mathbf{E}\left[f\left(x, z, \tilde{\Psi}_x\right)\right] M_{\tilde{\Phi}}\left(\mathrm{d}x \times \mathrm{d}z\right) \\
&\quad = \int_{\mathbb{G} \times \mathbb{K}} \mathbf{E}\left[f\left(x, z, \tilde{\Psi}_x\right)\right] M_{\Phi}\left(\mathrm{d}x\right) \tilde{p}(x, \mathrm{d}z) \\
&\quad = \int_{\mathbb{G} \times \mathbb{K}} \mathbf{E}\left[\int_{\mathbb{K}^{\mathbb{Z}}} f\left(x, z, \Phi_x^!, \hat{y}\right) \prod_{n \in \mathbb{Z}: X_n \in \Phi_x^!} \tilde{p}(X_n, \mathrm{d}y_n)\right] M_{\Phi}\left(\mathrm{d}x\right) \tilde{p}(x, \mathrm{d}z) \\
&\quad = \int_{\mathbb{G}} \mathbf{E}\left[\int_{\mathbb{K} \times \mathbb{K}^{\mathbb{Z}}} f\left(x, z, \Phi_x^!, \hat{y}\right) \tilde{p}(x, \mathrm{d}z) \prod_{n \in \mathbb{Z}: X_n \in \Phi_x^!} \tilde{p}(X_n, \mathrm{d}y_n)\right] M_{\Phi}\left(\mathrm{d}x\right) \\
&\quad = \int_{\mathbb{G}} \mathbf{E}\left[\varphi\left(x, \Phi_x^!\right)\right] M_{\Phi}\left(\mathrm{d}x\right),
\end{aligned} \tag{3.2.13}$$

where

$$\varphi\left(x, \Phi_x^!\right) = \int_{\mathbb{K} \times \mathbb{K}^{\mathbb{Z}}} f\left(x, z, \Phi_x^!, \hat{y}\right) \tilde{p}(x, \mathrm{d}z) \prod_{n \in \mathbb{Z}: X_n \in \Phi_x^!} \tilde{p}(X_n, \mathrm{d}y_n).$$

By (3.1.4), we have

$$\begin{aligned}
&\int_{\mathbb{G}} \mathbf{E}\left[\varphi\left(x, \Phi_x^!\right)\right] M_\Phi\left(\mathrm{d}x\right) \\
&\quad = \int_{\mathbb{G}} \mathbf{E}\left[\varphi\left(x, \Phi_x^!\right)\right] M_\Phi\left(\mathrm{d}x\right) \\
&\quad = \mathbf{E}\left[\int_{\mathbb{G}} \varphi\left(x, \Phi - \delta_x\right) \Phi\left(\mathrm{d}x\right)\right] \\
&\quad = \mathbf{E}\left[\int_{\mathbb{G}} \left(\int_{\mathbb{K}\times\mathbb{K}^{\mathbb{Z}}} f\left(x, z, \Phi - \delta_x, \hat{y}\right) \tilde{p}(x, \mathrm{d}z) \prod_{n\in\mathbb{Z}: X_n \in \Phi - \delta_x} \tilde{p}(X_n, \mathrm{d}y_n)\right) \Phi\left(\mathrm{d}x\right)\right] \\
&\quad = \mathbf{E}\left[\sum_{i\in\mathbb{Z}: X_i\in\Phi} \int_{\mathbb{K}\times\mathbb{K}^{\mathbb{Z}}} f\left(X_i, z, \Phi - \delta_{X_i}, \hat{y}\right) \tilde{p}(X_i, \mathrm{d}z) \prod_{n\in\mathbb{Z}: X_n\in\Phi - \delta_{X_i}} \tilde{p}(X_n, \mathrm{d}y_n)\right] \\
&\quad = \mathbf{E}\left[\sum_{i\in\mathbb{Z}: X_i\in\Phi} f\left(X_i, Z_i, \tilde{\Phi} - \delta_{(X_i, Z_i)}\right)\right] \\
&\quad = \mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} f\left(x, z, \tilde{\Phi} - \delta_{(x,z)}\right) \tilde{\Phi}\left(\mathrm{d}x \times \mathrm{d}z\right)\right]. \qquad (3.2.14)
\end{aligned}$$

Combining (3.2.12)-(3.2.14) we get

$$\int_{\mathbb{G}\times\mathbb{K}} \mathbf{E}\left[f\left(x, z, \tilde{\Phi}_{(x,z)}^!\right)\right] M_{\tilde{\Phi}}\left(\mathrm{d}x \times \mathrm{d}z\right) = \int_{\mathbb{G}\times\mathbb{K}} \mathbf{E}\left[f\left(x, z, \tilde{\Psi}_x\right)\right] M_{\tilde{\Phi}}\left(\mathrm{d}x \times \mathrm{d}z\right).$$

Taking $f\left(x, z, \mu\right) = \mathbf{1}\left\{(x, z) \in B\right\} h\left(\mu\right)$ where $B \in \mathcal{B}\left(\mathbb{G} \times \mathbb{K}\right)$ and h is a measurable nonnegative function on $\mathbb{M}\left(\mathbb{G} \times \mathbb{K}\right)$, we get

$$\int_B \mathbf{E}\left[h\left(\tilde{\Phi}_{(x,z)}^!\right)\right] M_{\tilde{\Phi}}\left(\mathrm{d}x \times \mathrm{d}z\right) = \int_B \mathbf{E}\left[h\left(\tilde{\Psi}_x\right)\right] M_{\tilde{\Phi}}\left(\mathrm{d}x \times \mathrm{d}z\right)$$

Hence

$$\mathbf{E}\left[h\left(\tilde{\Phi}_{(x,z)}^!\right)\right] = \mathbf{E}\left[h\left(\tilde{\Psi}_x\right)\right],$$

and consequently,

$$\tilde{\Phi}_{(x,z)}^! \stackrel{\text{dist}}{=} \tilde{\Psi}_x,$$

for $M_{\tilde{\Phi}}$-almost all (x, z). □

Example 3.2.15. Palm for i.i.d. marked point processes. *Let $\tilde{\Phi}$ be an i.i.d marked point process on $\mathbb{G} \times \mathbb{K}$ associated to the ground process Φ with mark distribution $F(\cdot)$; cf. Definition 2.2.18. Then, by Proposition 3.2.14, for $M_{\tilde{\Phi}}$-almost all (x, z), $\tilde{\Phi}_{(x,z)}^!$ is equal in distribution to an i.i.d. marked point process on $\mathbb{G} \times \mathbb{K}$ associated to the ground process $\Phi_x^!$ with the same mark distribution $F(\cdot)$.*

3.3 Higher order Palm and reduced Palm

3.3.1 Higher order Palm

Lemma 3.3.1. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$, $n \in \mathbb{N}^*$, Φ^n be the n-th power of Φ and $\mathcal{B}(\mathbb{G})^{\otimes n}$ be the product σ-algebra on $\mathbb{G}^n$. There exists a unique σ-finite measure $\mathcal{C}_\Phi^n$ on $\mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G})$ characterized by*

$$\mathcal{C}_\Phi^n(B \times L) = \mathbf{E}\left[\int_B 1\{\Phi \in L\}\Phi^n(\mathrm{d}x)\right], \quad B \in \mathcal{B}(\mathbb{G})^{\otimes n}, L \in \bar{\mathcal{M}}(\mathbb{G})$$

called the n-th Campbell measure *associated to Φ.*

Proof. The proof follows the same lines as that of Lemma 3.1.1. In particular, it relies of the fact that $\mathcal{C}_\Phi^n$ is σ-finite of the algebra of unions of sets $B \times L, B \in \mathcal{B}(\mathbb{G})^{\otimes n}, L \in \bar{\mathcal{M}}(\mathbb{G})$. Indeed, by Lemma 1.1.4, we may construct a partition of $\mathbb{G}$ into $B_0, B_1, \ldots \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$. Then $\mathbb{G}^n = \bigcup_{j_1,\ldots,j_n \in \mathbb{N}} B_{j_1} \times \ldots \times B_{j_n}$, and since the measures in $\bar{\mathbb{M}}(\mathbb{G})$ are locally finite we get

$$\mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G}) = \bigcup_{j_1,\ldots,j_n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} B_{j_1} \times \ldots \times B_{j_n} \times \left\{\mu \in \bar{\mathbb{M}}(\mathbb{G}) : \mu(B_{j_i}) \leq k, \forall i \in [1,n]\right\}.$$

If we apply $\mathcal{C}_\Phi^n$ to each term in the above union, we get

$$\mathbf{E}\left[\int_{B_{j_1} \times \ldots \times B_{j_n}} \mathbf{1}\{\Phi(B_{j_i}) \leq k, \forall i \in [1,n]\}\Phi^n(\mathrm{d}x)\right] \leq k^n < \infty.$$

□

Note that, for $B \in \mathcal{B}(\mathbb{G})^{\otimes n}$,

$$\mathcal{C}_\Phi^n\left(B \times \bar{\mathbb{M}}(\mathbb{G})\right) = \mathbf{E}[\Phi^n(B)] = M_{\Phi^n}(B).$$

That is, the mean measure of Φ^n is the projection of the n-th Campbell measure onto $\mathbb{G}^n$.

Assume that M_{Φ^n} is σ-finite. Since $\left(\bar{\mathbb{M}}(\mathbb{G}), \bar{\mathcal{M}}(\mathbb{G})\right)$ is Polish [52, §15.7.7] and $\mathcal{C}_\Phi^n$ is σ-finite from Lemma 3.3.1, it follows from the measure disintegration theorem 14.D.10 that $\mathcal{C}_\Phi^n$ admits a disintegration

$$\mathcal{C}_\Phi^n(B \times L) = \int_B \mathbf{P}_\Phi^x(L) M_{\Phi^n}(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{G})^{\otimes n}, L \in \bar{\mathcal{M}}(\mathbb{G}), \qquad (3.3.1)$$

where $\mathbf{P}_\Phi^{\cdot}(\cdot)$ is a probability kernel from $\mathbb{G}^n$ to $\bar{\mathbb{M}}(\mathbb{G})$. We call $\mathbf{P}_\Phi^x(\cdot)$ the *n-th Palm distribution* at $x \in \mathbb{G}^n$.

Theorem 3.3.2. Higher order Campbell-Little-Mecke. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ such that M_{Φ^n} is σ-finite and let $\{\mathbf{P}_\Phi^x\}_{x \in \mathbb{G}^n}$ be a family of n-th Palm distributions of Φ. Then for all measurable $f : \mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G}) \to \bar{\mathbb{R}}_+$, we have*

$$\mathbf{E}\left[\int_{\mathbb{G}^n} f(x,\Phi)\Phi^n(\mathrm{d}x)\right] = \int_{\mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G})} f(x,\mu)\mathbf{P}_\Phi^x(\mathrm{d}\mu) M_{\Phi^n}(\mathrm{d}x). \qquad (3.3.2)$$

Proof. We first show the announced equality for $f(x,\mu) = \mathbf{1}\{x \in B, \mu \in L\}$, where $B \in \mathcal{B}(\mathbb{G})^{\otimes n}, L \in \bar{\mathcal{M}}(\mathbb{G})$. To do so, note that

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{G}^n} f(x,\Phi)\,\Phi^n(\mathrm{d}x)\right] &= \mathbf{E}\left[\int_{\mathbb{G}^n} \mathbf{1}\{x \in B, \Phi \in L\}\Phi^n(\mathrm{d}x)\right] \\
&= \mathcal{C}_\Phi^n(B \times L) \\
&= \int_B \mathbf{P}_\Phi^x(L)\, M_{\Phi^n}(\mathrm{d}x) \\
&= \int_{\mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G})} \mathbf{1}\{x \in B, \mu \in L\}\mathbf{P}_\Phi^x(\mathrm{d}\mu)\, M_{\Phi^n}(\mathrm{d}x) \\
&= \int_{\mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G})} f(x,\mu)\,\mathbf{P}_\Phi^x(\mathrm{d}\mu)\, M_{\Phi^n}(\mathrm{d}x),
\end{aligned}$$

where we use (3.3.1) for the third equality For general measurable $f : \mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G}) \to \bar{\mathbb{R}}_+$, the proof relies on the functional monotone class theorem along the same lines as in the proof of Theorem 3.1.9. □

Corollary 3.3.3. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ such that M_{Φ^n} is σ-finite and let $\{\mathbf{P}_\Phi^x\}_{x\in\mathbb{G}^n}$ be a family of n-th Palm distributions of Φ. Let $f : \mathbb{G}^n \times \bar{\mathbb{M}}(\mathbb{G}) \to \bar{\mathbb{C}}$ be measurable. If either of the following conditions*

$$\mathbf{E}\left[\int_{\mathbb{G}^n} |f(x,\Phi)|\,\Phi^n(\mathrm{d}x)\right] < \infty, \quad \textit{or} \int_{\mathbb{G}^n\times\mathbb{M}} |f(x,\mu)|\,\mathbf{P}_\Phi^x(\mathrm{d}\mu)\, M_{\Phi^n}(\mathrm{d}x) < \infty$$

holds, then the other one holds, and Equality (3.3.2) is true.

Proof. The proof follows the same lines as that of Corollary 3.1.10. □

Similarly to the case of first order Palm distributions, for $x \in \mathbb{G}^n$, let Φ_x be a random measure on $\mathbb{G}$ with probability distribution $\mathbf{P}_\Phi^x(\cdot)$ in the probability space $(\Omega, \mathcal{A}, \mathbf{P})$; that is

$$\mathbf{P}(\Phi_x \in L) = \mathbf{P}_\Phi^x(L), \quad L \in \bar{\mathcal{M}}(\mathbb{G}).$$

Φ_x is called *Palm version* of Φ at $x \in \mathbb{G}^n$.

Proposition 3.3.4. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ such that M_{Φ^n} is σ-finite. Then for M_{Φ^n}-almost all $x = (x_1, \ldots, x_n) \in \mathbb{G}^n$*

$$\mathbf{P}(\forall i \in [1,n], \Phi_x(\{x_i\}) \geq m_i(x)) = 1,$$

where $m_i(x)$ is the multiplicity of x_i within x; that is

$$m_i(x) = \operatorname{card}\{j \in [1,n] : x_j = x_i\}.$$

Proof. Note first that

$$\begin{aligned}
\mathbf{P}(\exists i \in [1,n] : \Phi_x(\{x_i\}) < m_i(x)) &= \mathbf{E}[\mathbf{1}\{\exists i \in [1,n] : \Phi_x(\{x_i\}) < m_i(x)\}] \\
&= \int_{\mathbb{M}(\mathbb{G})} \mathbf{1}\{\exists i \in [1,n] : \mu(\{x_i\}) < m_i(x)\}\,\mathbf{P}_\Phi^x(\mathrm{d}\mu).
\end{aligned}$$

Following the lines of the proof of Lemma 3.1.11, one may prove that the mapping $\mathbb{G}^n \times \mathbb{M}(\mathbb{G}) \to \mathbb{R}_+; (x,\mu) \mapsto (\mu(\{x_1\}), \ldots, \mu(\{x_n\}))$ is measurable. Then we may integrate the above equation with respect to M_{Φ^n}, which gives

$$\begin{aligned}
&\int_{\mathbb{G}^n} \mathbf{P}\left(\exists i \in [1,n] : \Phi_x(\{x_i\}) < m_i(x)\right) M_{\Phi^n}(\mathrm{d}x) \\
&= \int_{\mathbb{G}^n \times \mathbb{M}(\mathbb{G})} \mathbf{1}\{\exists i \in [1,n] : \mu(\{x_i\}) < m_i(x)\} \mathbf{P}_\Phi^x(\mathrm{d}\mu) M_{\Phi^n}(\mathrm{d}x) \\
&= \mathbf{E}\left[\int_{\mathbb{G}^n} \mathbf{1}\{\exists i \in [1,n] : \Phi(\{x_i\}) < m_i(x)\} \Phi^n(\mathrm{d}x)\right] = 0,
\end{aligned}$$

where the second equality follows from the higher order C-L-M theorem 3.3.2. Therefore, $\mathbf{P}(\exists i \in [1,n] : \Phi_x(\{x_i\}) < m_i(x)) = 0$ for M_{Φ^n}-almost all $x \in \mathbb{G}^n$. $\square$

3.3.2 Higher order reduced Palm

Lemma 3.3.5. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$, $n \in \mathbb{N}^*$, and $\Phi^{(n)}$ be the n-th factorial power of Φ. There exists a unique σ-finite measure $\mathcal{C}_\Phi^{(n)}$ on $\mathbb{G}^n \times \mathbb{M}(\mathbb{G})$ characterized by*

$$\mathcal{C}_\Phi^{(n)}(B \times L) = \mathbf{E}\left[\int_B \mathbf{1}\left\{\Phi - \sum_{i=1}^n \delta_{x_i} \in L\right\} \Phi^{(n)}(\mathrm{d}x)\right], \quad B \in \mathcal{B}(\mathbb{G})^{\otimes n}, L \in \mathcal{M}(\mathbb{G})$$

called the n-th reduced Campbell measure *associated to Φ.*

Proof. The proof relies on the same arguments as that of Lemma 3.3.1. $\square$

Note that, for $B \in \mathcal{B}(\mathbb{G})^{\otimes n}$,

$$\mathcal{C}_\Phi^{(n)}(B \times \mathbb{M}(\mathbb{G})) = \mathbf{E}\left[\Phi^{(n)}(B)\right] = M_{\Phi^{(n)}}(B).$$

Assume that $M_{\Phi^{(n)}}$ is σ-finite. Since $\mathcal{C}_\Phi^{(n)}$ is σ-finite and $(\mathbb{M}(\mathbb{G}), \mathcal{M}(\mathbb{G}))$ is Polish [52, §15.7.7], it follows from the measure disintegration theorem 14.D.10 that $\mathcal{C}_\Phi^{(n)}$ admits a disintegration

$$\mathcal{C}_\Phi^{(n)}(B \times L) = \int_B \mathbf{P}_\Phi^{!x}(L) M_{\Phi^{(n)}}(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{G})^{\otimes n}, L \in \mathcal{M}(\mathbb{G}), \tag{3.3.3}$$

where $\mathbf{P}_\Phi^{!\cdot}(\cdot)$ is a probability kernel from $\mathbb{G}^n$ to $\mathbb{M}(\mathbb{G})$. We call $\mathbf{P}_\Phi^{!x}(\cdot)$ the *n-th reduced Palm distribution* at $x \in \mathbb{G}^n$.

We may state the Campbell-Little-Mecke theorem in terms of the higher order reduced Palm distributions as follows (which extends Corollary 3.1.14 to higher order).

Theorem 3.3.6. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ such that $M_{\Phi^{(n)}}$ is σ-finite for some $n \in \mathbb{N}^*$. For all measurable $f : \mathbb{G}^n \times \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{R}}_+$, we have*

$$\mathbf{E}\left[\int_{\mathbb{G}^n} f\left(x, \Phi - \sum_{i=1}^{n} \delta_{x_i}\right) \Phi^{(n)}(\mathrm{d}x)\right] = \int_{\mathbb{G}^n \times \mathbb{M}(\mathbb{G})} f(x, \mu)\, \mathbf{P}_{\Phi}^{!x}(\mathrm{d}\mu)\, M_{\Phi^{(n)}}(\mathrm{d}x). \tag{3.3.4}$$

Proof. The proof follows the same lines as that of Theorem 3.3.2. □

Corollary 3.3.7. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ such that $M_{\Phi^{(n)}}$ is σ-finite for some $n \in \mathbb{N}^*$ and let $f : \mathbb{G}^n \times \mathbb{M}(\mathbb{G}) \to \bar{\mathbb{C}}$ be measurable. If either of the following conditions*

$$\mathbf{E}\left[\int_{\mathbb{G}^n} \left| f\left(x, \Phi - \sum_{i=1}^{n} \delta_{x_i}\right)\right| \Phi^{(n)}(\mathrm{d}x)\right] < \infty,$$

or

$$\int_{\mathbb{G}^n \times \mathbb{M}(\mathbb{G})} |f(x, \mu)|\, \mathbf{P}_{\Phi}^{!x}(\mathrm{d}\mu)\, M_{\Phi^{(n)}}(\mathrm{d}x) < \infty$$

holds, then the other one holds, and Equality (3.3.4) holds true.

Proof. The proof follows the same lines as that of Corollary 3.1.10. □

For $x \in \mathbb{G}^n$ let $\Phi_x^!$ be a random measure on $\mathbb{G}$ with probability distribution $\mathbf{P}_{\Phi}^{!x}(\cdot)$ in the probability space $(\Omega, \mathcal{A}, \mathbf{P})$; that is

$$\mathbf{P}\left(\Phi_x^! \in L\right) = \mathbf{P}_{\Phi}^{!x}(L), \quad L \in \mathcal{M}(\mathbb{G}).$$

We call $\Phi_x^!$ the *n-th reduced Palm version* of Φ at x.

Corollary 3.3.8. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ such that $M_{\Phi^{(n)}}$ is σ-finite for some $n \in \mathbb{N}^*$. Then for $M_{\Phi^{(n)}}$-almost all $x \in \mathbb{G}^n$ with pairwise distinct coordinates,*

$$\Phi_x^! \overset{\text{dist.}}{=} \Phi_x - \sum_{i=1}^{n} \delta_{x_i}.$$

Proof. Let A be the subset of $x \in \mathbb{G}^n$ with pairwise distinct coordinates. For any measurable $f : A \times \mathbb{M}(\mathbb{G}) \to \mathbb{R}_+$, we get from Theorem 3.3.2

$$\mathbf{E}\left[\int_A f(x, \Phi)\, \Phi^n(\mathrm{d}x)\right] = \int_A \mathbf{E}\left[f(x, \Phi_x)\right] M_{\Phi^n}(\mathrm{d}x)$$

and from Theorem 3.3.6

$$\mathbf{E}\left[\int_A f(x, \Phi)\, \Phi^{(n)}(\mathrm{d}x)\right] = \int_A \mathbf{E}\left[f\left(x, \Phi_x^! + \sum_{i=1}^{n} \delta_{x_i}\right)\right] M_{\Phi^{(n)}}(\mathrm{d}x).$$

Since on A, $\Phi^n(\mathrm{d}x) = \Phi^{(n)}(\mathrm{d}x)$ and $M_{\Phi^n}(\mathrm{d}x) = M_{\Phi^{(n)}}(\mathrm{d}x)$, we deduce that the right-hand sides of the above two equations are equal. Then for $M_{\Phi^{(n)}}$-almost all $x \in \mathbb{G}^n$ with pairwise distinct coordinates,

$$\mathbf{E}\left[f\left(x, \Phi_x\right)\right] = \mathbf{E}\left[f\left(x, \Phi_x^! + \sum_{i=1}^n \delta_{x_i}\right)\right].$$

This being true for all measurable $f : A \times \mathbb{M}(\mathbb{G}) \to \mathbb{R}_+$, it follows that

$$\Phi_x \overset{\text{dist.}}{=} \Phi_x^! + \sum_{i=1}^n \delta_{x_i}.$$

□

Proposition 3.3.9. Palm algebra. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ such that $M_{\Phi^{(n+m)}}$ is σ-finite for some $n, m \in \mathbb{N}^*$. Then for any $A \in \mathcal{B}(\mathbb{G})^{\otimes n}, B \in \mathcal{B}(\mathbb{G})^{\otimes m}$,*

$$M_{\Phi^{(n+m)}}(A \times B) = \int_A M_{\Phi_x^{!(m)}}(B)\, M_{\Phi^{(n)}}(\mathrm{d}x). \tag{3.3.5}$$

Moreover, for $M_{\Phi^{(n+m)}}$-almost all $(x, y) \in \mathbb{G}^n \times \mathbb{G}^m$,

$$\left(\Phi_x^!\right)_y^! \overset{\text{dist.}}{=} \Phi_{(x,y)}^!. \tag{3.3.6}$$

Proof. Let $A \in \mathcal{B}(\mathbb{G})^{\otimes n}, B \in \mathcal{B}(\mathbb{G})^{\otimes m}$. It follows from (14.E.7) that

$$\begin{aligned}
M_{\Phi^{(n+m)}}(A \times B) &= \mathbf{E}\left[\Phi^{(n+m)}(A \times B)\right] \\
&= \mathbf{E}\left[\int_A \left(\Phi - \sum_{i=1}^n \delta_{x_i}\right)^{(m)}(B)\, \Phi^{(n)}(\mathrm{d}x)\right] \\
&= \int_{A \times \mathbb{M}(\mathbb{G})} \mu^{(m)}(B)\, \mathbf{P}_\Phi^{!x}(\mathrm{d}\mu)\, M_{\Phi^{(n)}}(\mathrm{d}x) \\
&= \int_A \mathbf{E}\left[\Phi_x^{!(m)}(B)\right] M_{\Phi^{(n)}}(\mathrm{d}x) = \int_A M_{\Phi_x^{!(m)}}(B)\, M_{\Phi^{(n)}}(\mathrm{d}x),
\end{aligned}$$

where the third equality is due to Theorem 3.3.6 with $f(x, \mu) = \mathbf{1}\{x \in A\}\, \mu^{(m)}(B)$.

By definition of the reduced Campbell measure, for all $L \in \mathcal{M}(\mathbb{G})$, we have

$$\begin{aligned}
&\mathcal{C}_{\Phi}^{(n+m)}((A \times B) \times L) \\
&= \mathbf{E}\left[\int_{A\times B} \mathbf{1}\left\{\Phi - \sum_{i=1}^{n}\delta_{x_i} - \sum_{j=1}^{m}\delta_{y_j} \in L\right\}\Phi^{(n+m)}(\mathrm{d}x \times \mathrm{d}y)\right] \\
&= \mathbf{E}\left[\int_{A}\left(\int_{B} \mathbf{1}\left\{\Phi - \sum_{i=1}^{n}\delta_{x_i} - \sum_{j=1}^{m}\delta_{y_j} \in L\right\}\left(\Phi - \sum_{i=1}^{n}\delta_{x_i}\right)^{(m)}(\mathrm{d}y)\right)\Phi^{(n)}(\mathrm{d}x)\right] \\
&= \int_{A\times L}\left(\int_{B} \mathbf{1}\left\{\mu - \sum_{j=1}^{m}\delta_{y_j} \in L\right\}\mu^{(m)}(\mathrm{d}y)\,\mathbf{P}_{\Phi}^{!x}(\mathrm{d}\mu)\right) M_{\Phi^{(n)}}(\mathrm{d}x) \\
&= \int_{A} \mathbf{E}\left[\int_{B} \mathbf{1}\left\{\Phi_x^! - \sum_{j=1}^{m}\delta_{y_j} \in L\right\}\Phi_x^{!(m)}(\mathrm{d}y)\right] M_{\Phi^{(n)}}(\mathrm{d}x) \\
&= \int_{A}\int_{B} \mathbf{1}\{\mu \in L\}\,\mathbf{P}_{\Phi_x^!}^{!y}(\mathrm{d}\mu)\, M_{\Phi_x^{!(m)}}(\mathrm{d}y)\, M_{\Phi^{(n)}}(\mathrm{d}x) \\
&= \int_{A\times B} \mathbf{P}_{\Phi_x^!}^{!y}(L)\, M_{\Phi^{(n+m)}}(\mathrm{d}x \times \mathrm{d}y),
\end{aligned}$$

where we use (14.E.7) for the second equality. The third one is due to Theorem 3.3.6 with $f(x,\mu) = 1\{x \in A\}\int_B \mathbf{1}\left\{\mu - \sum_{j=1}^{m}\delta_{y_j} \in L\right\}\mu^{(m)}(\mathrm{d}y)$, the fifth one is also due to Theorem 3.3.6, and the last one is due to (3.3.5). Comparing the above equation with (3.3.3), we deduce that

$$\mathbf{P}_{\Phi}^{!(x,y)} = \mathbf{P}_{\Phi_x^!}^{!y},$$

or, equivalently,

$$\mathbf{P}_{\Phi^!_{(x,y)}} = \mathbf{P}_{(\Phi_x^!)_y^!}.$$

□

Theorem 3.3.10. *If Φ is a Poisson point process, then for M_{Φ}^{n}-almost all $x \in \mathbb{G}^n$,*

$$\Phi_x^! \stackrel{\text{dist.}}{=} \Phi.$$

Proof. Recall first that, by Proposition 2.3.25, $M_{\Phi^{(n)}} = M_{\Phi}^{n}$. We will prove the announced result by induction on n. For $n = 1$, the announced identity follows from Corollary 3.3.8 and Slivnyak's theorem 3.2.4. Assume now that the announced identity holds for some $n \geq 1$. We deduce from (3.3.6) that, for

M_Φ^{n+1}-almost all $x \in \mathbb{G}^{n+1}$,

$$\begin{aligned}\Phi_x^! &= \left(\Phi_{(x_1,\ldots,x_n)}^!\right)_{x_{n+1}}^! \\ &= \left(\Phi_{(x_1,\ldots,x_n)}^!\right)_{x_{n+1}} - \delta_{x_{n+1}} \\ &= \Phi_{x_{n+1}} - \delta_{x_{n+1}} = \Phi\end{aligned}$$

in distribution. □

3.4 Further examples

3.4.1 For Section 3.1

Hard-core models

The following examples allow one to model some repulsion between points (modelled as atoms of a point process) in such a way that they may not be too close to each other.

Example 3.4.1. Matérn I hard-core point process. *Let $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure Λ and let $h > 0$. Consider the point process consisting of points of Φ which have no neighbors at distance h; that is*

$$\Phi_1 = \sum_{k\in\mathbb{Z}} \delta_{X_k} \mathbf{1}\left\{\Phi\left(B\left(X_k, h\right)\right) = 1\right\},$$

where $B(x,r) = \left\{y \in \mathbb{R}^d : |y-x| < r\right\}$. The point process Φ_1 of isolated Poisson points is called a Matérn I hard-core point process*. The mean measure of Φ_1 is given by, for $A \in \mathcal{B}(\mathbb{G})$,*

$$\begin{aligned}\mathbf{E}\left[\Phi_1(A)\right] &= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\left\{x \in A, \Phi\left(B(x,h)\right) = 1\right\} \Phi(\mathrm{d}x)\right] \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{M}(\mathbb{R}^d)} \mathbf{1}\left\{x \in A, \mu\left(B(x,h)\right) = 1\right\} \mathbf{P}_\Phi^x(\mathrm{d}\mu)\,\Lambda(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{M}(\mathbb{R}^d)} \mathbf{1}\left\{x \in A, \mu\left(B(x,h)\right) = 1\right\} \mathbf{P}_{\Phi+\delta_x}(\mathrm{d}\mu)\,\Lambda(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{M}(\mathbb{R}^d)} \mathbf{1}\left\{x \in A, \mu\left(B(x,h)\right) = 0\right\} \mathbf{P}_\Phi(\mathrm{d}\mu)\,\Lambda(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathbf{1}\left\{x \in A\right\} \mathbf{P}\left(\Phi\left(B(x,h)\right) = 0\right)\Lambda(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathbf{1}\left\{x \in A\right\} e^{-\Lambda(B(x,h))}\Lambda(\mathrm{d}x),\end{aligned}$$

where the second equality is due to the C-L-M theorem 3.1.9 and the third one follows from Slivnyak's theorem 3.2.4. If Φ is homogeneous on $\mathbb{R}^d$ with intensity

λ, then we may continue the above equalities as follows

$$\mathbf{E}\left[\Phi_1(A)\right] = e^{-\Lambda(B(0,h))}\Lambda(A) = \lambda e^{-\lambda\kappa_d h^d}|A|, \tag{3.4.1}$$

where κ_d is the volume of the unit ball in $\mathbb{R}^d$. Thus the mean measure of Φ_1 equals the Lebesgue measure multiplied by the constant $\lambda_1 = \lambda e^{-\lambda\kappa_d h^d}$, which is the intensity of Φ_1.

Note that, for given $h > 0$, the intensity $\lambda_1 = \lambda e^{-\lambda\kappa_d h^d}$ of isolated Poisson points, as a function of λ, first increases to attain its maximum value

$$\max_{\lambda\in\mathbb{R}_+^*} \lambda_1 = 1/(e\kappa_d h^d), \tag{3.4.2}$$

for $\lambda = 1/(\kappa_d h^d)$ and then decreases to 0 as $\lambda \to \infty$.

Example 3.4.2. Matérn II hard-core point process. *We present a more efficient strategy of dependent thinning of Poisson process, which leads to a model called the Matérn II hard-core point process. Let $\Phi = \sum_{k\in\mathbb{Z}}\delta_{X_k}$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure Λ and let*

$$\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k,U_k)}$$

be an independently marked point process associated to Φ such that U_k is, conditionally on X_k, uniformly distributed in $[0,1]$. For a given $h > 0$, we define the Matérn II hard-core point process *as*

$$\Phi_2 := \sum_{k\in\mathbb{Z}} \delta_{X_k}\mathbf{1}\left\{U_k \le U_j, \forall j : X_j \in B(X_k,h)\right\}.$$

Its mean measure is given by, for $A \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned}
&\mathbf{E}\left[\Phi_2(A)\right] \\
&\quad= \mathbf{E}\left[\sum_{k\in\mathbb{Z}}\mathbf{1}\{X_k\in A\}\mathbf{1}\{U_k\le U_j,\forall j: X_j\in B(X_k,h)\}\right] \\
&\quad= \mathbf{E}\left[\int_{\mathbb{R}^d\times[0,1]}\mathbf{1}\{x\in A\}\mathbf{1}\left\{u\le V,\forall (Y,V)\in\tilde{\Phi}: Y\in B(x,h)\right\}\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}u)\right] \\
&\quad= \int_{\mathbb{R}^d}\mathbf{1}\{x\in A\}\int_0^1\int_{\mathbb{M}(\mathbb{R}^d)}\mathbf{1}\{u\le v,\forall (y,v)\in\mu: y\in B(x,h)\}\mathbf{P}_{\tilde{\Phi}}^{(x,u)}(\mathrm{d}\mu)\,\mathrm{d}u\Lambda(\mathrm{d}x) \\
&\quad= \int_{\mathbb{R}^d}\mathbf{1}\{x\in A\}\int_0^1\int_{\mathbb{M}(\mathbb{R}^d)}\mathbf{1}\{u\le v,\forall (y,v)\in\mu: y\in B(x,h)\}\mathbf{P}_{\tilde{\Phi}+\delta_{(x,u)}}(\mathrm{d}\mu)\,\mathrm{d}u\Lambda(\mathrm{d}x),
\end{aligned}$$

where the third equality is due to the C-L-M theorem 3.1.9 and Theorem 2.2.21 and the fourth one follows from Slivnyak's theorem 3.2.4. Note that for $x \in \mathbb{R}^d$

and $u \in (0,1)$*, we have*

$$\begin{aligned}
&\int_{\mathbb{M}(\mathbb{R}^d)} \mathbf{1}\{u \le v, \forall (y,v) \in \mu : y \in B(x,h)\} \mathbf{P}_{\tilde{\Phi}+\delta_{(x,u)}}(\mathrm{d}\mu) \\
&= \mathbf{E}\left[\mathbf{1}\left\{u \le V, \forall (Y,V) \in \tilde{\Phi} + \delta_{(x,u)} : Y \in B(x,h)\right\}\right] \\
&= \mathbf{E}\left[\mathbf{1}\left\{u \le V, \forall (Y,V) \in \tilde{\Phi} : Y \in B(x,h)\right\}\right] \\
&= \mathbf{P}\left[\tilde{\Phi}(B(x,h) \times [0,u]) = 0\right] \\
&= e^{-\Lambda(B(x,h))u}.
\end{aligned}$$

Then we may continue the calculation of the mean measure of the Matérn II hard-core point process as follows

$$\begin{aligned}
\mathbf{E}[\Phi_2(A)] &= \int_{\mathbb{R}^d} \mathbf{1}\{x \in A\} \int_0^1 e^{-\Lambda(B(x,h))u} \mathrm{d}u \Lambda(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \mathbf{1}\{x \in A\} \frac{1 - e^{-\Lambda(B(x,h))}}{\Lambda(B(x,h))} \Lambda(\mathrm{d}x).
\end{aligned}$$

It follows from this formula that the mean measure of the Matérn II hard-core point process is absolutely-continuous w.r.t. Λ *with Radon-Nikodym derivative*

$$\frac{1 - e^{-\Lambda(B(x,h))}}{\Lambda(B(x,h))}.$$

Since for all $a > 0$, $\exp(-a) \le \frac{1-\exp(-a)}{a}$*, it follows that the density of the Matérn II hard-core point process is always larger than that of the Matérn I hard-core point process of Example 3.4.1.*

If Φ *is homogeneous on* $\mathbb{R}^d$ *with intensity* λ*, then the mean measure of* Φ_2 *equals the Lebesgue measure multiplied by the constant*

$$\lambda_2 = \frac{1 - e^{-\lambda \kappa_d h^d}}{\kappa_d h^d}$$

which is the intensity of Φ_2*. Note that, for given* $h > 0$*, the intensity* λ_2 *is an increasing function of* λ *and*

$$\lim_{\lambda \to \infty} \lambda_2 = \sup_{\lambda \in \mathbb{R}_+^*} \lambda_2 = \frac{1}{\kappa_d h^d}. \tag{3.4.3}$$

Aloha and CSMA

Example 3.4.3. Transmission collision model on the plane. *Let* $\Phi = \sum_{i \in \mathbb{Z}} \delta_{X_i}$ *be a homogeneous Poisson point process on* $\mathbb{R}^2$ *of intensity* $\lambda \in \mathbb{R}_+^*$ *representing the locations of the transmitters. Each transmitter* X_i *serves a receiver located at* $y_i \in \mathbb{R}^2$*. There is no* collision *at* X_i *from* X_j *(*$j \neq i$*) if*

$$\frac{\textit{Power received at } y_i \textit{ from } X_i}{\textit{Power received at } y_i \textit{ from } X_j} \ge T_{\mathrm{c}},$$

for some constant $T_{\mathrm{c}} > 0$.

Assume the propagation model

$$P_{\mathrm{rec}} = P_{\mathrm{tr}} \times (Kr)^{-\beta}, \quad r \in \mathbb{R}_+, \tag{3.4.4}$$

where $K > 0, \beta > 2$ are two given constants, P_{tr} is the transmitted power, P_{rec} is the received power and r is the distance between the transmitter and the receiver.

Assume that the receiver y_i is very close to the corresponding transmitter X_i in such a way that the received power is approximately equal to the transmitted one. The interference at y_i from another transmitter X_j ($j \neq i$) is calculated by assuming the above propagation model and by approximating $|y_i - X_j|$ by $|X_i - X_j|$. Then the condition to avoid collision becomes

$$\frac{P_{\mathrm{tr}}}{P_{\mathrm{tr}} \times (K\,|X_i - X_j|)^{-\beta}} \geq T_{\mathrm{c}} \quad \text{which is equivalent to} \quad |X_i - X_j| \geq r := \frac{T_{\mathrm{c}}^{1/\beta}}{K}.$$

We say that the transmission of X_i to y_i is successful *if $B(X_i, r)$ does not contain any other transmitter X_j ($j \neq i$). It follows from Example 3.4.1 that the density of successful transmissions equals*

$$\sigma = \lambda \times e^{-\lambda \pi r^2},$$

which may be seen as the product of the density of transmitters λ by the probability that a transmission is successful $e^{-\lambda\pi r^2}$.

Maximizing the above quantity, we get the optimal density of transmitters

$$\lambda_{\mathrm{opt}} = \frac{1}{\pi r^2} = \frac{K^2}{\pi T_{\mathrm{c}}^{2/\beta}} \tag{3.4.5}$$

and the corresponding density of successful transmissions equals

$$\sigma_{\mathrm{opt}} = \lambda_{\mathrm{opt}} \times e^{-\lambda_{\mathrm{opt}}\pi r^2} = \frac{1}{e\pi r^2} = \frac{K^2}{e\pi T_{\mathrm{c}}^{2/\beta}}.$$

Example 3.4.4. Spatial Aloha with collision model. *Let $\Phi = \sum_{i\in\mathbb{Z}} \delta_{X_i}$ be a homogeneous Poisson point process on $\mathbb{R}^2$ of intensity $\lambda \in \mathbb{R}_+^*$ representing the locations of the transmitters also called* nodes. *Suppose the density of nodes λ is fixed. The idea is then to authorize (at a given time) only some fraction of nodes to transmit. At a given timeslot, each node decides to transmit with probability p, or to be silent with probability $1-p$, independently from the other nodes. This protocol, which is called* Aloha, *is mathematically equivalent to a* thinning *of the point process. By Corollary 2.2.7, the transmitting nodes form a homogeneous Poisson point process on $\mathbb{R}^2$ of intensity λp. Assuming the collision model described in Example 3.4.3, the optimal density of the transmitting nodes is deduced from (3.4.5)*

$$\lambda p_{\mathrm{opt}} = \frac{1}{\pi r^2}, \quad \text{which implies} \quad p_{\mathrm{opt}} = \max\left(1, \frac{1}{\lambda\pi r^2}\right).$$

Aloha	*Replication number*
Successful transmission density $\sigma = \lambda p \times e^{-\lambda p \pi r^2}$	Authorized transmissions density $\sigma = \frac{1-e^{-\lambda \pi r^2}}{\pi r^2}$
σ is maximal when $\lambda p_{\mathrm{opt}} = \frac{1}{\pi r^2}$	σ increases with λ
Optimal density $\sigma_{\mathrm{opt}} = \frac{1}{e \pi r^2}$	Supremum density $\sigma_{\sup} = \frac{1}{\pi r^2}$

Table 3.1: Comparison of Aloha and CSMA.

Example 3.4.5. Carrier sense multiple access (CSMA). *Let $\Phi = \sum_{i \in \mathbb{Z}} \delta_{X_i}$ be a homogeneous Poisson point process on $\mathbb{R}^2$ of intensity $\lambda \in \mathbb{R}_+^*$ representing the locations of the transmitters also called* nodes*. Consider the collision model described in Example 3.4.3, where receiver y_i is assumed very close to the corresponding transmitter X_i and where the transmission of X_i to y_i is successful if $B(X_i, r)$ does not contain any other transmitter X_j $(j \neq i)$. In order to avoid collisions of transmissions, we give independent marks U_i uniformly distributed in $[0,1]$ to points $X_i \in \Phi$. Transmission at X_i takes place if*

$$U_i \leq U_j, \forall j : X_j \in B(X_i, r).$$

The transmitting nodes constitute a point process

$$\Phi_2 := \sum_{i \in \mathbb{Z}} \delta_{X_i} \mathbf{1}\{U_i \leq U_j, \forall j : X_j \in B(X_i, r)\}.$$

We recognize the Matérn II hard-core Model studied in Example 3.4.2. In particular, we deduce that

$$\mathbf{E}[\Phi_2(A)] = \lambda |A| \frac{1 - e^{-\lambda \pi r^2}}{\lambda \pi r^2} = |A| \frac{1 - e^{-\lambda \pi r^2}}{\pi r^2}.$$

Then the density of the authorized transmissions equals

$$\sigma = \frac{1 - e^{-\lambda \pi r^2}}{\pi r^2},$$

which converges to $\sigma_{\sup} = \frac{1}{\pi r^2}$ when $\lambda \to \infty$. Table 3.1 summarizes the results of CSMA compared to those of Aloha. Note that $\sigma_{\sup}$ for CSMA is larger than σ_{opt} of Aloha with a ratio equal to $e \simeq 2.72$. Note moreover that CSMA is better than optimal Aloha when $\frac{1-e^{-\lambda \pi r^2}}{\pi r^2} > \frac{1}{e \pi r^2}$ which is equivalent to

$$\lambda > \frac{-\log\left(1 - e^{-1}\right)}{\pi r^2} \simeq \frac{0.46}{\pi r^2}.$$

3.4.2 For Section 3.3

Example 3.4.6. SINR in wireless networks *[14]. On $\mathbb{R}^2$, we model the base stations (BS) with a homogeneous Poisson point process Φ with intensity $\lambda > 0$.*

We define the signal to interference and noise ratio *(SINR) of the typical user located at the origin with respect to the station $X \in \Phi$ by*

$$\mathrm{SINR}\left(X,\Phi\right)=\frac{\mathbf{P}/l\left(|X|\right)}{N+\mathbf{P}\sum_{Z\in\Phi\setminus\{X\}}1/l\left(|Z|\right)},\quad X\in\Phi,$$

where $\mathbf{P}\in\mathbb{R}_+$ is the power transmitted by each BS, $N\in\mathbb{R}_+$ is the noise power, and $l:\mathbb{R}_+\to\mathbb{R}_+$ is a measurable function representing the propagation loss due to distance. The user is served by the BS with the highest received power; or, equivalently, largest SINR. We are interested in the probability distribution of the SINR of the typical user with his serving BS; that is

$$\begin{aligned}\mathbf{P}_{\mathrm{c}}\left(T\right)&=\mathbf{P}\left(\inf_{X\in\Phi}\mathrm{SINR}\left(X,\Phi\right)>T\right)\\&=\mathbf{P}\left(\bigcup_{X\in\Phi}\left\{\mathrm{SINR}\left(X,\Phi\right)>T\right\}\right),\end{aligned}$$

where $T\in\mathbb{R}_+$. Using Poincaré's formula $\mathbf{P}(\bigcup_i A_i)=\sum_i\mathbf{P}(A_i)-\sum_{i\neq j}\mathbf{P}(A_i\cap A_j)+\sum_{i\neq j\neq k}\mathbf{P}(A_i\cap A_j\cap A_k)-\cdots$ *where $i\neq j\neq k$ means that i,j and k are pairwise different, it follows that*

$$\mathbf{P}_{\mathrm{c}}\left(T\right)=\sum_{n=1}^{\infty}\left(-1\right)^n S_n\left(T\right),$$

where

$$\begin{aligned}S_n\left(T\right)&=\mathbf{E}\left[\sum_{(X_1,\ldots,X_n)\in\Phi^{(n)}}\mathbf{1}\left(\bigcap_{i=1}^{n}\left\{\mathrm{SINR}\left(X_i,\Phi\right)>T\right\}\right)\right]\\&=\frac{1}{n!}\mathbf{E}\left[\int_{(\mathbb{R}^2)^n}f\left(x,\Phi-\sum_{i=1}^{n}\delta_{x_i}\right)\Phi^{(n)}\left(\mathrm{d}x\right)\right],\end{aligned}$$

where

$$f\left(x,\mu\right)=\mathbf{1}\left(\bigcap_{i=1}^{n}\left\{\mathrm{SINR}\left(x_i,\mu+\sum_{j=1}^{n}\delta_{x_j}\right)>T\right\}\right).$$

By the Campbell-Little-Mecke theorem 3.3.6, we get

$$S_n\left(T\right)=\frac{1}{n!}\int_{(\mathbb{R}^2)^n\times\mathbb{M}(\mathbb{R}^2)}f\left(x,\mu\right)\mathbf{P}_{\Phi}^{!x}\left(\mathrm{d}\mu\right)M_{\Phi^{(n)}}\left(\mathrm{d}x\right).$$

Moreover, by Proposition 2.3.25 $M_{\Phi^{(n)}}\left(\mathrm{d}x\right)=\lambda^n\mathrm{d}x$ and by Slivnyak's theorem 3.2.4 $\mathbf{P}_{\Phi}^{!x}\left(\mathrm{d}\mu\right)=\mathbf{P}_{\Phi}\left(\mathrm{d}\mu\right)$, then

$$\begin{aligned}S_n\left(T\right)&=\frac{\lambda^n}{n!}\int_{(\mathbb{R}^2)^n}\mathbf{E}\left[f\left(x,\Phi\right)\right]\mathrm{d}x\\&=\frac{\lambda^n}{n!}\int_{(\mathbb{R}^2)^n}\mathbf{P}\left(\bigcap_{i=1}^{n}\left\{\mathrm{SINR}'\left(x_i,\Phi\right)>T'\right\}\right)\mathrm{d}x,\end{aligned}$$

where the last equality is obtained by simple algebraic manipulations with $T' = \frac{T}{1+T}$ *and*

$$\mathrm{SINR}'(x_i, \Phi) = \frac{1/l(|x|)}{N/\mathbf{P} + \sum_{Y\in\Phi} 1/l(|Y|) + \sum_{j=1}^{n} 1/l(|x_j|)}.$$

Cf. [14] *for the continuation of the calculations to get an explicit expression of* $S_n(T)$*, and therefore for* $\mathbf{P}_{\mathrm{c}}(T)$*.*

3.5 Exercises

3.5.1 For Section 3.1

Exercise 3.5.1. *Let* X *be an integer-valued random variable with positive expectation,* μ *be a counting measure on a l.c.s.h. space* $\mathbb{G}$ *and define a point process* $\Phi = X\mu$*. Let* A *be the support of* μ*.*

1. *Show that for any* $x \in A$*, a Palm version of* Φ *at* x *is* $\Phi_x := Y\mu,$*, where* Y *has the size biased distribution of* X*; that is*

$$\mathbf{P}(Y = k) = \frac{k\mathbf{P}(X = k)}{\mathbf{E}[X]}, \quad k \in \mathbb{N}. \tag{3.5.1}$$

2. *In particular, if* X *is a Poisson random variable of mean* λ*, show that* $Y = X + 1$*.*

3. *Show that for any* $x \in A$*,*

$$\mathbf{P}(\Phi = k\mu \mid x \in \Phi) = \mathbf{P}(X = k \mid X > 0).$$

Observe that in general this last conditional law of X *given* $X > 0$ *is not equal to the size biased law of* X *in* (3.5.1)*. It follows that the Palm distributions of a point process may not be seen in general as the conditional distributions of* Φ*, given it has an atom at* x*.*

Solution 3.5.1. *1. Clearly* $M_\Phi = \mathbf{E}[X]\mu$*. Moreover, for* $L = \{k\mu : k \in \mathbb{N}\}$*,*

$$\begin{aligned}\mathcal{C}_\Phi(B \times L) &= \mathbf{E}[X\mu(B)\mathbf{1}\{X = k\}]\\ &= \mu(B)\,k\mathbf{P}(X = k)\\ &= \frac{k\mathbf{P}(X = k)}{\mathbf{E}[X]}M_\Phi(B).\end{aligned}$$

Letting A *be the support of* μ*, we deduce from the above equality that, for any* $x \in A$*,*

$$\mathbf{P}(\Phi_x = k\mu) = \mathbf{P}_\Phi^x(L) = \frac{k\mathbf{P}(X = k)}{\mathbf{E}[X]}.$$

Therefore, $\Phi_x = Y\mu$, *where*

$$\mathbf{P}(Y = k) = \frac{k\mathbf{P}(X = k)}{\mathbf{E}[X]}.$$

2. In particular, if X *is a Poisson random variable of mean* λ, *then*

$$\mathbf{P}(Y = k) = e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}, \quad k \in \mathbb{N}^*,$$

i.e., $Y = X + 1$.

3. For any $x \in A$,

$$\begin{aligned}\mathbf{P}(\Phi = k\mu \mid x \in \Phi) &= \mathbf{P}(X\mu = k\mu \mid x \in X\mu) \\ &= \mathbf{P}(X = k \mid X > 0),\end{aligned}$$

which is in general different from (3.5.1).

Exercise 3.5.2. *Let* Y *be a nonnegative random variable and let* Φ *be a Cox point process on* $\mathbb{R}$ *directed by* $\Lambda(\mathrm{d}x) = Y \times \mathrm{d}x$. *In the canonical probability space associated to* Φ, *show that the cumulative distribution function of* Y *under the Palm distribution* $\mathbf{P}_\Phi^x$ *at* x *equals*

$$\mathbf{P}_\Phi^x(Y \le y) = \frac{\mathbf{E}[Y \times \mathbf{1}\{Y \le y\}]}{\mathbf{E}[Y]}.$$

Solution 3.5.2. *It follows from (2.3.1) that the mean measure of* Φ *is*

$$M_\Phi(B) = \mathbf{E}[Y] \times |B|.$$

Let $\mathcal{C}_\Phi$ *be the Campbell measure associated to a point process* Φ. *Then, for* $L = \mathbf{1}\{Y \le y\}, B \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned}\mathcal{C}_\Phi(B \times L) &= \mathbf{E}[\Phi(B)\mathbf{1}\{\Phi \in L\}] \\ &= \mathbf{E}[\Phi(B)\mathbf{1}\{Y \le y\}] \\ &= \mathbf{E}[\mathbf{E}[\Phi(B)\mathbf{1}\{Y \le y\} \mid Y]] \\ &= \mathbf{E}[Y \times \mathbf{1}\{Y \le y\}] \times |B|.\end{aligned}$$

Thus

$$\mathbf{P}_\Phi^x(Y \le y) = \frac{\mathcal{C}_\Phi(B \times L)}{M_\Phi(B)} = \frac{\mathbf{E}[Y \times \mathbf{1}\{Y \le y\}]}{\mathbf{E}[Y]}.$$

Exercise 3.5.3. Discrete analogue of Slivnyak's theorem. *Let* $\lambda > 0$ *and* N *be a random variable taking values in* $\{0, 1, \ldots\}$. *Show that*

$$\mathbf{E}[Nf(N)] = \lambda\mathbf{E}[f(N+1)],$$

for all (say nonnegative) functions f *if and only if* N *is a Poisson random variable of parameter* λ. *Enough to consider* $f(N) = \mathbf{1}\{N = k\}$ *for* $k \ge 1$.

Solution 3.5.3. *Let* $f(N) = \mathbf{1}\{N = k\}$ *for some* $k \geq 1$*. Observe that*

$$\mathbf{E}[Nf(N)] = \mathbf{E}[N\mathbf{1}\{N = k\}] = k\mathbf{P}(N = k)$$

and

$$\begin{aligned}\mathbf{E}[f(N+1)] &= \mathbf{E}[\mathbf{1}\{N+1 = k\}] \\ &= \mathbf{P}(N+1 = k) = \mathbf{P}(N = k-1).\end{aligned}$$

Then $\mathbf{E}[Nf(N)] = \lambda\mathbf{E}[f(N+1)]$ *iff*

$$\mathbf{P}(N = k) = \frac{\lambda}{k}\mathbf{P}(N = k-1).$$

Sufficiency is obvious since $\mathbf{P}(N = k) = \frac{\lambda^k}{k!}e^{-\lambda}$*. For necessity, note that the above equation implies by induction that*

$$\mathbf{P}(N = k) = \frac{\lambda^k}{k!}\mathbf{P}(N = 0).$$

Since $\sum_{k=0}^{\infty}\mathbf{P}(N = k) = 1$ *we deduce that* $\mathbf{P}(N = 0) = e^{-\lambda}$*. Then* N *is a Poisson random variable of parameter* λ*.*

Exercise 3.5.4. *Let* Φ *be a Poisson point process on* $\mathbb{R}^d$ *with intensity measure* Λ*. Assume that* Λ *has a density with respect to the Lebesgue measure. Show that,* $\mathbf{P}$*-almost surely,* Φ *has no two points equidistant from* 0*.*

Solution 3.5.4. *Observe that*

$$\begin{aligned}\mathbf{P}(\{\exists X \neq Y \in \Phi : |X| = |Y|\}) &\leq \mathbf{E}[\mathrm{card}(\{X \neq Y \in \Phi : |X| = |Y|\})] \\ &= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\{\exists Y \in \Phi, Y \neq x : |x| = |Y|\}\Phi(\mathrm{d}x)\right] \\ &\leq \mathbf{E}\left[\int_{\mathbb{R}^d} \Phi(\{y \neq x : |x| = |y|\})\Phi(\mathrm{d}x)\right].\end{aligned}$$

Using the Campbell-Little-Mecke formula (3.1.8) and then using Slivnyak's theorem 3.2.4 we get

$$\begin{aligned}\mathbf{E}\left[\int_{\mathbb{R}^d} \Phi(\{y \neq x : |x| = |y|\})\Phi(\mathrm{d}x)\right] &= \int_{\mathbb{R}^d} \mathbf{E}^x[\Phi(\{y \neq x : |x| = |y|\})]\Lambda(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathbf{E}[(\Phi + \delta_x)(\{y \neq x : |x| = |y|\})]\Lambda(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \Lambda(\{y \neq x : |x| = |y|\})\Lambda(\mathrm{d}x) = 0,\end{aligned}$$

where the last equality follows from the fact that Λ *has a density with respect to the Lebesgue measure which implies that the function to integrate vanishes.*

Exercise 3.5.5. Nearest neighbor for Poisson point processes. *Consider a Poisson point process Φ of intensity measure $\Lambda(\mathrm{d}x)$ on $\mathbb{R}^d$.*

1. *Under Palm probability $\mathbf{P}^x$ of Φ, what is the distribution of the distance between the point $x \in \Phi$ and its nearest neighbor in Φ? Comment on the case when Λ has an atom at x.*
2. *What is the difference between the distribution of Φ under $\mathbf{P}^x$ and the distribution of Poisson point process Φ' of intensity $\Lambda + \delta_x$.*

Solution 3.5.5. *1. By Slivnyak's theorem 3.2.4*

$$\mathbf{P}^x(\Phi \in \Gamma) = \mathbf{P}(\Phi + \delta_x \in \Gamma).$$

Then

$$\begin{aligned}\mathbf{P}^x\left(\inf_{X \in \Phi \setminus \{x\}} |x - X| \geq t\right) &= \mathbf{P}\left(\inf_{X \in (\Phi + \delta_x) \setminus \{x\}} |x - X| \geq t\right)\\ &= \mathbf{P}\left(\inf_{X \in \Phi \setminus \{x\}} |x - X| \geq t\right)\\ &= \mathbf{P}(\Phi(B(x,t) \setminus \{x\}) = 0) = e^{-\Lambda(B(x,t)\setminus\{x\})}.\end{aligned}$$

When Λ has an atom at x, we should stress that we consider the nearest neighbor of x in Φ other than x itself.

2. The distribution of Φ under $\mathbf{P}^x$ equals that of $\Phi + \delta_x$. The distribution of a Poisson point process Φ' of intensity $\Lambda + \delta_x$ equals that of $\Phi + N\delta_x$ where N is a Poisson random variable of mean 1.

Exercise 3.5.6. *Consider a Poisson point process$\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ of intensity measure Λ on $\mathbb{R}^d$.*

1. *Using the Campbell-Little-Mecke and Slivnyak's theorems, prove the following equations*

$$\begin{aligned}\mathbf{E}\left[\sum_{X_i, X_j \in \Phi} f(X_i, X_j, \Phi)\right] &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{E}\left[f(x, y, \Phi + \delta_x + \delta_y)\right]\Lambda(\mathrm{d}x)\Lambda(\mathrm{d}y)\\ &\quad + \int_{\mathbb{R}^d} \mathbf{E}\left[f(x, x, \Phi + \delta_x)\right]\Lambda(\mathrm{d}x),\end{aligned} \tag{3.5.2}$$

$$\mathbf{E}\left[\sum_{X_i \neq X_j \in \Phi} f(X_i, X_j, \Phi)\right] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}\{x \neq y\}\,\mathbf{E}\left[f(x, y, \Phi + \delta_x + \delta_y)\right]\Lambda(\mathrm{d}x)\Lambda(\mathrm{d}y). \tag{3.5.3}$$

2. *Using (3.5.2) calculate the second moment $\mathbf{E}[I_\epsilon^2]$ and variance $\mathrm{Var}(I_\epsilon)$ of the total received power I_ϵ defined in Exercise 2.7.14. Consider also random i.i.d. transmitted powers independent of Φ.*

Solution 3.5.6. *1. Let $\mathbb{G}=\mathbb{R}^d$ and observe that*

$$\sum_{X_i,X_j\in\Phi} f(X_i,X_j,\Phi)=\int_{\mathbb{G}} g\left(x,\Phi\right)\Phi\left(\mathrm{d}x\right),$$

where

$$g\left(x,\Phi\right)=\sum_{X_j\in\Phi} f(x,X_j,\Phi).$$

By the Campbell-Little-Mecke formula (3.1.8), we have

$$\mathbf{E}\left[\int_{\mathbb{G}} g\left(x,\Phi\right)\Phi\left(\mathrm{d}x\right)\right]=\int_{\mathbb{G}}\mathbf{E}^x\left[g\left(x,\Phi\right)\right]\Lambda\left(\mathrm{d}x\right).$$

By Slivnyak's theorem 3.2.4

$$\begin{aligned}
\mathbf{E}^x\left[g\left(x,\Phi\right)\right]&=\mathbf{E}\left[g\left(x,\Phi+\delta_x\right)\right]\\
&=\mathbf{E}\left[\sum_{X_j\in\Phi+\delta_x} f(x,X_j,\Phi+\delta_x)\right]\\
&=\mathbf{E}\left[f(x,x,\Phi+\delta_x)+\sum_{X_j\in\Phi} f(x,X_j,\Phi+\delta_x)\right]\\
&=\mathbf{E}\left[f(x,x,\Phi+\delta_x)\right]+\mathbf{E}\left[\int_{\mathbb{G}} f(x,y,\Phi+\delta_x)\Phi\left(\mathrm{d}y\right)\right]\\
&=\mathbf{E}\left[f(x,x,\Phi+\delta_x)\right]+\int_{\mathbb{G}}\mathbf{E}^y\left[f(x,y,\Phi+\delta_x)\right]\Lambda\left(\mathrm{d}y\right)\\
&=\mathbf{E}\left[f(x,x,\Phi+\delta_x)\right]+\int_{\mathbb{G}}\mathbf{E}\left[f(x,y,\Phi+\delta_x+\delta_y)\right]\Lambda\left(\mathrm{d}y\right).
\end{aligned}$$

Then the announced equality follows.

2. Consider first the case of a deterministic transmitted power. Recall that

$$I_\epsilon=\sum_{n\in\mathbb{Z}} P_{\mathrm{tr}}(K\left|X_n-y\right|)^{-\beta}\mathbf{1}\left\{\left|X_n-y\right|>\epsilon\right\}=\int_{\mathbb{R}^2} f\left(x\right)\Phi\left(\mathrm{d}x\right),$$

where $f\left(x\right)=P_{\mathrm{tr}}(K\left|x-y\right|)^{-\beta}\mathbf{1}\left\{\left|x-y\right|>\epsilon\right\}$. Then

$$\begin{aligned}
\mathbf{E}[I_\epsilon^2]&=\mathbf{E}\left[\left(\int_{\mathbb{R}^2} f\left(x\right)\Phi\left(\mathrm{d}x\right)\right)^2\right]\\
&=\mathbf{E}\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\left(x\right)f\left(y\right)\Phi\left(\mathrm{d}x\right)\Phi\left(\mathrm{d}y\right)\right]\\
&=\lambda^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\left(x\right)f\left(y\right)\mathrm{d}x\mathrm{d}y+\lambda\int_{\mathbb{R}^d} f\left(x\right)^2\mathrm{d}x\\
&=\mathbf{E}[I_\epsilon]^2+\lambda\int_{\mathbb{R}^d} f\left(x\right)^2\mathrm{d}x.
\end{aligned}$$

Thus

$$\mathrm{Var}(I_\epsilon) = \lambda \int_{\mathbb{R}^d} f(x)^2 \,\mathrm{d}x = \frac{2\pi\lambda P_{\mathrm{tr}}^2}{(2\beta-2)\,\epsilon^{2\beta-2}K^{2\beta}},$$

where the second equality follows by analogy to (2.7.3) with P_{tr} replaced by P_{tr}^2 and β by 2β.

Consider now the case of random i.i.d. transmitted powers independent of Φ. Observe that

$$\tilde{I}_\epsilon = \sum_{n\in\mathbb{Z}} P_n(K\,|X_n-y|)^{-\beta}\mathbf{1}\{|X_n-y|>\epsilon\} = \int_{\mathbb{R}^2} p\times \tilde{f}(x)\,\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}p),$$

where $\tilde{f}(x) = (K\,|x-y|)^{-\beta}\mathbf{1}\{|x-y|>\epsilon\}$ and

$$\tilde{\Phi} := \sum_{n\in\mathbb{Z}} \delta_{(X_n,P_n)},$$

which is a Poisson point process with intensity measure

$$\tilde{\Lambda}(\mathrm{d}x\times\mathrm{d}p) = \lambda\mathrm{d}x\times\mathbf{P}_{P_1}(\mathrm{d}p).$$

Then

$$\begin{aligned}\mathbf{E}[\tilde{I}_\epsilon] &= \lambda\int_{\mathbb{R}^d\times\mathbb{R}_+} p\tilde{f}(x)\,\mathrm{d}x\mathbf{P}_{P_1}(\mathrm{d}p)\\ &= \lambda\mathbf{E}[P_1]\int_{\mathbb{R}^d}\tilde{f}(x)\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}[\tilde{I}_\epsilon^2] &= \mathbf{E}\left[\left(\int_{\mathbb{R}^2\times\mathbb{R}_+} p\tilde{f}(x)\,\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}p)\right)^2\right]\\ &= \mathbf{E}\left[\int_{\mathbb{R}^2\times\mathbb{R}_+}\int_{\mathbb{R}^2\times\mathbb{R}_+} p\tilde{f}(x)\,q\tilde{f}(y)\,\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}p)\,\tilde{\Phi}(\mathrm{d}y\times\mathrm{d}q)\right]\\ &= \lambda^2\int_{\mathbb{R}^d\times\mathbb{R}_+}\int_{\mathbb{R}^d\times\mathbb{R}_+} p\tilde{f}(x)\,q\tilde{f}(y)\,\mathrm{d}x\mathrm{d}y\mathbf{P}_{P_1}(\mathrm{d}p)\,\mathbf{P}_{P_1}(\mathrm{d}q)\\ &\quad+\lambda\int_{\mathbb{R}^d\times\mathbb{R}_+}\left(p\tilde{f}(x)\right)^2\mathrm{d}x\mathbf{P}_{P_1}(\mathrm{d}p)\\ &= \lambda^2\mathbf{E}[P_1]^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{f}(x)\,\tilde{f}(y)\,\mathrm{d}x\mathrm{d}y + \lambda\mathbf{E}\left[P_1^2\right]\int_{\mathbb{R}^d}\tilde{f}(x)^2\,\mathrm{d}x\\ &= \mathbf{E}[\tilde{I}_\epsilon]^2 + \lambda\mathbf{E}\left[P_1^2\right]\int_{\mathbb{R}^d}\tilde{f}(x)^2\,\mathrm{d}x.\end{aligned}$$

Thus

$$\mathrm{Var}(\tilde{I}_\epsilon) = \lambda\mathbf{E}\left[P_1^2\right]\int_{\mathbb{R}^d}\tilde{f}(x)^2\,\mathrm{d}x = \frac{2\pi\lambda\mathbf{E}\left[P_1^2\right]}{(2\beta-2)\,\epsilon^{2\beta-2}K^{2\beta}}.$$

Exercise 3.5.7. Mixed Palm version. *Let Φ be a point process on $\mathbb{R}^d$ and $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$. Let g and h be two nonnegative measurable functions defined respectively on $\mathbb{R}^d$ and $\mathbb{M}\left(\mathbb{R}^d\right)$ such that g vanishes outside B and $0 < M_\Phi(g) < \infty$. Let X be a random variable on $\mathbb{R}^d$ such that, given Φ, X is chosen uniformly among the point of Φ in B when $\Phi(B) > 0$ (X is arbitrary if $\Phi(B) = 0$).*

1. *Show that*

$$\mathbf{E}\left[g(X)\, h(\Phi)\, \mathbf{1}\{\Phi(B) \geq 1\}\right] = M_\Phi(g)\ \mathbf{E}\left[\frac{h(\Phi_g)\, \mathbf{1}\{\Phi_g(B) \geq 1\}}{\Phi_g(B)}\right],$$

where Φ_g is the mixed Palm version of Φ with respect to g (cf. Definition 3.1.15).

2. *Taking $g \equiv 1$ and $h \equiv 1$, we get*

$$\mathbf{P}(\Phi(B) \geq 1) = M_\Phi(B)\ \mathbf{E}\left[\frac{\mathbf{1}\{\Phi_g(B) \geq 1\}}{\Phi_g(B)}\right].$$

Check the above equality from the very definition of Φ_g; i.e., from Equation (3.1.5).

Solution 3.5.7. *1. Observe that*

$$\begin{aligned}
\mathbf{E}\left[g(X)\, h(\Phi)\, \mathbf{1}\{\Phi(B) \geq 1\}\right] &= \mathbf{E}\left[\frac{1}{\Phi(B)} \sum_{n\in\mathbb{Z}} g(X_n)\, h(\Phi)\, \mathbf{1}\{\Phi(B) \geq 1\}\right] \\
&= \mathbf{E}\left[\int_B g(x)\, \frac{h(\Phi)\, \mathbf{1}\{\Phi(B) \geq 1\}}{\Phi(B)} \Phi(\mathrm{d}x)\right] \\
&= \int_B g(x)\, \mathbf{E}\left[\frac{h(\Phi_x)\, \mathbf{1}\{\Phi_x(B) \geq 1\}}{\Phi_x(B)}\right] \Phi(\mathrm{d}x) \\
&= M_\Phi(g)\ \mathbf{E}\left[\frac{h(\Phi_g)\, \mathbf{1}\{\Phi_g(B) \geq 1\}}{\Phi_g(B)}\right],
\end{aligned}$$

where the third equality follows from the Campbell-Little-Mecke formula (3.1.3) and the fourth one is due to (3.1.5).

2. Applying (3.1.5) with $g \equiv 1$, we get, for any $L \in \bar{\mathcal{M}}(\mathbb{G})$,

$$\begin{aligned}
\mathbf{P}^g(L) &= \frac{1}{M_\Phi(g)} \int_B \mathbf{P}_\Phi^x(L)\, M_\Phi(\mathrm{d}x) \\
&= \frac{1}{M_\Phi(g)} \mathcal{C}_\Phi(B \times L) \\
&= \frac{1}{M_\Phi(g)} \mathbf{E}\left[\Phi(B)\, \mathbf{1}\{\Phi \in L\}\right].
\end{aligned}$$

where the second equality is due to (3.1.2) and the third one follows from (3.1.1). Then

$$\begin{aligned}
\mathbf{E}\left[\frac{\mathbf{1}\{\Phi_g(B) \geq 1\}}{\Phi_g(B)}\right] &= \frac{1}{M_\Phi(g)} \mathbf{E}\left[\Phi(B)\, \frac{\mathbf{1}\{\Phi(B) \geq 1\}}{\Phi(B)}\right] \\
&= \frac{1}{M_\Phi(g)} \mathbf{P}(\Phi(B) \geq 1).
\end{aligned}$$

Chapter 4

Transforms and moment measures

The framework and the notation of this chapter are those of Section 1.1.

4.1 Characteristic function

For any measure μ on $\mathbb{G}$, recall the notation $L^1_{\mathbb{R}}(\mu, \mathbb{G})$ for the set of measurable functions $f : \mathbb{G} \to \mathbb{R}$ which are integrable with respect to μ.

Definition 4.1.1. *The* characteristic function *of a random measure Φ on a l.c.s.h. space $\mathbb{G}$, denoted by Ψ_Φ, is defined for all $f \in L^1_{\mathbb{R}}(M_\Phi, \mathbb{G})$ by*

$$\Psi_\Phi(f) = \mathbf{E}\left[\exp\left(\boldsymbol{i}\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right],$$

where $\boldsymbol{i}$ is the imaginary unit complex number.

Lemma 4.1.2. *Let Φ be a random measure on a l.c.s.h space $\mathbb{G}$. Its characteristic function $\Psi_\Phi : L^1_{\mathbb{R}}(M_\Phi, \mathbb{G}) \to \mathbb{C}$ is continuous.*

Proof. Cf. [78, p.12]. Recall that for all $x, y \in \mathbb{R}$, $\left|e^{\boldsymbol{i}x} - e^{\boldsymbol{i}y}\right| \leq |x - y|$. Then, for all $f, g \in L^1_{\mathbb{R}}(M_\Phi, \mathbb{G})$,

$$\begin{aligned}
|\Psi_\Phi(f) - \Psi_\Phi(g)| &\leq \mathbf{E}\left[\left|\exp\left(\boldsymbol{i}\int_{\mathbb{G}} f\mathrm{d}\Phi\right) - \exp\left(\boldsymbol{i}\int_{\mathbb{G}} g\mathrm{d}\Phi\right)\right|\right] \\
&\leq \mathbf{E}\left[\left|\int_{\mathbb{G}} f\mathrm{d}\Phi - \int_{\mathbb{G}} g\mathrm{d}\Phi\right|\right] \\
&\leq \mathbf{E}\left[\int_{\mathbb{G}} |f - g|\,\mathrm{d}\Phi\right] \leq \int_{\mathbb{G}} |f - g|\,\mathrm{d}M_\Phi,
\end{aligned}$$

which shows that Ψ_Φ is Lipschitz continuous, and thus continuous. $\square$

Example 4.1.3. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with intensity measure Λ. We shall show that, for all $f \in L^1_{\mathbb{R}}(\Lambda, \mathbb{G})$,*

$$\Psi_\Phi(f) = \exp\left(\int_{\mathbb{G}} (e^{\boldsymbol{i}f} - 1)\mathrm{d}\Lambda\right).$$

The announced equality holds for $f = a1_B$ where $a \in \mathbb{R}_+, B \in \mathcal{B}_c(\mathbb{G})$, since

$$\Psi_\Phi(f) = \mathbf{E}\left[e^{\boldsymbol{i}a\Phi(B)}\right] = e^{\Lambda(B)\left(e^{\boldsymbol{i}a}-1\right)}$$

and

$$\exp\left(\int_{\mathbb{G}} (e^{\boldsymbol{i}f} - 1)\mathrm{d}\Lambda\right) = \exp\left(\int_B \left(e^{\boldsymbol{i}a} - 1\right)\mathrm{d}\Lambda\right) = e^{\Lambda(B)\left(e^{\boldsymbol{i}a}-1\right)}.$$

For all simple functions $f = \sum_{j=1}^n a_j \mathbf{1}_{B_j}$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $B_1, \ldots, B_n \in \mathcal{B}_c(\mathbb{G})$ are pairwise disjoint, we have

$$\begin{aligned}\Psi_\Phi(f) &= \mathbf{E}\left[\exp\left(\boldsymbol{i}\sum_{j=1}^n a_j\Phi(B_j)\right)\right]\\ &= \prod_{j=1}^n \mathbf{E}\left[e^{\boldsymbol{i}a_j\Phi(B_j)}\right]\\ &= \prod_{j=1}^n \exp\left[\Lambda(B_j)(e^{\boldsymbol{i}a_j} - 1)\right] = \exp\left(\int_{\mathbb{G}} (e^{\boldsymbol{i}f} - 1)\mathrm{d}\Lambda\right).\end{aligned}$$

Such simple functions are dense in $L^1_{\mathbb{R}}(\Lambda, \mathbb{G})$, and therefore for any $f \in L^1_{\mathbb{R}}(\Lambda, \mathbb{G})$, there exists a sequence f_n of simple functions such that $\lim_{n\to\infty} \int_{\mathbb{G}} |f_n - f|\mathrm{d}\Lambda = 0$. For each f_n we have

$$\Psi_\Phi(f_n) = \exp\left(\int_{\mathbb{G}} (e^{\boldsymbol{i}f_n} - 1)\mathrm{d}\Lambda\right).$$

Note that by arguments similar to those of the proof of Lemma 4.1.2, the function $L^1_{\mathbb{R}}(M_\Phi, \mathbb{G}) \to \mathbb{C}, f \mapsto \int_{\mathbb{G}}(e^{\boldsymbol{i}f} - 1)\mathrm{d}\Lambda$ is continuous. Then letting $n \to \infty$ in the above equality gives the announced result.

4.1.1 Cumulant measures

Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$. Recall that we denote by $\mathcal{B}_c(\mathbb{G})$ the set of measurable relatively compact sets in $\mathbb{G}$, by M_{Φ^n} the n-th moment measure of Φ, and if Φ is a point process we denote by $M_{\Phi^{(n)}}$ the n-th factorial moment measure of Φ. In the remaining part of this section, we will assume that, for some $n \in \mathbb{N}^*$, M_{Φ^n} is locally finite.

Lemma 4.1.4. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ such that M_{Φ^n} is locally finite for some $n \in \mathbb{N}^*$. Then M_{Φ^k} and $M_{\Phi^{(k)}}$ are locally finite for all $k \in \{1, \ldots, n\}$.*

Proof. Note that M_{Φ^n} is locally finite iff for all $B_1, \ldots, B_n \in \mathcal{B}_c(\mathbb{G})$,

$$M_{\Phi^n}(B_1 \times \cdots \times B_n) = \mathbf{E}[\Phi(B_1) \cdots \Phi(B_n)] < \infty,$$

or, equivalently, for all $B \in \mathcal{B}_c(\mathbb{G})$,

$$M_{\Phi^n}(B^n) = \mathbf{E}[\Phi(B)^n] < \infty.$$

It follows from the moments inequality (13.A.4) that, for all $k \in \{1, \ldots, n\}$,

$$\mathbf{E}\left[\Phi(B)^k\right] < \infty, \quad B \in \mathcal{B}_c(\mathbb{G}),$$

and therefore M_{Φ^k} is locally finite. Using the above inequality and (13.A.22), we deduce that, for all $k \in \{1, \ldots, n\}$,

$$\mathbf{E}\left[\Phi(B)^{(k)}\right] < \infty, \quad B \in \mathcal{B}_c(\mathbb{G}),$$

which, combined with (14.E.6), implies

$$M_{\Phi^{(k)}}\left(B^k\right) < \infty, \quad B \in \mathcal{B}_c(\mathbb{G}),$$

and therefore $M_{\Phi^{(k)}}$ is locally finite. $\square$

The following lemma shows that the high-order moment measures M_{Φ^n} of a random measure Φ may be deduced from its characteristic function.

Lemma 4.1.5. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$. For all $B_1, \ldots, B_n \in \mathcal{B}_c(\mathbb{G})$,*

$$M_{\Phi^n}(B_1 \times \cdots \times B_n) = \frac{1}{\boldsymbol{i}^n} \frac{\partial^n}{\partial_{t_1} \ldots \partial_{t_n}} \Psi_\Phi(t_1 1_{B_1} + \cdots + t_n 1_{B_n})\Big|_{t_1, \ldots, t_n = 0}.$$

Moreover, the function $\log \Psi_\Phi(t_1 1_{B_1} + \ldots + t_n 1_{B_n})$ of $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ is well-defined and C^n (i.e., n times differentiable and its n-th derivative is continuous) on a neighborhood of 0.

Proof. For any $B_1, \ldots, B_n \in \mathcal{B}_c(\mathbb{G})$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$,

$$\begin{aligned}\Psi_\Phi(t_1 1_{B_1} + \cdots + t_n 1_{B_n}) &= \mathbf{E}\left[e^{\boldsymbol{i}t_1\Phi(B_1) + \ldots + \boldsymbol{i}t_n\Phi(B_n)}\right] \\ &= \mathbf{E}\left[e^{\boldsymbol{i}t^{\mathrm{T}}Y}\right] = \Psi_Y(t),\end{aligned}$$

where $Y = (\Phi(B_1), \ldots, \Phi(B_n))$ and t^{T} is the transpose of t. Since the n-th moment measure of Φ is assumed locally finite, then $\mathbf{E}[Y_1 \ldots Y_n] < \infty$, thus by Lemma 13.C.1,

$$\begin{aligned}M_{\Phi^n}(B_1 \times \ldots \times B_n) &= \mathbf{E}[Y_1 \ldots Y_n] \\ &= \frac{1}{\boldsymbol{i}^n} \frac{\partial^n}{\partial_{t_1} \ldots \partial_{t_n}} \Psi_Y(0) \\ &= \frac{1}{\boldsymbol{i}^n} \frac{\partial^n}{\partial_{t_1} \ldots \partial_{t_n}} \Psi_\Phi(t_1 1_{B_1} + \cdots + t_n 1_{B_n})\Big|_{t_1, \ldots, t_n = 0}.\end{aligned}$$

By Corollary 13.C.2, the function $\Psi_Y(t)$ is C^n on $\mathbb{R}^n$. Moreover, $\Psi_Y(0) = 1 > 0$, then $\log \Psi_Y(t)$ is well-defined and C^n on a neighborhood of 0. □

Definition 4.1.6. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$. The n-th* cumulant measure *of Φ, denoted by , is defined by*

$$C_n(B_1 \times \cdots \times B_n) = \frac{1}{\boldsymbol{i}^n} \frac{\partial^n}{\partial_{t_1} \ldots \partial_{t_n}} \log \Psi_\Phi(t_1 1_{B_1} + \cdots + t_n 1_{B_n})\bigg|_{t_1,\ldots,t_n=0},$$

for all $B_1, \ldots, B_n \in \mathcal{B}_c(\mathbb{G})$. Note the analogy between the above equation and (13.C.1).

By Corollary 13.C.5, the cumulant measures and moment measures are related by,

$$C_n(B_1 \times \ldots \times B_n) = \sum_{q=1}^{n} (-1)^{q-1}(q-1)! \sum_{\{J_1,\ldots,J_q\}} \prod_{p=1}^{q} M_{\Phi^p}\left(\prod_{j\in J_p} B_j\right), \tag{4.1.1}$$

$$M_{\Phi^n}(B_1 \times \ldots \times B_n) = \sum_{q=1}^{n} \sum_{\{J_1,\ldots,J_q\}} \prod_{p=1}^{q} C_p\left(\prod_{j\in J_p} B_j\right), \tag{4.1.2}$$

for all $B_1, \ldots, B_n \in \mathcal{B}_c(\mathbb{G})$, where the summation is over all partitions $\{J_1, \ldots, J_q\}$ of $\{1, \ldots, n\}$.

Example 4.1.7. *It follows from (4.1.2) that, for all $B, B_1, B_2, \ldots \in \mathcal{B}_c(\mathbb{G})$,*

$$\begin{aligned} M_\Phi(B) &= C_1(B) \\ M_{\Phi^2}(B_1 \times B_2) &= C_2(B_1 \times B_2) + C_1(B_1)C_1(B_2) \\ M_{\Phi^3}(B_1 \times B_2 \times B_3) &= C_3(B_1 \times B_2 \times B_3) \\ &+ C_2(B_1 \times B_2)C_1(B_3) + C_2(B_1 \times B_3)C_1(B_2) + C_2(B_2 \times B_3)C_1(B_1) \\ &+ C_1(B_1)C_1(B_2)C_1(B_3) \end{aligned} \tag{4.1.3}$$

and from (4.1.1) that

$$C_2(B_1 \times B_2) = M_{\Phi^2}(B_1 \times B_2) - M_\Phi(B_1)M_\Phi(B_2) \tag{4.1.4}$$

$$\begin{aligned} C_3(B_1 \times B_2 \times B_3) &= M_{\Phi^3}(B_1 \times B_2 \times B_3) \\ - M_{\Phi^2}(B_1 \times B_2)M_\Phi(B_3) &- M_{\Phi^2}(B_1 \times B_3)M_\Phi(B_2) - M_{\Phi^2}(B_2 \times B_3)M_\Phi(B_1) \\ &+ 2M_\Phi(B_1)M_\Phi(B_2)M_\Phi(B_3). \end{aligned}$$

Note the analogy with Example 13.C.6. Note in particular that the first cumulant and moment measures are equal. Moreover, Equation (4.1.4) reads

$$C_2(B_1 \times B_2) = \operatorname{cov}(\Phi(B_1), \Phi(B_2)). \tag{4.1.5}$$

Example 4.1.8. *Taking $n = 4$ in (4.1.1), we get*

$$\begin{aligned} C_4(B_1 \times \cdots \times B_4) = {} & M_{\Phi^4}(B_1 \times \cdots \times B_4) \\ & - \sum\nolimits^* M_\Phi(B_1) M_{\Phi^3}(B_2 \times B_3 \times B_4) \\ & - \sum\nolimits^* M_{\Phi^2}(B_1 \times B_2) M_{\Phi^2}(B_3 \times B_4) \\ & + 2 \sum\nolimits^* M_\Phi(B_1) M_\Phi(B_2) M_{\Phi^2}(B_3 \times B_4) \\ & - 6 M_\Phi(B_1) \ldots M_\Phi(B_4), \end{aligned}$$

where $\sum^*$ denotes the sum of all the terms of the same type (there are 4 terms in the first sum, 3 terms in the second one and 6 terms in the last sum).

Example 4.1.9. *The second cumulant measure is useful to get the covariance of two shot-noise processes. Indeed, let $f, g : \mathbb{G} \to \bar{\mathbb{R}}_+$ be integrable with respect to M_Φ (or $f, g \in L^2_\Phi$; cf. Definition 2.4.9) and let $\Phi(f) = \int_{\mathbb{G}} f \mathrm{d}\Phi$ and $\Phi(g) = \int_{\mathbb{G}} g \mathrm{d}\Phi$ be the corresponding shot-noise processes, then*

$$\begin{aligned} \operatorname{cov}(\Phi(f), \Phi(g)) &= \mathbf{E}[\Phi(f)\Phi(g)] - \mathbf{E}[\Phi(f)]\mathbf{E}[\Phi(g)] \\ &= \int_{\mathbb{G}\times\mathbb{G}} f(x) g(y) M_{\Phi^2}(\mathrm{d}x \times \mathrm{d}y) \\ &\quad - \int_{\mathbb{G}} f(x) M_\Phi(\mathrm{d}x) \int_{\mathbb{G}} g(y) M_\Phi(\mathrm{d}y) \\ &= \int_{\mathbb{G}\times\mathbb{G}} f(x) g(y) C_2(\mathrm{d}x \times \mathrm{d}y), \end{aligned} \tag{4.1.6}$$

where the second equality follows from the Campbell averaging formula and the third one follows from (4.1.4).

4.1.2 Factorial cumulant measures

The following lemma shows that the higher-order factorial moment measures $M_{\Phi^{(n)}}$ of a point process Φ may be deduced from its generating function; cf. Defintion 1.6.18.

Lemma 4.1.10. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$. For any $k \in \{1, \ldots, n\}$, any pairwise disjoint $B_1, \ldots, B_k \in \mathcal{B}_c(\mathbb{G})$, and any $\nu_1, \ldots, \nu_k \in \mathbb{N}^*$ such that $\nu_1 + \ldots + \nu_k = n$,*

$$M_{\Phi^{(n)}}(B_1^{\nu_1} \times \ldots \times B_k^{\nu_k}) = \lim_{x_1, \ldots, x_k \uparrow 1} \frac{\partial^n}{\partial_{x_1}^{\nu_1} \ldots \partial_{x_k}^{\nu_k}} \mathcal{G}_\Phi\left(1 - \sum_{j=1}^k (1 - x_j) \mathbf{1}_{B_j}\right),$$

where $\mathcal{G}_\Phi$ is the generating function of Φ. Moreover, the function $\log \mathcal{G}_\Phi\left(1 - \sum_{j=1}^k (1 - x_j) \mathbf{1}_{B_j}\right)$ of $x \in \mathbb{R}^k$ is well-defined on $(0, 1]^k$ and is C^∞ on $(0, 1)^k$.

Proof. Let $Y = (\Phi(B_1), \ldots, \Phi(B_k))$. For any $y = (y_1, \ldots, y_k) \in [0,1]^k$,

$$\mathcal{G}_\Phi \left(1 - \sum_{j=1}^{k} (1 - x_j) \mathbf{1}_{B_j}\right) = \mathbf{E}\left[\prod_{j=1}^{k} x_j^{\Phi(B_j)}\right] = \mathcal{G}_Y(x).$$

On the other hand, for all pairwise disjoint $B_1, \ldots, B_k \in \mathcal{B}_c(\mathbb{G})$, and all $\nu_1, \ldots, \nu_k \in \mathbb{N}^*$ such that $\nu_1 + \cdots + \nu_k = n$,

$$\begin{aligned}
M_{\Phi^{(n)}}\left(B_1^{\nu_1} \times \ldots \times B_k^{\nu_k}\right) &= \mathbf{E}\left[\Phi(B_1)^{(\nu_1)} \ldots \Phi(B_k)^{(\nu_k)}\right] \\
&= \mathbf{E}\left[Y_1^{(\nu_1)} \ldots Y_k^{(\nu_k)}\right] \\
&= \lim_{x_1, \ldots, x_k \uparrow 1} \frac{\partial^{\nu_1 + \cdots + \nu_k}}{\partial_{x_1}^{\nu_1} \ldots \partial_{x_k}^{\nu_k}} \mathcal{G}_Y(x) \\
&= \lim_{x_1, \ldots, x_k \uparrow 1} \frac{\partial^{\nu_1 + \cdots + \nu_k}}{\partial_{x_1}^{\nu_1} \ldots \partial_{x_k}^{\nu_k}} \mathcal{G}_\Phi \left(1 - \sum_{j=1}^{k} (1 - x_j) \mathbf{1}_{B_j}\right).
\end{aligned}$$

□

Definition 4.1.11. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$. The n-th* factorial cumulant measure *of Φ, denoted by $C_{(n)}$, is defined by*

$$C_{(n)}\left(B_1^{\nu_1} \times \cdots \times B_k^{\nu_k}\right) = \lim_{x_1, \ldots, x_k \uparrow 1} \frac{\partial^{\nu_1 + \cdots + \nu_k}}{\partial_{x_1}^{\nu_1} \ldots \partial_{x_k}^{\nu_k}} \log \mathcal{G}_\Phi \left(1 - \sum_{j=1}^{k} (1 - x_j) \mathbf{1}_{B_j}\right), \tag{4.1.7}$$

for all pairwise disjoint $B_1, \ldots, B_k \in \mathcal{B}_c(\mathbb{G})$, and all $\nu_1, \ldots, \nu_k \in \mathbb{N}$ such that $\nu_1 + \ldots + \nu_k = n$. (Note the analogy with (13.C.2).)

By Corollary 13.C.5, the factorial cumulant measures and factorial moment measures are related by, for all $B_1, \ldots, B_n \in \mathcal{B}_c(\mathbb{G})$,

$$C_{(n)}(B_1 \times \ldots \times B_n) = \sum_{q=1}^{n} (-1)^{q-1} (q-1)! \sum_{\{J_1, \ldots, J_q\}} \prod_{p=1}^{q} M_{\Phi^{(p)}}\left(\prod_{j \in J_p} B_j\right) \tag{4.1.8}$$

$$M_{\Phi^{(n)}}(B_1 \times \ldots \times B_n) = \sum_{q=1}^{n} \sum_{\{J_1, \ldots, J_q\}} \prod_{p=1}^{q} C_{(p)}\left(\prod_{j \in J_p} B_j\right),$$

where the summation is over all partitions $\{J_1, \ldots, J_q\}$ of $\{1, \ldots, n\}$. Indeed, the above identities hold true for all $B_1, \ldots, B_n$ which are pairwise either identical or disjoint. Two measures which coincide on such $B_1 \times \ldots \times B_n$ are identical by standard results of measure theory.

Example 4.1.12. *By analogy to (4.1.4), we have, for any* $B_1, B_2 \in \mathcal{B}_c(\mathbb{G})$,

$$\begin{aligned} C_{(2)}(B_1 \times B_2) &= M_{\Phi^{(2)}}(B_1 \times B_2) - M_\Phi(B_1) M_\Phi(B_2) \\ &= M_{\Phi^2}(B_1 \times B_2) - M_\Phi(B_1 \cap B_2) - M_\Phi(B_1) M_\Phi(B_2) \\ &= \operatorname{cov}(\Phi(B_1), \Phi(B_2)) - M_\Phi(B_1 \cap B_2), \end{aligned}$$

where the second equality is due to (14.E.5).

Example 4.1.13. Factorial cumulant measure of Poisson point processes. *Let* Φ *be a Poisson point process. Its generating function is given by (2.1.3), then*

$$\log \mathcal{G}_\Phi \left(1 - \sum_{j=1}^{k} (1 - x_j) \mathbf{1}_{B_j}\right) = -\sum_{j=1}^{k} (1 - x_j) M_\Phi(B_j).$$

Then, by (4.1.7), the first factorial cumulant measure of Φ *equals its mean measure; that is* $C_{(1)} = M_\Phi$. *Moreover, for* $n \geq 2$, *the* n*-th factorial cumulant measure of* Φ *is null.*

4.2 Finite series transform expansions

4.2.1 Characteristic function expansion

Proposition 4.2.1. Moment expansion of characteristic functions. *Let* Φ *be a random measure on a l.c.s.h. space* $\mathbb{G}$ *and let* $f : \mathbb{G} \to \mathbb{R}$ *be a measurable function such that the function defined on* $\mathbb{G}^n$ *by* $(x_1, \ldots, x_n) \mapsto f(x_1) \ldots f(x_n)$ *is integrable with respect to* M_{Φ^n}; *that is*

$$\int_{\mathbb{G}^n} |f(x_1) \cdots f(x_n)| M_{\Phi^n}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n) < \infty.$$

Then the characteristic function of Φ *admits the following expansion, for* $t \in \mathbb{R}$,

$$\Psi_\Phi(tf) = 1 + \sum_{r=1}^{n} \frac{(\boldsymbol{i}t)^r}{r!} \int_{\mathbb{G}^r} f(x_1) \cdots f(x_r) M_{\Phi^r}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_r) + \frac{(\boldsymbol{i}t)^n}{n!} \varepsilon_n(t), \tag{4.2.1}$$

where $|\varepsilon_n(t)| \leq 3 \int_{\mathbb{G}^n} |f(x_1) \cdots f(x_n)| M_{\Phi^n}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n)$ *and* $\lim_{t \to 0} \varepsilon_n(t)$. *Moreover, for* $t \in \mathbb{R}$,

$$\log \Psi_\Phi(tf) = \sum_{r=1}^{n} \frac{(\boldsymbol{i}t)^r}{r!} \int_{\mathbb{G}^r} f(x_1) \cdots f(x_r) C_r(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_r) + o(t^n), \tag{4.2.2}$$

where C_r *is the* r*-th cumulant measure of* Φ.

Proof. Observe that

$$\begin{aligned} \mathbf{E}\left[\left(\int |f| \mathrm{d}\Phi\right)^n\right] &= \mathbf{E}\left[\int_{\mathbb{G}^n} |f(x_1) \cdots f(x_n)| \Phi^n(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n)\right] \\ &= \int_{\mathbb{G}^n} |f(x_1) \cdots f(x_n)| M_{\Phi^n}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n) < \infty, \end{aligned}$$

where the second equality follows from the Campbell averaging formula (1.2.2). If follows form Lyapunov's inequality ($\mathbf{E}[X^r]^{1/r} \leq \mathbf{E}[X^n]^{1/n}$ for $r \leq n$) that $\mathbf{E}\left[\int |f| \, \mathrm{d}\Phi\right] < \infty$, then $X = \int f \mathrm{d}\Phi$ is a well defined random variable and is almost surely finite. Its characteristic function equals

$$\begin{aligned}\Psi_X(t) &= \mathbf{E}\left[e^{\boldsymbol{i}tX}\right] \\ &= \mathbf{E}\left[e^{\boldsymbol{i}t\int f \mathrm{d}\Phi}\right] = \Psi_\Phi(tf),\end{aligned}$$

for $t \in \mathbb{R}$. Moreover $\mathbf{E}[|X|^n] < \infty$ then it follows from Lemma 13.A.1 that

$$\Psi_X(t) = \sum_{r=0}^{n} \mathbf{E}[X^r] \frac{(\boldsymbol{i}t)^r}{r!} + \frac{(\boldsymbol{i}t)^n}{n!} \varepsilon_n(t), \quad t \in \mathbb{R},$$

where $|\varepsilon_n(t)| \leq 3\mathbf{E}[|X|^n]$ and $\lim_{t\to 0} \varepsilon_n(t)$. Observing that, for all $r \leq n$,

$$\begin{aligned}\mathbf{E}[X^r] &= \mathbf{E}\left[\int_{\mathbb{G}^r} f(x_1) \ldots f(x_r) \, \Phi^r(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r)\right] \\ &= \int_{\mathbb{G}^r} f(x_1) \ldots f(x_r) \, M_{\Phi^r}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r)\end{aligned}$$

completes the proof of (4.2.1). Consider the cumulant function of X; that is

$$\zeta_X(t) = \log \Psi_X(t) = \log \Psi_\Phi(tf), \quad t \in \mathbb{R}.$$

Note that ζ_X is C^n on a neighborhood of 0, then by Taylor-Young theorem

$$\zeta_X(t) = \sum_{r=0}^{n} \frac{(\boldsymbol{i}t)^r}{r!} c_r + o(t^n),$$

where c_r are the cumulants of X. They are related to the moments $m_r = \mathbf{E}[X^r]$ by (13.A.34), that is

$$\begin{aligned}c_r &= \sum_{q=1}^{r} (-1)^{q-1}(q-1)! \sum_{\nu=\{\nu_1,\ldots,\nu_q\}} \prod_{p=1}^{q} m_{|\nu_p|} \\ &= \sum_{q=1}^{r} (-1)^{q-1}(q-1)! \sum_{\nu=\{\nu_1,\ldots,\nu_q\}} \prod_{p=1}^{q} \int_{\mathbb{G}^{|\nu_p|}} f(x_1) \cdots f\left(x_{|\nu_p|}\right) \\ &\qquad M_{\Phi^{|\nu_p|}}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_{|\nu_p|}\right) \\ &= \int_{\mathbb{G}^r} f(x_1) \cdots f(x_r) \, C_r(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_r),\end{aligned}$$

where the last equality follows from (4.1.8). Combining the above three equations we get (4.2.2). □

4.2.2 Laplace transform expansion

Recall that the Laplace transform of random measure Φ is defined by

$$\mathcal{L}_\Phi(f) = \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f \mathrm{d}\Phi\right)\right],$$

for all measurable $f : \mathbb{G} \to \mathbb{R}_+$.

Proposition 4.2.2. Moment expansion of Laplace transforms. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ and let $f : \mathbb{G} \to \mathbb{R}_+$ be a measurable function such that the function defined on $\mathbb{G}^n$ by $(x_1, \ldots, x_n) \mapsto f(x_1) \ldots f(x_n)$ is integrable with respect to M_{Φ^n}; that is*

$$\int_{\mathbb{G}^n} f(x_1) \ldots f(x_n) M_{\Phi^n}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n) < \infty.$$

Then, for $t \in \mathbb{R}_+$,

$$\mathcal{L}_\Phi(tf) = 1 + \sum_{r=1}^{n} \frac{(-t)^r}{r!} \int_{\mathbb{G}^r} f(x_1) \ldots f(x_r) M_{\Phi^r}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r) + \frac{t^n}{n!}\varepsilon_n(t), \tag{4.2.3}$$

where $|\varepsilon_n(t)| \leq \int_{\mathbb{G}^n} f(x_1) \ldots f(x_n) M_{\Phi^n}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n)$ and $\lim_{t\to 0} \varepsilon_n(t) = 0$. Moreover, for $t \in \mathbb{R}_+$,

$$\log \mathcal{L}_\Phi(tf) = \sum_{r=1}^{n} \frac{(-t)^r}{r!} \int_{\mathbb{G}^r} f(x_1) \ldots f(x_r) C_r(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r) + o(t^n), \tag{4.2.4}$$

where C_r is the r-th cumulant measure of Φ.

Proof. The proof follows the same lines as in Proposition 4.2.1. It relies on the expansion (13.B.3) of the Laplace transform of the random variable $X = \int f \mathrm{d}\Phi$. □

Example 4.2.3. Cumulant measures of the Poisson point process. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$. It follows from (2.3.18) that all its factorial moment measures are locally finite. Then all its moment measures are also locally finite by Lemma 14.E.4. Let $f : \mathbb{G} \to \mathbb{R}_+$ be measurable and integrable with respect to M_Φ; that is $\int_{\mathbb{G}} f \mathrm{d}M_\Phi < \infty$. Then, for all $n \in \mathbb{N}^*$,*

$$\int_{\mathbb{G}^n} f(x_1) \ldots f(x_n) M_{\Phi^{(n)}}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n) = \left(\int_{\mathbb{G}} f \mathrm{d}M_\Phi\right)^n < \infty,$$

where we use (2.3.18). It follows that $f(x_1) \ldots f(x_n)$ is integrable with respect to M_{Φ^n}. Then (4.2.4) holds true.

On the other hand, taking f bounded with support in $\mathcal{B}_c(\mathbb{G})$,

$$\begin{aligned}\log \mathcal{L}_\Phi(tf) &= \int_{\mathbb{G}} \left(e^{-tf} - 1\right) \mathrm{d}M_\Phi \\ &= \int_{\mathbb{G}} \sum_{r=1}^{\infty} \frac{(-tf)^r}{r!} \mathrm{d}M_\Phi = \sum_{r=1}^{\infty} \frac{(-t)^r}{r!} \int_{\mathbb{G}} f^r \mathrm{d}M_\Phi,\end{aligned}$$

where the third equality follows from the dominated convergence theorem. Comparing the above equation with (4.2.4) shows that

$$\int_{\mathbb{G}^r} f(x_1)\dots f(x_r)\, C_r(\mathrm{d}x_1 \times \dots \times \mathrm{d}x_r) = \int_{\mathbb{G}} f^r \mathrm{d}M_\Phi .$$

Thus the cumulant measures of a Poisson point process are given by

$$C_r(\mathrm{d}x_1 \times \dots \times \mathrm{d}x_r) = M_\Phi(\mathrm{d}x_1)\, \delta_{x_1}(\mathrm{d}x_2) \dots \delta_{x_1}(\mathrm{d}x_r), \tag{4.2.5}$$

(which may be compared to (13.A.27)). We see that the cumulant measures of a Poisson point process are concentrated on the diagonal where they reduce to the intensity measure.

The second cumulant measure allows one to get the covariance of two shot-noise processes. Indeed, let $f, g : \mathbb{G} \to \bar{\mathbb{R}}_+$ be integrable with respect to M_Φ (or $f, g \in L^2_\Phi$; cf. Definition 2.4.9) and let $\Phi(f) = \int_{\mathbb{G}} f \mathrm{d}\Phi$ and $\Phi(g) = \int_{\mathbb{G}} g \mathrm{d}\Phi$ be the corresponding shot-noise processes, then it follows from (4.1.6) that

$$\begin{aligned}\operatorname{cov}(\Phi(f), \Phi(g)) &= \int_{\mathbb{G}\times\mathbb{G}} f(x)\, g(y)\, C_2(\mathrm{d}x \times \mathrm{d}y) \\ &= \int_{\mathbb{G}\times\mathbb{G}} f(x)\, g(y)\, M_\Phi(\mathrm{d}x)\, \delta_x(\mathrm{d}y) = \int_{\mathbb{G}} f(x)\, g(x)\, M_\Phi(\mathrm{d}x),\end{aligned}$$

which has already been proven in Proposition 2.4.6.

4.2.3 Generating function expansion

Proposition 4.2.4. Factorial moment expansion of generating functions. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ with generating function $\mathcal{G}_\Phi$ and let $h : \mathbb{G} \to [0,1]$ be a measurable function whose support is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$. Then, for $\rho \in [0,1)$,*

$$\mathcal{G}_\Phi(1-\rho h) = 1 + \sum_{r=1}^{n} \frac{(-\rho)^r}{r!} \int_{\mathbb{G}^r} h(x_1) \cdots h(x_r)\, M_{\Phi^{(r)}}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_r) + o(\rho^n), \tag{4.2.6}$$

where $M_{\Phi^{(r)}}$ is the r-th factorial moment measure of Φ; and

$$\log \mathcal{G}_\Phi(1-\rho h) = \sum_{r=1}^{n} \frac{(-\rho)^r}{r!} \int_{\mathbb{G}^r} h(x_1) \cdots h(x_r)\, C_{(r)}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_r) + o(\rho^n), \tag{4.2.7}$$

where $C_{(r)}$ is the r-th factorial cumulant measure of Φ. Moreover, if $M_{\Phi^{n+1}}$ is locally finite, then the remainder term $o(\rho^n)$ in (4.2.6) is dominated by

$$\frac{\rho^{n+1}}{(n+1)!} \int_{\mathbb{G}^{n+1}} h(x_1) \cdots h(x_{n+1})\, M_{\Phi^{(n+1)}}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_{n+1}). \tag{4.2.8}$$

Proof. Cf. [34, §5] or [31, Proposition 9.5.VI]. □

We will show later that there exists also an infinite series expansion of Laplace transform; cf. Proposition 4.3.15 below.

Example 4.2.5. Mixed Poisson point process. *Consider a mixed Poisson point process* Φ *as in Example 2.3.6; that is a Cox point process* Φ *directed by* $\Lambda = X\mu$ *where* X *is a nonnegative random variable and* μ *is a locally finite measure on* $\mathbb{G}$*. Assume that* $\mathbf{E}[X^n] < \infty$ *for some* $n \in \mathbb{N}^*$*. For any* $r \leq n$*, the* r*-th factorial moment measure of* Φ *equals*

$$\begin{aligned} M_{\Phi^{(r)}}(B) &= \mathbf{E}\left[\Phi^{(r)}(B)\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\Phi_{\Lambda}^{(r)}(B) \mid \Lambda\right]\right] \\ &= \mathbf{E}\left[\Lambda^{r}(B)\right] \\ &= \mathbf{E}\left[X^{r}\right]\mu^{r}(B), \end{aligned}$$

where the third equality follows from (2.3.18). Then the factorial moment measures of Φ *up to order* n *are locally finite. By Equation (14.E.8)* M_{Φ^n}*, being a combination of factorial moment measures of order up to* n*, is itself locally finite. On the other hand, if follows from (2.3.4) that the generating function of* Φ *equals*

$$\mathcal{G}_{\Phi}(v) = \mathcal{L}_X\left(\int_{\mathbb{G}}\left[1 - v(t)\right]\mu(\mathrm{d}t)\right), \quad v \in \mathcal{V}(\mathbb{G}).$$

In particular, for $1 - h \in \mathcal{V}(\mathbb{G})$ *and* $\rho \in (0,1)$*,*

$$\mathcal{G}_{\Phi}(1 - \rho h) = \mathcal{L}_X\left(\rho\int_{\mathbb{G}} h(t)\,\mu(\mathrm{d}t)\right).$$

Since $\mathbf{E}[X^n] < \infty$*, it follows from Lemma 13.B.1 that*

$$\mathcal{L}_X(t) = \sum_{r=0}^{n}\frac{(-t)^r}{r!}\mathbf{E}\left[X^r\right] + o(t^n), \quad t \in \mathbb{R}_+.$$

Combining the above two equalities, we get

$$\mathcal{G}_{\Phi}(1 - \rho h) = \sum_{r=0}^{n}\frac{(-\rho)^r}{r!}\mathbf{E}\left[X^r\right]\left(\int_{\mathbb{G}} h(t)\,\mu(\mathrm{d}t)\right)^r + o(\rho^n), \quad \rho \in (0,1),$$

which compared to (4.2.6) implies

$$M_{\Phi^{(r)}} = \mathbf{E}\left[X^r\right]\mu^r.$$

Then the factorial moment measures of a mixed Poisson point process has a power form similar to that of a Poisson point process (2.3.18), but multiplied with the moments of the random variable X*.*

4.3 Infinite series transform expansions

We will develop in the present section some further expansions of transforms of point processes. The first result concerning the void probability is immediate. For further results, we need to introduce some specific tools for finite point processes, including Janossy measures. These tools may be applied to any point process restricted to a relatively compact set.

4.3.1 Void probability expansion

Proposition 4.3.1. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and let $B \in \mathcal{B}(\mathbb{G})$ such that the radius of convergence of the generating function $\mathcal{G}_{\Phi(B)}$ (cf. Definition 13.A.11) is strictly larger than 2. Then the void probability admits the following expansion*

$$\mathbf{P}(\Phi(B) = 0) = 1 + \sum_{k=1}^{\infty} \mathbf{E}\left[\Phi(B)^{(k)}\right] \frac{(-1)^k}{k!},$$

and the series in the right-hand side converges absolutely.

Proof. This follows from Corollary 13.A.14, applied to the integer-valued random variable $X = \Phi(B)$. □

4.3.2 Symmetric enumeration of atoms of finite point processes

Recall that a point process Φ on a l.c.s.h space $\mathbb{G}$ is said to be *finite* if $\Phi(\mathbb{G}) < \infty$ almost surely; that is it takes its values in the set of finite counting measures on $\mathbb{G}$. For example, a point process with finite mean measure is finite.

Any finite point process Φ on $\mathbb{G}$ may be written by Corollary 1.6.12 as

$$\Phi = \sum_{i=1}^{N} \delta_{X_i},$$

where $N = \Phi(\mathbb{G}) < \infty$ and $(X_1, \ldots, X_N)$ is the sequence of atoms of Φ enumerated in a particular way.

Remark 4.3.2. Symmetric enumeration of atoms. *We may enumerate the atoms of a finite point process $\Phi = \sum_{i=1}^{N} \delta_{\hat{X}_i}$ on a l.c.s.h. space $\mathbb{G}$ in a* symmetric *way; that is*

$$\begin{aligned}\mathbf{P}&\left(\left(\hat{X}_1(\Phi), \ldots, \hat{X}_n(\Phi)\right) \in B \mid \Phi(\mathbb{G}) = n\right) \\ &= \mathbf{P}\left(\left(\hat{X}_{\sigma(1)}(\Phi), \ldots, \hat{X}_{\sigma(n)}(\Phi)\right) \in B \mid \Phi(\mathbb{G}) = n\right),\end{aligned} \quad (4.3.1)$$

for any $n \in \mathbb{N}^*$, $B \in \mathcal{B}(\mathbb{G}^n)$ and any permutation σ of the set $\{1, \ldots, n\}$.

Indeed, consider an arbitrary atoms enumeration $\Phi \mapsto (X_1(\Phi), \ldots, X_N(\Phi))$ *where* $N = \Phi(\mathbb{G})$*. Let* $n \in \mathbb{N}^*$*. Given that* $\Phi(\mathbb{G}) = n$*, consider a permutation* σ *of the set* $\{1, \ldots, n\}$ *uniformly distributed amoung the* $n!$ *possible permutations. Then let* $\hat{X}_i(\Phi) = X_{\sigma(i)}(\Phi)$*, for all* $i \in \{1, \ldots, n\}$*. (The permutation* σ *shuffles the atoms so that no particular enumeration is privileged.) Clearly,*

$$\begin{aligned}&\mathbf{P}\left(\left(\hat{X}_1(\Phi), \ldots, \hat{X}_n(\Phi)\right) \in B \mid \Phi(\mathbb{G}) = n\right)\\&= \frac{1}{n!}\sum_{\sigma}\mathbf{P}\left(\left(X_{\sigma(1)}(\Phi), \ldots, X_{\sigma(n)}(\Phi)\right) \in B \mid \Phi(\mathbb{G}) = n\right), \quad B \in \mathcal{B}(\mathbb{G}^n),\end{aligned}$$

where the summation is over all permutations σ *of the set* $\{1, \ldots, n\}$*. Therefore, the required property (4.3.1) holds true.*

Let Φ be a finite point process. For any $n \in \mathbb{N}^*$, let Π_n be a probability measure on $\mathbb{G}^n$ defined by

$$\Pi_n(B) = \frac{1}{n!}\sum_{\sigma}\mathbf{P}\left(\left(X_{\sigma(1)}(\Phi), \ldots, X_{\sigma(n)}(\Phi)\right) \in B \mid \Phi(\mathbb{G}) = n\right), \quad B \in \mathcal{B}(\mathbb{G}^n), \tag{4.3.2}$$

where the summation is over all permutations σ of the set $\{1, \ldots, n\}$. Observe that by construction, the measure Π_n is *symmetric*; i.e., invariant with respect to the permutation of coordinates in $\mathbb{G}^n$.

In view of Remark 4.3.2, the probability measure Π_n is the distribution of the atoms of Φ enumerated in a symmetric way, given that $\Phi(\mathbb{G}) = n$.

Lemma 4.3.3. *Let* Φ *be a point process on a l.c.s.h. space* $\mathbb{G}$*. Then for any* $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{G})$ *forming a partition of* $\mathbb{G}$ *and any* $n_1, \ldots, n_k \in \mathbb{N}$*,*

$$\mathbf{P}(\Phi(A_1) = n_1, \ldots, \Phi(A_k) = n_k \mid \Phi(\mathbb{G}) = n) = \binom{n}{n_1, \ldots, n_k}\Pi_n\left(A_1^{n_1} \times \cdots \times A_k^{n_k}\right), \tag{4.3.3}$$

where Π_n *is defined by (4.3.2) and* $n = n_1 + \cdots + n_k$*.*

Proof. Let $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{G})$ be such that $A_{m+1} = \cdots = A_n = A$. Then (4.3.2) gives

$$\begin{aligned}&\Pi_n\left(A_1 \times \cdots \times A_m \times A^{n-m}\right)\\&= \frac{(n-m)!}{n!}\sum_{\sigma}\mathbf{P}\left(\left(X_{\sigma(1)}, \ldots, X_{\sigma(m)}, X_{m+1}, \ldots, X_n\right) \in A_1 \times \cdots \times A_m \times A^{n-m}\right.\\&\qquad\left.\mid \Phi(\mathbb{G}) = n\right)\\&= \frac{(n-m)!}{n!}\sum_{\sigma}\mathbf{P}\left(\left(X_{\sigma(1)}, \ldots, X_{\sigma(m)}\right) \in A_1 \times \cdots \times A_m, \Phi(A) = n-m\right.\\&\qquad\left.\mid \Phi(\mathbb{G}) = n\right),\end{aligned}$$

where the summation is over all permutations σ of the set $\{1, \ldots, m\}$. Proceeding recursively, we get

$$\Pi_n\left(A_1^{n_1} \times \cdots \times A_k^{n_k}\right) = \frac{n_1! \ldots n_k!}{n!}\mathbf{P}(\Phi(A_1) = n_1, \ldots, \Phi(A_k) = n_k \mid \Phi(\mathbb{G}) = n).$$

□

Lemma 4.3.4. Construction of a finite point process. *We may construct (generate a realization of) a finite point process Φ on a l.c.s.h. space $\mathbb{G}$ as follows:*

1. *Generate the number of points according to the distribution*

$$p_n = \mathbf{P}\left(\Phi\left(\mathbb{G}\right) = n\right), \quad n \in \mathbb{N}. \tag{4.3.4}$$

2. *Given the number of points $n \geq 1$, draw the sequence of atoms according to the distribution Π_n defined by (4.3.2).*

This construction leads to a symmetric enumeration of the atoms.

Proof. This follows immediately from Remark 4.3.2. □

Example 4.3.5. Mixed Binomial point process. *Let $\Phi = \sum_{j=1}^{N} \delta_{X_j}$ be a mixed Binomial point process as in Example 2.2.28. Let λ be the probability distribution of the atom X_1. Then Φ is a finite point process with associated atoms distributions*

$$\Pi_n\left(A_1 \times \ldots \times A_n\right) = \lambda\left(A_1\right) \ldots \lambda\left(A_n\right), \quad n \in \mathbb{N}^*, A_1, \ldots, A_n \in \mathcal{B}\left(\mathbb{G}\right). \tag{4.3.5}$$

In this case, Equation (4.3.3) reads, for any $A_1, \ldots, A_k \in \mathcal{B}\left(\mathbb{G}\right)$ forming a partition of $\mathbb{G}$ and any $n_1, \ldots, n_k \in \mathbb{N}$,

$$\mathbf{P}\left(\Phi\left(A_1\right) = n_1, \ldots, \Phi\left(A_k\right) = n_k \mid \Phi\left(\mathbb{G}\right) = n\right) = \binom{n}{n_1, \ldots, n_k} \lambda\left(A_1\right)^{n_1} \ldots \lambda\left(A_k\right)^{n_k}.$$

Thus, conditionally to $\Phi(\mathbb{G}) = n$, the random vector $(\Phi(A_1), \ldots, \Phi(A_k))$ has a multinomial distribution.

4.3.3 Janossy measures

Definition 4.3.6. Janossy measures. *Let Φ be a finite point process on a l.c.s.h space $\mathbb{G}$. For any $n \in \mathbb{N}^*$, the* Janossy measure *on $\mathbb{G}^n$ is defined by*

$$J_n\left(B\right) = n! p_n \Pi_n\left(B\right), \quad B \in \mathcal{B}\left(\mathbb{G}^n\right), \tag{4.3.6}$$

where p_n and Π_n are defined by (4.3.4) and (4.3.2), respectively.

The Janossy measure J_n is symmetric since Π_n is so. Moreover,

$$p_0 + \sum_{n \in \mathbb{N}^*} \frac{1}{n!} J_n\left(\mathbb{G}^n\right) = 1. \tag{4.3.7}$$

Remark 4.3.7. Janossy measures of restriction. *The Janossy measures of the restriction of a finite point process Φ to $D \in \mathcal{B}(\mathbb{G})$ are not the projections of the corresponding Janossy measures of Φ on D; in contrast to the moment and factorial moment measures.*

Corollary 4.3.8. *Let $\mathbb{G}$ be a l.c.s.h. space, $p_0 \in [0,1]$, and for each $n \in \mathbb{N}^*$, let J_n be a measures on $\mathbb{G}^n$ such that (4.3.7) holds true. Then there exists a finite point process Φ on $\mathbb{G}$ with Janossy measures $\{J_n\}_{n\in\mathbb{N}^*}$ and such that $\mathbf{P}(\Phi(\mathbb{G})=0)=p_0$.*

Proof. For any $n \in \mathbb{N}^*$, let

$$p_n := \frac{1}{n!} J_n(\mathbb{G}^n)$$

and, if $p_n \neq 0$,

$$\Pi_n(B) := \frac{J_n(B)}{J_n(\mathbb{G}^n)}, \quad B \in \mathcal{B}(\mathbb{G}^n).$$

Then construct Φ as in Lemma 4.3.4. □

We will now express the finite-dimensional distributions of Φ in function of its Janossy measures.

Corollary 4.3.9. Finite-dimentional distributions versus Janossy measures. *Let Φ be a finite point process on a l.c.s.h. space $\mathbb{G}$. Then for any $A_1,\ldots,A_k \in \mathcal{B}(\mathbb{G})$ forming a partition of $\mathbb{G}$ and any $n_1,\ldots,n_k \in \mathbb{N}$,*

$$\mathbf{P}(\Phi(A_1)=n_1,\ldots,\Phi(A_k)=n_k) = \frac{J_n(A_1^{n_1}\times\cdots\times A_k^{n_k})}{n_1!\ldots n_k!}, \tag{4.3.8}$$

where p_n and J_n are given by (4.3.4) and (4.3.6) respectively, and $n = n_1+\cdots+n_k$. More generally, for any disjoint $A_1,\ldots,A_k \in \mathcal{B}(\mathbb{G})$ and all $n_1,\ldots,n_k \in \mathbb{N}$,

$$\mathbf{P}(\Phi(A_1)=n_1,\ldots,\Phi(A_k)=n_k) = \frac{1}{n_1!\ldots n_k!}\sum_{r\in\mathbb{N}} \frac{J_{n+r}(A_1^{n_1}\times\cdots\times A_k^{n_k}\times B^r)}{r!},$$

where $B = (A_1\cup\ldots\cup A_k)^{\mathrm{c}}$ and $n = n_1+\cdots+n_k$.

Proof. The first equalities in the corollary are immediate from Lemma 4.3.3 and Equation (4.3.6). Moreover, for any $A \in \mathcal{B}(\mathbb{G})$ and $n \in \mathbb{N}$,

$$\mathbf{P}(\Phi(A)=n) = \sum_{r\in\mathbb{N}} \mathbf{P}(\Phi(A)=n, \Phi(A^{\mathrm{c}})=r) = \frac{1}{n!}\sum_{r\in\mathbb{N}} \frac{J_{n+r}(A^n\times(A^{\mathrm{c}})^r)}{r!}.$$

where the second equality follows from (4.3.8). More generally, for all disjoint $A_1,\ldots,A_k \in \mathcal{B}(\mathbb{G})$ and all $n_1,\ldots,n_k \in \mathbb{N}$, letting $B = (A_1\cup\ldots\cup A_k)^{\mathrm{c}}$ and $n = n_1+\ldots+n_k$, we get

$$\begin{aligned}
&\mathbf{P}(\Phi(A_1)=n_1,\ldots,\Phi(A_k)=n_k)\\
&\quad= \sum_{r\in\mathbb{N}} \mathbf{P}(\Phi(A_1)=n_1,\ldots,\Phi(A_k)=n_k,\Phi(B)=r)\\
&\quad= \frac{1}{n_1!\ldots n_k!}\sum_{r\in\mathbb{N}} \frac{J_{n+r}(A_1^{n_1}\times\cdots\times A_k^{n_k}\times B^r)}{r!},
\end{aligned}$$

which combined with (4.3.8) concludes the proof. □

4.3.4 Moment versus Janossy measures

We now express the factorial moment measure in terms of the Janossy measures.

Proposition 4.3.10. Moment versus Janossy measures. *For any finite point process Φ on a l.c.s.h. space $\mathbb{G}$, the k-th factorial moment measure can be expressed in terms of the Janossy measures as follows*

$$M_{\Phi^{(k)}}(B) = \sum_{n\in\mathbb{N}} \frac{J_{n+k}(B\times\mathbb{G}^n)}{n!}, \quad B\in\mathcal{B}(\mathbb{G})^{\otimes k}. \tag{4.3.9}$$

Proof. For any $A_1,\ldots,A_r\in\mathcal{B}(\mathbb{G})$ forming a partition of $\mathbb{G}$ and any $k_1,\ldots,k_r\in\mathbb{N}$ such that $k_1+\cdots+k_r=k$,

$$\begin{aligned}
M_{\Phi^{(k)}}&\left(A_1^{k_1}\times\cdots\times A_r^{k_r}\right)\\
&=\mathbf{E}\left[\Phi^{(k)}\left(A_1^{k_1}\times\cdots\times A_r^{k_r}\right)\right]\\
&=\mathbf{E}\left[\Phi(A_1)^{(k_1)}\ldots\Phi(A_r)^{(k_r)}\right]\\
&=\sum_{j_1=k_1}^{\infty}\cdots\sum_{j_r=k_r}^{\infty} j_1^{(k_1)}\ldots j_r^{(k_r)}\mathbf{P}(\Phi(A_1)=j_1,\ldots,\Phi(A_r)=j_r)\\
&=\sum_{j_1=k_1}^{\infty}\cdots\sum_{j_r=k_r}^{\infty}\frac{J_{j_1+\cdots+j_r}\left(A_1^{j_1}\times\cdots\times A_r^{j_r}\right)}{(j_1-k_1)!\ldots(j_r-k_r)!}\\
&=\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{\substack{n_1,\ldots,n_r\in\mathbb{N}\\ n_1+\cdots+n_r=n}}\binom{n}{n_1,\ldots,n_k}J_{n+k}\left(A_1^{k_1+n_1}\times\cdots\times A_r^{k_r+n_r}\right),
\end{aligned}$$

where the fourth equality follows from Corollary 4.3.9, and the fifth equality follows by the change of variable $n_i=j_i-k_i$ and grouping together the terms such that $n_1+\cdots+n_r=n$. Applying Lemma 14.A.2 to the measure

$$S(B)=J_{n+k}\left(A_1^{k_1}\times\cdots\times A_r^{k_r}\times B\right),\quad B\in\mathcal{B}(\mathbb{G}^n),$$

shows that

$$\begin{aligned}
S(\mathbb{G}^n)&=\sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=n}}\binom{n}{n_1,\ldots,n_k}S\left(A_1^{n_1}\times\cdots\times A_k^{n_k}\right)\\
&=\sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=n}}\binom{n}{n_1,\ldots,n_k}J_{n+k}\left(A_1^{k_1+n_1}\times\cdots\times A_r^{k_r+n_r}\right),
\end{aligned}$$

where for the second equality we use the symmetry of the Janossy measures. Combining the above equality with the equation at the beginning of the proof shows that

$$M_{\Phi^{(k)}}\left(A_1^{k_1}\times\cdots\times A_r^{k_r}\right)=\sum_{n\in\mathbb{N}}\frac{J_{n+k}\left(A_1^{k_1}\times\cdots\times A_r^{k_r}\times\mathbb{G}^n\right)}{n!}.$$

Two measures on $\mathbb{G}^k$ which coincide on sets of the form $A_1^{k_1}\times\ldots\times A_r^{k_r}$ are equal by [11, Theorem 10.3 p.163]. This concludes the proof. □

Example 4.3.11. Factorial moment measures of a mixed Binomial point process . *Let $\Phi=\sum_{j=1}^N\delta_{X_j}$ be a mixed Binomial point process as in Example 2.2.28. Then the k-th factorial moment measure of Φ equals*

$$M_{\Phi^{(k)}}\left(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k\right)=\mathbf{E}\left[N^{(k)}\right]\lambda\left(\mathrm{d}x_1\right)\ldots\lambda\left(\mathrm{d}x_k\right). \tag{4.3.10}$$

where λ is the probability distribution of the atom X_1.
Indeed, using (4.3.5) and (4.3.6) we deduce that the Janossy measures equal

$$J_n\left(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_n\right)=n!p_n\lambda\left(\mathrm{d}x_1\right)\ldots\lambda\left(\mathrm{d}x_n\right).$$

where $\{p_n\}_{n\in\mathbb{N}}$ be the probability distribution of $N=\Phi(\mathbb{G})$. Then it follows from (4.3.9) that

$$M_{\Phi^{(k)}}\left(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k\right)=\sum_{n\in\mathbb{N}}p_{n+k}\frac{(n+k)!}{n!}\lambda\left(\mathrm{d}x_1\right)\ldots\lambda\left(\mathrm{d}x_k\right),$$

which concludes the proof.

4.3.5 Janossy versus moment measures

Conversely, we will express the Janossy measure in terms of the factorial moment measures. We first extend Definition 1.6.18 of the generating function in the particular case of finite point processes so that it operates on a wider class of measurable functions than the class $\mathcal{V}(\mathbb{G})$.

For any *finite* integer-valued random variable X, its generating function may be written as a series

$$\mathcal{G}_X(z)=\mathbf{E}\left[z^X\right]=\sum_{n=0}^{\infty}\mathbf{P}(X=n)\,z^n, \tag{4.3.11}$$

which is absolutely convergent (at least) for any $z\in\mathbb{C}$ such that $|z|\le 1$ since $\sum_{n=0}^{\infty}\mathbf{P}(X=n)=1$. Let $R_{\mathcal{G}_X}$ be the *radius of convergence* of the above series (cf. Definition 13.A.11). Observe that, by (13.A.12) in the same chapter, $R_{\mathcal{G}_X}\ge 1$ but it may be strictly larger than 1 in some cases.

Definition 4.3.12. Generating function of finite point processes. *Let Φ be a finite point process on a l.c.s.h. space $\mathbb{G}$, and let $\mathcal{V}_\Phi(\mathbb{G})$ be the set of measurable functions $v:\mathbb{G}\to\mathbb{C}$ such that $\|v\|_\infty:=\sup_{x\in\mathbb{G}}|v(x)|<R_{\mathcal{G}_{\Phi(\mathbb{G})}}$. Then, the*

generating function of Φ *introduced in Definition 1.6.18 may be extended to* $\mathcal{V}_\Phi(\mathbb{G})$ *as follows*

$$\mathcal{G}_\Phi(v) = \mathbf{E}\left[\prod_{X\in\Phi} v(X)\right], \quad v \in \mathcal{V}_\Phi(\mathbb{G}).$$

Moreover, the Laplace transform of Φ *introduced in Definition 1.2.1 may be extended to all measurable functions* $f : \mathbb{G} \to \mathbb{R}$ *such that*

$$\left\|e^{-f}\right\|_\infty := \sup_{x\in\mathbb{G}} \left|e^{-f(x)}\right| < R_{\mathcal{G}_{\Phi(\mathbb{G})}}$$

as follows

$$\mathcal{L}_\Phi(f) := \mathcal{G}_\Phi\left(e^{-f}\right) = \mathbf{E}\left[\exp\left(-\int_{\mathbb{G}} f\mathrm{d}\Phi\right)\right]. \tag{4.3.12}$$

Observe that, for any $v \in \mathcal{V}_\Phi(\mathbb{G})$,

$$\prod_{X\in\Phi} v(X) \leq (\|v\|_\infty)^{\Phi(\mathbb{G})}.$$

Then taking expectation shows that $\mathcal{G}_\Phi(v) < \infty$ since $\|v\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}}$.

Remark 4.3.13. *Let* Φ *be a finite point process on a l.c.s.h. space* $\mathbb{G}$*. For any* $v \in \mathcal{V}(\mathbb{G})$*,* $\|v\|_\infty \leq 1$ *(cf. Definition 1.6.18). Thus when* $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 1$*, we have* $\mathcal{V}(\mathbb{G}) \subset \mathcal{V}_\Phi(\mathbb{G})$*.*

Lemma 4.3.14. Generating function expansion versus Janossy. *Let* Φ *be a finite point process on a l.c.s.h. space* $\mathbb{G}$*. Then the following results hold.*

(i) For any $v \in \mathcal{V}(\mathbb{G}) \cup \mathcal{V}_\Phi(\mathbb{G})$*,*

$$\mathcal{G}_\Phi(v) = p_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{G}^n} \left(\prod_{i=1}^{n} v(x_i)\right) J_n(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n). \tag{4.3.13}$$

Moreover, the above series is absolutely convergent.

(ii) For any measurable function $f : \mathbb{G} \to \mathbb{R}$ *such that* $\left\|e^{-f}\right\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}}$*,*

$$\mathcal{L}_\Phi(f) = p_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{G}^n} e^{-\sum_{i=1}^n f(x_i)} J_n(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n).$$

Moreover, the above series is absolutely convergent.

(iii) The result in (ii) holds true for any measurable function $f : \mathbb{G} \to \bar{\mathbb{R}}_+$*.*

Proof. (i) Consider first some nonnegative $v \in \mathcal{V}(\mathbb{G}) \cup \mathcal{V}_\Phi(\mathbb{G})$,

$$\begin{aligned}\mathcal{G}_\Phi(v) &= \mathbf{E}\left[\prod_{X\in\Phi} v(X)\right]\\ &= \sum_{n=0}^{\infty} p_n \mathbf{E}\left[\prod_{X\in\Phi} v(X) \middle| \Phi(\mathbb{G}) = n\right]\\ &= p_0 + \sum_{n=1}^{\infty} p_n \mathbf{E}\left[\prod_{i=1}^{n} v(X_i) \middle| \Phi(\mathbb{G}) = n\right]\\ &= p_0 + \sum_{n=1}^{\infty} p_n \int_{\mathbb{G}^n}\left(\prod_{i=1}^{n} v(x_i)\right)\Pi_n(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_n)\\ &= p_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\int_{\mathbb{G}^n}\left(\prod_{i=1}^{n} v(x_i)\right) J_n(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_n).\end{aligned}$$

For general $v \in \mathcal{V}_\Phi(\mathbb{G})$, applying the above equality for the function $|v|$ shows that

$$\sum_{n=1}^{\infty} \frac{1}{n!}\int_{\mathbb{G}^n}\left(\prod_{i=1}^{n} |v(x_i)|\right) J_n(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_n) = \mathcal{G}_\Phi(|v|) < \infty.$$

Then we may rewrite the same equalities as in the beginning of the proof for general $v \in \mathcal{V}_\Phi(\mathbb{G})$ since all the series there are absolutely convergent. (ii) This follows from (4.3.12) and (i) applied to $v = e^{-f}$ which is in $\mathcal{V}_\Phi(\mathbb{G})$. (iii) Same argument as in (ii) with $v = e^{-f}$ being in $\mathcal{V}(\mathbb{G})$. □

We will now give expansions of the generating function which extend (4.2.6) to a wider class of functions. These expansions are given in [74, p.27] but the conditions for them to hold are not explicitly stated there. These expansions are also stated in [30, Chapter 5] without detailed proof.

Proposition 4.3.15. Generating function expansion for finite point processes. *Let Φ be a finite point process on a l.c.s.h. space $\mathbb{G}$.*

(i) If $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 1$, then for all measurable functions $v : \mathbb{G} \to \mathbb{C}$ such that $\|v\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}} - 1$,

$$\mathcal{G}_\Phi(v+1) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\int_{\mathbb{G}^k}\left(\prod_{i=1}^{k} v(x_i)\right) M_{\Phi^{(k)}}(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k). \tag{4.3.14}$$

(ii) If $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 2$, then

$$J_n(B) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M_{\Phi^{(n+k)}}\left(B\times\mathbb{G}^k\right), \quad B \in \mathcal{B}(\mathbb{G}^n). \tag{4.3.15}$$

Moreover, the above two series are absolutely convergent.

Proof. (i) Assume that $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 1$. Note first that, by Lemma 13.A.13, for all $n \in \mathbb{N}$, $\mathbf{E}\left[\Phi\left(\mathbb{G}\right)^{(n)}\right] < \infty$, that is, all factorial moment measures are finite. Consider a measurable function $v : \mathbb{G} \to \mathbb{C}$ such that $\|v\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}} - 1$, the function $v + 1$ is in $\mathcal{V}_\Phi\left(\mathbb{G}\right)$ and by (4.3.13), we get

$$
\begin{aligned}
\mathcal{G}_\Phi\left(v+1\right) &= p_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{G}^n} \left(\prod_{i=1}^{n} \left[v\left(x_i\right)+1\right]\right) J_n\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n\right) \\
&= p_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{G}^n} \sum_{k=0}^{n} \binom{n}{k} \left(\prod_{i=1}^{k} v\left(x_i\right)\right) J_n\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} \int_{\mathbb{G}^n} \left(\prod_{i=1}^{k} v\left(x_i\right)\right) J_n\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{G}^k} \left(\prod_{i=1}^{k} v\left(x_i\right)\right) M_{\Phi^{(k)}}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k\right),
\end{aligned}
$$

where the third equality is due to the dominated convergence theorem and the last one is due to Proposition 4.3.10. (ii) Assume now that $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 2$ and consider a measurable functions $v : \mathbb{G} \to \mathbb{C}$ such that $\|v\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}} - 2$. Then $\||v|+1\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}} - 1$, thus by (4.3.14)

$$
\begin{aligned}
\mathcal{G}_\Phi\left(|v|+2\right) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{G}^k} \left(\prod_{i=1}^{k} \left[\left|v\left(x_i\right)\right|+1\right]\right) M_{\Phi^{(k)}}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{G}^k} \sum_{n=0}^{k} \binom{k}{n} \left(\prod_{i=1}^{n} \left|v\left(x_i\right)\right|\right) M_{\Phi^{(k)}}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k\right) \\
&= a + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{G}^n} \left(\prod_{i=1}^{n} \left|v\left(x_i\right)\right|\right) \sum_{k=n}^{\infty} \frac{1}{(k-n)!} \\
&\qquad M_{\Phi^{(k)}}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n \times \mathbb{G}^{k-n}\right),
\end{aligned}
\tag{4.3.16}
$$

with

$$
a = 1 + \sum_{k=1}^{\infty} \frac{\mathbf{E}\left[\Phi\left(\mathbb{G}\right)^{(k)}\right]}{k!} = \mathcal{G}_{\Phi(\mathbb{G})}\left(2\right),
$$

where the second equality is due to (13.A.16). Since $|v| + 2 \in \mathcal{V}_\Phi\left(\mathbb{G}\right)$, then all the terms in Equation (4.3.16) are finite. Note that $\|v-1\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}} - 1$, thus by (4.3.14)

$$\begin{aligned}
\mathcal{G}_\Phi(v) &= \mathcal{G}_\Phi((v-1)+1)\\
&= 1+\sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{G}^k}\left(\prod_{i=1}^{k}\left[v(x_i)-1\right]\right)M_{\Phi^{(k)}}(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k)\\
&= 1+\sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{G}^k}\sum_{n=0}^{k}(-1)^{k-n}\binom{k}{n}\left(\prod_{i=1}^{n}v(x_i)\right)M_{\Phi^{(k)}}(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k)\\
&= b+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\mathbb{G}^n}\left(\prod_{i=1}^{n}v(x_i)\right)\sum_{k=n}^{\infty}\frac{(-1)^{k-n}}{(k-n)!}\\
&\qquad M_{\Phi^{(k)}}\left(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_n\times\mathbb{G}^{k-n}\right),
\end{aligned}\tag{4.3.17}$$

where the fourth equality is due to the dominated convergence theorem and the fact that the quantities in Equation (4.3.16) are finite and with

$$b = 1+\sum_{k=1}^{\infty}\frac{(-1)^k\,\mathbf{E}\left[\Phi(\mathbb{G})^{(k)}\right]}{k!} = \mathcal{G}_{\Phi(\mathbb{G})}(0) = p_0,$$

where the second equality is due to (13.A.16). Comparing the expansion (4.3.17) of $\mathcal{G}_\Phi(v)$ with (4.3.13) completes the proof. □

We deduce the Laplace transform expansion for finite point processes.

Corollary 4.3.16. Laplace transform expansion for finite point processes. *Let Φ be a finite point process on a l.c.s.h. space $\mathbb{G}$. Then for all measurable functions $f:\mathbb{G}\to\bar{\mathbb{C}}$ such that $\left\|1-e^{-f}\right\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}}-1$,*

$$\mathcal{L}_\Phi(f) = 1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int_{\mathbb{G}^k}\left(\prod_{i=1}^{k}\left(1-e^{-f(x_i)}\right)\right)M_{\Phi^{(k)}}(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k).\tag{4.3.18}$$

Moreover, the above series is absolutely convergent.

Proof. This follows from Proposition 4.3.15 applied to $v:=e^{-f}-1$ and (4.3.12). □

We deduce now the Laplace transform expansion for general point processes and functions with bounded support.

Corollary 4.3.17. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and let $f:\mathbb{G}\to\bar{\mathbb{C}}$ be a measurable function with support $D\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ and such that $\left\|1-e^{-f}\right\|_\infty < R_{\mathcal{G}_{\Phi(D)}}-1$. Then the expansion (4.3.18) holds true and the series in the right-hand side is absolutely convergent.*

Proof. This is immediate from Corollary 4.3.16 applied to the restriction of the point process Φ to D which is finite by the very definition of a point process (since $D\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$). □

We now get rid of the bounded support assumption.

Corollary 4.3.18. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and let $f : \mathbb{G} \to \bar{\mathbb{R}}_+$ be a measurable function such that the series in the right-hand side of (4.3.18) is absolutely convergent and such that for any $D \in \mathcal{B}_c(\mathbb{G})$,*

$$\sup_{x \in D} \left(1 - e^{-f(x)}\right) \leq R_{\mathcal{G}_{\Phi(D)}} - 1.$$

Then the expansion (4.3.18) holds true.

Proof. Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of locally compact sets increasing to $\mathbb{G}$ which exists by Lemma 1.1.4. For any $n \in \mathbb{N}$, let $f_n : \mathbb{G} \to \bar{\mathbb{R}}_+$ be defined by $f_n(x) = f(x)\mathbf{1}_{D_n}(x)$. Observe that, for any $x \in \mathbb{G}$, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ increases to $f(x)$. Then, by the monotone convergence theorem,

$$\lim_{n\to\infty} \int f_n \mathrm{d}\Phi = \int f \mathrm{d}\Phi.$$

Observe that $e^{-\int f_0 \mathrm{d}\Phi} - e^{-\int f_n \mathrm{d}\Phi}$ is nonnegative, nondecreasing and converges as $n \to \infty$ to $e^{-\int f_0 \mathrm{d}\Phi} - e^{-\int f \mathrm{d}\Phi}$. Again by the monotone convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\left[e^{-\int f_0 \mathrm{d}\Phi} - e^{-\int f_n \mathrm{d}\Phi}\right] &= \mathbf{E}\left[\lim_{n\to\infty}\left(e^{-\int f_0 \mathrm{d}\Phi} - e^{-\int f_n \mathrm{d}\Phi}\right)\right] \\ &= \mathbf{E}\left[e^{-\int f_0 \mathrm{d}\Phi} - e^{-\int f \mathrm{d}\Phi}\right].\end{aligned}$$

Then

$$\lim_{n\to\infty} \mathcal{L}_\Phi(f_n) = \mathcal{L}_\Phi(f). \tag{4.3.19}$$

For each $n \in \mathbb{N}$, it follows from Corollary 4.3.17 that

$$\mathcal{L}_\Phi(f_n) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left(\prod_{i=1}^{k}\left(1 - e^{-f_n(x_i)}\right)\right) M_{\Phi^{(k)}}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k).$$

We decompose the series in the right-hand side into two series E_n and O_n corresponding to even values of k and odd values of k respectively. Applying the monotone convergence theorem to each of these series, we get

$$\lim_{n\to\infty} E_n = \sum_{k\in\mathbb{N}^* \text{ even}} \frac{1}{k!} \int_{\mathbb{G}^k} \left(\prod_{i=1}^{k}\left(1 - e^{-f(x_i)}\right)\right) M_{\Phi^{(k)}}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k),$$

$$\lim_{n\to\infty} O_n = -\sum_{k\in\mathbb{N}^* \text{ odd}} \frac{1}{k!} \int_{\mathbb{G}^k} \left(\prod_{i=1}^{k}\left(1 - e^{-f(x_i)}\right)\right) M_{\Phi^{(k)}}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k).$$

The right-hand sides of the above two equations are finite by assumption, then adding them gives

$$\lim_{n\to\infty} \mathcal{L}_\Phi(f_n) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left(\prod_{i=1}^{k}\left(1 - e^{-f(x_i)}\right)\right) M_{\Phi^{(k)}}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k).$$

Invoking (4.3.19) concludes the proof. □

Remark 4.3.19. *For any integer-valued random variable X, let $R'_{\mathcal{G}_X}$ denote the radius of convergence of the series*

$$\sum_{n=1}^{\infty} \mathbf{E}\left[X^{(n)}\right] \frac{z^n}{n!}.$$

By Lemma 13.A.16, $R_{\mathcal{G}_X} > 1$ iff $R'_{\mathcal{G}_X} > 0$ in which case we have $R'_{\mathcal{G}_X} = R_{\mathcal{G}_X} - 1$. Then the conditions $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 1$ and $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 2$ in Proposition 4.3.15 are respectively equivalent to $R'_{\mathcal{G}_{\Phi(\mathbb{G})}} > 0$ and $R'_{\mathcal{G}_{\Phi(\mathbb{G})}} > 1$.

Proposition 4.3.20. *Let Φ be a finite point process on a l.c.s.h. space $\mathbb{G}$ such that $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 1$. Then the following results hold true.*

(i) All the moments and factorial moments of $\Phi(\mathbb{G})$ are finite.

(ii) The distribution of Φ is characterized by its factorial moment measures.

Proof. (i) Let $R'_{\mathcal{G}_{\Phi(\mathbb{G})}}$ be the radius of convergence of the series

$$\sum_{n=1}^{\infty} \mathbf{E}\left[\Phi(\mathbb{G})^{(n)}\right] \frac{z^n}{n!}.$$

By Lemma 13.A.16, $R'_{\mathcal{G}_{\Phi(\mathbb{G})}} > 0$. Then $\mathbf{E}\left[\Phi(\mathbb{G})^{(n)}\right] < \infty$ for all $n \in \mathbb{N}^*$. Thus all the moments of $\Phi(\mathbb{G})$ are also finite by (13.A.21). (ii) **Case** $R_{\mathcal{G}_{\Phi(\mathbb{G})}} > 2$. By Proposition 4.3.15(ii), the factorial moment measures uniquely determine the Janossy measures; and the latter characterize the distribution of Φ by Corollary 4.3.9. **Case** $R_{\mathcal{G}_{\Phi(\mathbb{G})}} \in (1,2]$. Let $\tilde{\Phi}$ be another point process with the same factorial moment measures that Φ. We have to show that $\tilde{\Phi}$ equals Φ in distribution. By Proposition 1.3.11, it is enough to show that $\mathcal{L}_{\tilde{\Phi}}(f) = \mathcal{L}_{\Phi}(f)$ for all measurable functions $f : \mathbb{G} \to \mathbb{R}_+$ which are bounded with support in $\mathcal{B}_c(\mathbb{G})$. Let f be such function and let $M = \|f\|_\infty$. Observe that

$$\mathbf{E}\left[\Phi(f)^{(n)}\right] \leq M^n \mathbf{E}\left[\Phi(\mathbb{G})^{(n)}\right], \quad n \in \mathbb{N}^*.$$

Then $R'_{\mathcal{G}_{\Phi(f)}} > 0$, thus by Lemma 13.A.16,

$$R_{\mathcal{G}_{\Phi(f)}} = R'_{\mathcal{G}_{\Phi(f)}} + 1 > 1.$$

Therefore, by Proposition 13.B.4(iii)

$$R_{\mathcal{L}_{\Phi(f)}} = \log\left(R_{\mathcal{G}_{\Phi(f)}}\right) > 0.$$

By Proposition 13.A.7(iv), the distribution of $\Phi(f)$ is characterized by its moments, then $\mathcal{L}_{\tilde{\Phi}}(f) = \mathcal{L}_{\Phi}(f)$ which concludes the proof. □

Remark 4.3.21. *Table 4.1 summarizes the series expansions we established for the different transforms in terms of specific measures. This table may be completed by an expansion of $\log \mathcal{G}_\Phi(v)$ in terms of the* Khinchin measures *[30, Eq (5.5.6)].*

Transform	Measures	Equation
Characteristic function $\Psi_\Phi(tf)$	Moment measures	(4.2.1)
$\log \Psi_\Phi(tf)$	Cumulant measures	(4.2.2)
Laplace transform $\mathcal{L}_\Phi(tf)$	Moment measures	(4.2.3)
$\log \mathcal{L}_\Phi(tf)$	Cumulant measures	(4.2.4)
Generating function $\mathcal{G}_\Phi(1-\rho h)$	Factorial moment measures	(4.2.6)
$\log \mathcal{G}_\Phi(1-\rho h)$	Factorial cumulant measures	(4.2.7)
$\mathcal{G}_\Phi(v)$	Janossy measures	(4.3.13)

Table 4.1: Series expansions of different transforms.

4.3.6 Distribution of a finite point process

Let $\mathbb{G}^* = \bigcup_{n=0}^{\infty} \mathbb{G}^n$ be the *set of finite ordered sequences* of points of $\mathbb{G}$ with the convention that $\mathbb{G}^0$ is the empty sequence. We induce $\mathbb{G}^*$ with the σ-algebra $\mathcal{G}^* = \left\{A \subset \mathbb{G}^* : A \cap \mathbb{G}^n \in \mathcal{B}(\mathbb{G})^{\otimes n} \text{ for all } n \in \mathbb{N}\right\}$ where $\mathcal{B}(\mathbb{G})^{\otimes n}$ be the product σ-algebra on $\mathbb{G}^n$. Let $\mathbb{M}_{\mathrm{f}}(\mathbb{G})$ be the *set of finite counting measures* on $\mathbb{G}$. We will use the arguments of Section 4.3.2 to show that there is a bijective mapping between the set of symmetric probability measures on $\mathbb{G}^*$ and the set of probability measures on $\mathbb{M}_{\mathrm{f}}(\mathbb{G})$.

To this end, we introduce a mapping $u : \mathbb{G}^* \to \mathbb{M}_{\mathrm{f}}(\mathbb{G})$ defined by

$$u : (x_1, \ldots, x_n) \mapsto \sum_{k=1}^{n} \delta_{x_k} \tag{4.3.20}$$

Consider a finite point process Φ on $\mathbb{G}$ and let p_n and Π_n be defined respectively by (4.3.4) and (4.3.2). Moreover let Π_0 be the probability measure on $\mathbb{G}^0$ with all its mass on the empty sequence. Then

$$\Pi = \sum_{n=0}^{\infty} p_n \Pi_n \tag{4.3.21}$$

defines a probability measure on $\mathbb{G}^*$ which is symmetric (i.e., its restriction to each $\mathbb{G}^n$ is invariant with respect to permutation of coordinates).

Lemma 4.3.22. *The distribution $\mathbf{P}_\Phi$ of the finite point process Φ on a l.c.s.h. space $\mathbb{G}$ is given by*

$$\mathbf{P}_\Phi = \Pi \circ u^{-1}$$

where Π and u are respectively defined by (4.3.21) and (4.3.20).

Proof. The measure $\Pi \circ u^{-1}$ is a probability on $\mathbb{M}_{\mathrm{f}}(\mathbb{G})$. Then it is the probability measure of some point process $\tilde{\Phi}$. Observe that for any measurable mapping

$f : \mathbb{M}_{\mathrm{f}}(\mathbb{G}) \to \mathbb{R}_+$,

$$\begin{aligned}\mathbf{E}\left[f\left(\tilde{\Phi}\right)\right] &= \int_{\mathbb{M}_{\mathrm{f}}(\mathbb{G})} f(\mu)\,\mathbf{P}_{\tilde{\Phi}}(\mathrm{d}\mu) \\ &= \int_{\mathbb{M}_{\mathrm{f}}(\mathbb{G})} f(\mu)\,\Pi\circ u^{-1}(\mathrm{d}\mu) \\ &= \int_{\mathbb{G}^*} f\circ u(x)\,\Pi(\mathrm{d}x) \\ &= \sum_{n=0}^{\infty} p_n \int_{\mathbb{G}^n} f\circ u(x)\,\Pi_n(\mathrm{d}x) = \mathbf{E}\left[f(\Phi)\right],\end{aligned}$$

where the third equality is due to the change of variable theorem for measures and the last equality is due to Lemma 4.3.4. □

Proposition 4.3.23. *[79, Prop. I.10] Let $\mathbb{G}$ be a l.c.s.h. space. The mapping associating to each symmetric probability measure Π on $\mathbb{G}^*$ the probability measure $\Pi\circ u^{-1}$ on $\mathbb{M}_{\mathrm{f}}(\mathbb{G})$ is bijective.*

Proof. Lemma 4.3.22 shows that the mapping of the proposition is surjective. It remains to show that it is injective. Let $\Pi = \sum_{n=0}^{\infty} p_n \Pi_n$ and $\tilde{\Pi} = \sum_{n=0}^{\infty} \tilde{p}_n \tilde{\Pi}_n$ be two symmetric probability measures on $\mathbb{G}^*$ such that $\Pi\circ u^{-1} = \tilde{\Pi}\circ u^{-1}$. Let $A = \{\mu \in \mathbb{M}_{\mathrm{f}}(\mathbb{G}) : \mu(\mathbb{G}) = n\}$. Since $\Pi\circ u^{-1}(A) = \Pi(\mathbb{G}^n) = p_n$ with a similar equality for $\tilde{\Pi}$, it follows that $p_n = \tilde{p}_n$ for all $n \in \mathbb{N}$. Moreover, for any measurable function $f : A \to \mathbb{R}_+$,

$$\int_A f(\mu)\,\Pi\circ u^{-1}(\mathrm{d}\mu) = p_n \int_{\mathbb{G}^n} f\circ u(x)\,\Pi_n(\mathrm{d}x),$$

with a similar equality for $\tilde{\Pi}$. Thus

$$\int_{\mathbb{G}^n} f\circ u(x)\,\Pi_n(\mathrm{d}x) = \int_{\mathbb{G}^n} f\circ u(x)\,\tilde{\Pi}_n(\mathrm{d}x).$$

Applying the above equality with $f(\mu) = \mathbf{1}\{x_1(\mu) \in B_1, \ldots, x_n(\mu) \in B_n\}$ where $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{G})$, it follows that

$$\Pi_n(B_1 \times \cdots \times B_n) = \tilde{\Pi}_n(B_1 \times \cdots \times B_n).$$

Then $\Pi_n = \tilde{\Pi}_n$ for all $n \in \mathbb{N}$, which concludes the proof. □

In view of the above proposition, we may identify the probability distributions of finite point processes with the symmetric probability measures on $\mathbb{G}^*$.

Example 4.3.24. *The probability distribution of the mixed Binomial point process $\Phi = \sum_{j=1}^{N} \delta_{X_j}$ as in Example 2.2.28 is*

$$Q = \sum_{n=0}^{\infty} q_n \lambda^n,$$

where $\{q_n\}_{n\in\mathbb{N}}$ is the probability distribution of $N = \Phi(\mathbb{G})$, λ is the probability distribution of the atom X_1 and λ^n is the n-th power of λ in the sense of products of measures.

Proposition 4.3.25. *Let $\{q_n\}_{n\in\mathbb{N}}$ be a probability distribution on $\mathbb{N}$, let λ be a probability measure on a l.c.s.h. space $\mathbb{G}$ and let Q be a probability measure on $\mathbb{G}^*$ defined by*

$$Q = \sum_{n=0}^{\infty} q_n \lambda^n, \tag{4.3.22}$$

where λ^n is the n-th power of λ in the sense of products of measures. Consider a measurable symmetric function $f : \mathbb{G}^* \to \bar{\mathbb{R}}_+$ such that $\int f \mathrm{d}Q = 1$ and let P be a probability measure on $\mathbb{G}^*$ defined by

$$P(\mathrm{d}x) = f(x)\, Q(\mathrm{d}x).$$

Then

$$P = \sum_{n=0}^{\infty} p_n \pi_n \lambda^n, \tag{4.3.23}$$

where

$$p_n = q_n \int_{\mathbb{G}^n} f(x)\, \lambda^n(\mathrm{d}x),$$

and π_n is a function defined on $\mathbb{G}^n$ by

$$\pi_n(x) = \frac{f(x)}{\int_{\mathbb{G}^n} f(y)\, \lambda^n(\mathrm{d}y)}, \quad x \in \mathbb{G}^n.$$

Proof. Let $\tilde{P} := \sum_{n=0}^{\infty} p_n \pi_n \lambda^n$ where p_n and π_n are as in the proposition. For any measurable function $g : \mathbb{G}^* \to \mathbb{R}_+$

$$\begin{aligned}
\int_{\mathbb{G}^*} g(x)\, \tilde{P}(\mathrm{d}x) &= \sum_{n=0}^{\infty} p_n \int_{\mathbb{G}^n} g(x)\, \pi_n(x)\, \lambda^n(\mathrm{d}x) \\
&= \sum_{n=0}^{\infty} p_n \frac{\int_{\mathbb{G}^n} g(x) f(x)\, \lambda^n(\mathrm{d}x)}{\int_{\mathbb{G}^n} f(y)\, \lambda^n(\mathrm{d}y)} \\
&= \sum_{n=0}^{\infty} q_n \int_{\mathbb{G}^n} g(x) f(x)\, \lambda^n(\mathrm{d}x) \\
&= \int_{\mathbb{G}^*} g(x) f(x)\, Q(\mathrm{d}x) = \int_{\mathbb{G}^*} g(x)\, P(\mathrm{d}x).
\end{aligned}$$

Then $P = \tilde{P}$, which concludes the proof. □

The above proposition allows one to construct from a *reference* distribution Q, a new one with a specified density f which may characterize some interactions (for example attraction or repulsion) between the atoms of the point process.

Example 4.3.26. Gibbs point process. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with finite intensity measure Λ. Then the distribution of Φ is given by (4.3.22) where*

$$q_n = \frac{e^{-\Lambda(\mathbb{G})}\Lambda(\mathbb{G})^n}{n!}, \quad \lambda(\mathrm{d}x) = \frac{\Lambda(\mathrm{d}x)}{\Lambda(\mathbb{G})}.$$

Let $f : \mathbb{G}^* \to \bar{\mathbb{R}}_+$ be some symmetric measurable function such that $\mathbf{E}[f(\Phi)] = 1$. Let $\tilde{\Phi}$ be a Gibbs point process with density f with respect to Φ (cf. Definition 2.3.9). Then the distribution of $\tilde{\Phi}$ is given by (4.3.23) where

$$p_n = \frac{e^{-\Lambda(\mathbb{G})}}{n!}\int_{\mathbb{G}^n} f(x)\Lambda^n(\mathrm{d}x),$$

and π_n is the function defined on $\mathbb{G}^n$ by

$$\pi_n(x) = \Lambda(\mathbb{G})^n \frac{f(x)}{\int_{\mathbb{G}^n} f(y)\Lambda^n(\mathrm{d}y)}, \quad x \in \mathbb{G}^n.$$

Example 4.3.27. Hard-core point process. *We continue Example 4.3.26 by specifying*

$$f(x) = \alpha\beta^n \prod_{1\le i<j\le n} \mathbf{1}_{\{|x_i - x_j| > 2R\}}, \quad n \in \mathbb{N}, x \in \mathbb{G}^n,$$

where $\alpha > 0$ is chosen to ensure that $\mathbf{E}[f(\Phi)] = 1$ and $\beta > 0$. The Gibbs point process $\tilde{\Phi}$ with the above density f with respect to Φ is called a hard-core point process.

4.3.7 Order statistics on $\mathbb{R}$

The *order statistics* of a point process on the real line are its points sorted in the increasing or decreasing order. The following proposition gives the distributions of these order statistics.

Proposition 4.3.28. *Let Φ be a simple point process on $\mathbb{R}$ such that*

$$R_{\mathcal{G}_{\Phi([a,+\infty))}} > 2, \quad \textit{for all } a \in \mathbb{R}, \tag{4.3.24}$$

where $R_{\mathcal{G}_{\Phi([a,+\infty))}}$ is the radius of convergence of the generating function $\mathcal{G}_{\Phi([a,+\infty))}$ (or, equivalently, $R'_{\mathcal{G}_{\Phi([a,+\infty))}} > 1$ by Lemma 13.A.16). Then the points of Φ may be sorted in the decreasing order $X_{(1)} > X_{(2)} > \ldots$. Moreover, for all $x \in \mathbb{R}$,

$$\mathbf{P}\left(X_{(k)} \ge x\right) = \sum_{n=k}^{\infty} (-1)^{n-k}\binom{n-1}{k-1}\frac{1}{n!}M_{\Phi^{(n)}}\left([x,+\infty)^n\right).$$

If moreover for all $n \in \mathbb{N}^*$, $M_{\Phi^{(n)}}$ admits a density ρ_n with respect to the Lebesgue measure, then $X_{(k)}$ admits the following probability density function

$$f(x) = \frac{1}{(k-1)!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[x,+\infty)^{k+n-1}} \rho_{k+n}(x, y_1 \ldots, y_{k+n-1})\,\mathrm{d}y_1 \ldots \mathrm{d}y_{k+n-1}$$

and $\left(X_{(1)},\ldots,X_{(k)}\right)$ *admits the following probability density function*

$$f_k\left(x_1,\ldots,x_k\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[x_k,+\infty)^n}\rho_{k+n}\left(x_1,\ldots,x_k,y_1\ldots,y_n\right)\mathrm{d}y_1\ldots\mathrm{d}y_n,$$

for all $x_1>\ldots>x_k\in\mathbb{R}$.

Proof. Note that since $R_{([a,+\infty))}>1$ then, by Lemma 13.A.13, $\mathbf{E}\left[\Phi\left([a,+\infty)\right)\right]<\infty$ which shows in particular that $\Phi\left([a,+\infty)\right)$ is almost surely finite and therefore the points of Φ may be sorted in the decreasing order. (i) **Distribution of** $X_{(k)}$. Let $x\in\mathbb{R}$. By analogy to (14.A.1), for $n\in\mathbb{N}^*$, define the n-th *symmetric sum* as

$$\mathcal{S}_n:=\mathbf{E}\left[\sum_{\{X_1,\ldots,X_n\}\subset\Phi}\prod_{j=1}^{n}\mathbf{1}\left\{X_j\geq x\right\}\right]$$

and let $\mathcal{S}_0:=1$. Note first that

$$\mathcal{S}_n=\frac{1}{n!}\mathbf{E}\left[\sum_{(X_1,\ldots,X_n)\in\Phi^{(n)}}\prod_{j=1}^{n}\mathbf{1}\left\{X_j\geq x\right\}\right]=\frac{1}{n!}M_{\Phi^{(n)}}\left([x,+\infty)^n\right).$$

Let $\tilde{\Phi}$ be the restriction of Φ to $[x,+\infty)$, which is a finite point process, say $\tilde{\Phi}=\sum_{j=1}^{m}\delta_{X_j}$, where m is an integer-valued random variable. Note that

$$\begin{aligned}\mathcal{S}_n&=\mathbf{E}\left[\sum_{J\subset\{1,\ldots,m\}:|J|=n}\prod_{j\in J}\mathbf{1}\left\{X_j\geq x\right\}\right]\\&=\mathbf{E}\left[\sum_{J\subset\{1,\ldots,m\}:|J|=n}\mathbf{1}\left(\bigcap_{j\in J}\left\{X_j\geq x\right\}\right)\right].\end{aligned}$$

Given $\tilde{\Phi}$, we may apply Lemma 14.A.3 with $B_j=\{X_j\geq x\}$ and

$$S_n=\sum_{J\subset\{1,\ldots,m\}:|J|=n}\mathbf{1}\left(\bigcap_{j\in J}\left\{X_j\geq x\right\}\right).$$

Observe that $N=\sum_{i=1}^{m}\mathbf{1}\left\{X_j\geq x\right\}\geq k$ iff $X_{(k)}\geq x$; thus Equation (14.A.2) writes

$$\mathbf{1}\left\{X_{(k)}\geq x\right\}=\sum_{n=k}^{m}(-1)^{n-k}\binom{n-1}{k-1}S_n.$$

Taking the expectation of the above equality with respect to $\tilde{\Phi}$ implies

$$\begin{aligned}\mathbf{P}\left(X_{(k)} \geq x\right) &= \mathbf{E}\left[\sum_{n=k}^{\tilde{\Phi}([x,+\infty))} (-1)^{n-k} \binom{n-1}{k-1} S_n\right] \\ &= \mathbf{E}\left[\sum_{n=k}^{\infty} (-1)^{n-k} \binom{n-1}{k-1} S_n\right],\end{aligned}$$

since if $n > \tilde{\Phi}([x,+\infty))$, then $S_n = 0$. Swapping the expectation and the sum in the right-hand side of the above equation gives the announced result. This inversion is justified by the monotone convergence theorem and the fact that

$$\begin{aligned}\sum_{n=k}^{\infty} \binom{n-1}{k-1} \mathbf{E}[S_n] &= \sum_{n=k}^{\infty} \binom{n-1}{k-1} \mathcal{S}_n \\ &= \sum_{n=k}^{\infty} \binom{n-1}{k-1} \frac{1}{n!} M_{\Phi^{(n)}}\left([x,+\infty)^n\right) \\ &= \frac{1}{(k-1)!} \sum_{n=k}^{\infty} \frac{1}{n(n-k)!} M_{\Phi^{(n)}}\left([x,+\infty)^n\right) \\ &\leq \frac{1}{(k-1)!} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} M_{\Phi^{(n)}}\left([x,+\infty)^n\right) < \infty.\end{aligned}$$

Indeed, the above series is convergent since so is $\sum_{n=k}^{\infty} \frac{1}{n!} M_{\Phi^{(n)}}\left([x,+\infty)^n\right) = \sum_{n=k}^{\infty} \frac{1}{n!} \mathbf{E}\left[\left(\Phi\left([x,+\infty)\right)\right)^{(n)}\right]$ which is finite by assumption (4.3.24). (ii) **Density of $X_{(k)}$.** The announced result follows from (i) and the fact that

$$\frac{\partial}{\partial x} \int_{[x,+\infty)^k} \rho_k(y_1 \ldots, y_k) \mathrm{d}y_1 \ldots \mathrm{d}y_k = k \int_{[x,+\infty)^{k-1}} \rho_k(x, y_1 \ldots, y_k) \mathrm{d}y_1 \ldots \mathrm{d}y_{k-1}.$$

(iii) **Density of $\left(X_{(1)}, \ldots, X_{(k)}\right)$.** For any $x_1 > \ldots > x_k \in \mathbb{R}$,

$$\begin{aligned}&\mathbf{P}\left(X_{(1)} \geq x_1 > X_{(2)} \geq x_2 > \cdots > X_{(k)} \geq x_k\right) \\ &\quad = \mathbf{P}\left(\Phi([x_1,+\infty)) = 1, \Phi([x_2,x_1)) = 1, \ldots, \Phi([x_k,x_{k-1})) = 1\right) \\ &\quad = \mathbf{P}\left(\bigcap_{i=1}^{k} \Phi([x_i,x_{i-1})) = 1\right),\end{aligned}$$

where $x_0 := +\infty$. Let J_k be the Janossy measures of the restriction of Φ to $[x_k,+\infty)$ which is a finite point process. Then by Corollary 4.3.9

$$\mathbf{P}\left(\bigcap_{i=1}^{k} \Phi([x_i,x_{i-1})) = 1\right) = J_k\left(\prod_{i=1}^{k} [x_i,x_{i-1})\right).$$

Since $R_{\Phi([x_k,+\infty))} > 2$, then by Proposition 4.3.15

$$J_k\left(\prod_{i=1}^{k}[x_i,x_{i-1})\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}M_{\Phi^{(k+n)}}\left(\prod_{i=1}^{k}[x_i,x_{i-1})\times[x_k,+\infty)^n\right).$$

Since $M_{\Phi^{(n)}}$ admits a density ρ_n with respect to the n-dimensional Lebesgue measure, then

$$\begin{aligned}J_k&\left(\prod_{i=1}^{k}[x_i,x_{i-1})\right)\\&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{\prod_{i=1}^{k}[x_i,x_{i-1})\times[x_k,+\infty)^n}\rho_{k+n}(y_1,\ldots,y_{k+n})\,\mathrm{d}y_1\ldots\mathrm{d}y_{k+n}.\end{aligned}$$

Then

$$\begin{aligned}&\mathbf{P}\left(\bigcap_{i=1}^{k}\mathbf{1}\left\{X_{(i)}\in[x_i,x_{i-1})\right\}\right)\\&=\int_{\prod_{i=1}^{k}[x_i,x_{i-1})}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[x_k,+\infty)^n}\rho_{k+n}(y_1,\ldots,y_{k+n})\,\mathrm{d}y_{k+1}\cdots\mathrm{d}y_{k+n}\right)\\&\qquad\mathrm{d}y_1\cdots\mathrm{d}y_k,\end{aligned}$$

and therefore $\left(X_{(1)},\ldots,X_{(k)}\right)$ admits the probability density function

$$f_k(x_1,\ldots,x_k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[x_k,+\infty)^n}\rho_{k+n}(x_1,\ldots,x_k,y_1\ldots,y_n)\,\mathrm{d}y_1\ldots\mathrm{d}y_n.$$

□

Remark 4.3.29. Bibliographic notes. *The distribution of* $\left(X_{(1)},\ldots,X_{(k)}\right)$ *given in Proposition 4.3.28 is stated and proved in [45, Lemma 5.3]. The distribution of* $X_{(k)}$ *is stated in [45, Lemma 5.1] without neither the condition (4.3.24) nor a detailed proof.*

Example 4.3.30. *Let* Φ *be a Poisson point process on* $\mathbb{R}$ *with intensity measure* $M_\Phi(\mathrm{d}x)=\lambda(x)\,\mathrm{d}x$ *where* $\lambda:\mathbb{R}\to\mathbb{R}_+$ *is integrable on* $[a,+\infty)$ *for all* $a\in\mathbb{R}$. *By Proposition 2.3.25, for all* $n\in\mathbb{N}^*$,

$$M_{\Phi^{(n)}}([a,+\infty)^n)=\left(\int_a^{\infty}\lambda(y)\,\mathrm{d}y\right)^n.$$

On the other hand, by (14.E.6)

$$M_{\Phi^{(n)}}([a,+\infty)^n)=\mathbf{E}\left[\Phi([a,+\infty))^{(n)}\right].$$

Then

$$\sum_{n=0}^{\infty}\frac{\mathbf{E}\left[\Phi\left([a,+\infty)\right)^{(n)}\right]}{n!}z^{n}=\sum_{n=0}^{\infty}\frac{\mathbf{1}}{n!}\left(z\int_{a}^{\infty}\lambda\left(y\right)\mathrm{d}y\right)^{n}$$
$$=\exp\left(z\int_{a}^{\infty}\lambda\left(y\right)\mathrm{d}y\right),$$

thus $R'_{\mathcal{G}_{\Phi([a,+\infty))}}=\infty$*. Then, by Proposition 4.3.28, the points of* Φ *may be sorted in the decreasing order* $X_{(1)}>X_{(2)}>\ldots$*, and for all* $x\in\mathbb{R}$*,*

$$\mathbf{P}\left(X_{(k)}\geq x\right)=\sum_{n=k}^{\infty}(-1)^{n-k}\binom{n-1}{k-1}\frac{1}{n!}\left(\int_{x}^{\infty}\lambda\left(y\right)\mathrm{d}y\right)^{n}.$$

Moreover, by Proposition 2.3.25, for all $n\in\mathbb{N}^{*}$*,* $M_{\Phi^{(n)}}$ *admits the following density with respect to the Lebesgue measure*

$$\rho_{n}\left(x_{1},\ldots,x_{n}\right)=\prod_{i=1}^{n}\lambda\left(x_{i}\right),\quad x_{1},\ldots,x_{n}\in\mathbb{R}.$$

Then $X_{(k)}$ *admits the probability density function given by, for all* $x\in\mathbb{R}$*,*

$$f\left(x\right)=\frac{\lambda\left(x\right)}{(k-1)!}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\left(\int_{x}^{\infty}\lambda\left(y\right)\right)^{k+n-1}$$
$$=\frac{\lambda\left(x\right)}{(k-1)!}\left(\int_{x}^{\infty}\lambda\left(y\right)\mathrm{d}y\right)^{k-1}\exp\left(-\int_{x}^{\infty}\lambda\left(y\right)\mathrm{d}y\right)$$

and $\left(X_{(1)},\ldots,X_{(k)}\right)$ *admits the probability density function given by, for all* $x_{1}>\ldots>x_{k}\in\mathbb{R}$*,*

$$f_{k}\left(x_{1},\ldots,x_{k}\right)=\left(\prod_{i=1}^{n}\lambda\left(x_{i}\right)\right)\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\left(\int_{x_{k}}^{\infty}\lambda\left(y\right)\mathrm{d}y\right)^{n}$$
$$=\prod_{i=1}^{n}\lambda\left(x_{i}\right)\exp\left(-\int_{x_{k}}^{\infty}\lambda\left(y\right)\mathrm{d}y\right).$$

Proposition 4.3.31. *Let* Φ *be a simple point process on* $\mathbb{R}$ *such that*

$$R_{\mathcal{G}_{\Phi((-\infty,a])}}>2,\quad \text{for all } a\in\mathbb{R},\tag{4.3.25}$$

(or, equivalently, $R'_{\mathcal{G}_{\Phi((-\infty,a])}}>1$*). Then the points of* Φ *may be sorted in the increasing order* $X_{(1)}<X_{(2)}<\ldots$*. Moreover, for all* $x\in\mathbb{R}$

$$\mathbf{P}\left(X_{(k)}\leq x\right)=\sum_{n=k}^{\infty}(-1)^{n-k}\binom{n-1}{k-1}\frac{1}{n!}M_{\Phi^{(n)}}\left((-\infty,x]^{n}\right).$$

If moreover for all $n \in \mathbb{N}^*$, $M_{\Phi^{(n)}}$ admits a density ρ_n with respect to the Lebesgue measure, then $X_{(k)}$ admits the following probability density function

$$f(x) = \frac{1}{(k-1)!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{(-\infty,x]^{k+n-1}}\rho_{k+n}(x,y_1\ldots,y_{k+n-1})\,\mathrm{d}y_1\ldots\mathrm{d}y_{k+n-1}$$

and $\left(X_{(1)},\ldots,X_{(k)}\right)$ admits the following probability density function

$$f_k(x_1,\ldots,x_k) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{(-\infty,x_k]^n}\rho_{k+n}(x_1,\ldots,x_k,y_1\ldots,y_n)\,\mathrm{d}y_1\ldots\mathrm{d}y_n,$$

for all $x_1 < \ldots < x_k \in \mathbb{R}$.

Proof. The proof follows the same lines as that of Proposition 4.3.28. Note that since $R_{\mathcal{G}_{\Phi((-\infty,a])}} > 1$ then, by Lemma 13.A.13, $\mathbf{E}[\Phi((-\infty,a])] < \infty$ which shows in particular that $\Phi((-\infty,a])$ is almost surely finite and therefore the points of Φ may be sorted in the increasing order. (i) **Distribution of** $X_{(k)}$. Let $x \in \mathbb{R}$. By analogy to (14.A.1), define for $n \in \mathbb{N}^*$ the n-th *symmetric sum* as

$$\mathcal{S}_n := \mathbf{E}\left[\sum_{\{X_1,\ldots,X_n\}\subset\Phi}\prod_{j=1}^{n}\mathbf{1}\{X_j \le x\}\right]$$

and let $\mathcal{S}_0 := 1$. Note first that

$$\mathcal{S}_n = \frac{1}{n!}\mathbf{E}\left[\sum_{(X_1,\ldots,X_n)\in\Phi^{(n)}}\prod_{j=1}^{n}\mathbf{1}\{X_j \le x\}\right] = \frac{1}{n!}M_{\Phi^{(n)}}\left((-\infty,x]^n\right).$$

Let $\tilde{\Phi}$ be the restriction of Φ to $(-\infty,x]$ which is a finite point process, say $\tilde{\Phi} = \sum_{j=1}^{m}\delta_{X_j}$ where m is an integer-valued random variable. Note that

$$\begin{aligned}\mathcal{S}_n &= \mathbf{E}\left[\sum_{J\subset\{1,\ldots,m\}:|J|=n}\prod_{j\in J}\mathbf{1}\{X_j \le x\}\right]\\ &= \mathbf{E}\left[\sum_{J\subset\{1,\ldots,m\}:|J|=n}\mathbf{1}\left(\bigcap_{j\in J}\{X_j \le x\}\right)\right].\end{aligned}$$

Given $\tilde{\Phi}$, we may apply Lemma 14.A.3 with $B_j = \{X_j \le x\}$ and

$$S_n = \sum_{J\subset\{1,\ldots,m\}:|J|=n}\mathbf{1}\left(\bigcap_{j\in J}\{X_j \le x\}\right).$$

Observe that $N = \sum_{i=1}^{m}\mathbf{1}\{X_j \le x\} \ge k$ iff $X_{(k)} \ge x$; thus Equation (14.A.2) writes

$$\mathbf{1}\left\{X_{(k)} \le x\right\} = \sum_{n=k}^{m}(-1)^{n-k}\binom{n-1}{k-1}S_n.$$

Taking the expectation of the above equality with respect to $\tilde{\Phi}$ implies

$$\begin{aligned}\mathbf{P}\left(X_{(k)}\leq x\right)&=\mathbf{E}\left[\sum_{n=k}^{\tilde{\Phi}((-\infty,x])}(-1)^{n-k}\binom{n-1}{k-1}S_n\right]\\&=\mathbf{E}\left[\sum_{n=k}^{\infty}(-1)^{n-k}\binom{n-1}{k-1}S_n\right],\end{aligned}$$

since if $n>\tilde{\Phi}\left((-\infty,x]\right)$, then $S_n=0$. Swapping the expectation and the sum in the right-hand side of the above equation gives the announced result. This is justified by the monotone convergence theorem and the fact that

$$\begin{aligned}\sum_{n=k}^{\infty}\binom{n-1}{k-1}\mathbf{E}\left[S_n\right]&=\sum_{n=k}^{\infty}\binom{n-1}{k-1}\mathcal{S}_n\\&=\sum_{n=k}^{\infty}\binom{n-1}{k-1}\frac{1}{n!}M_{\Phi^{(n)}}\left((-\infty,x]^n\right)\\&=\frac{1}{(k-1)!}\sum_{n=k}^{\infty}\frac{1}{n\,(n-k)!}M_{\Phi^{(n)}}\left((-\infty,x]^n\right)\\&\leq\frac{1}{(k-1)!}\sum_{n=k}^{\infty}\frac{1}{(n-k)!}M_{\Phi^{(n)}}\left((-\infty,x]^n\right)<\infty.\end{aligned}$$

Indeed, the above series is convergent since so is $\sum_{n=k}^{\infty}\frac{1}{n!}M_{\Phi^{(n)}}\left((-\infty,x]^n\right)=\sum_{n=k}^{\infty}\frac{1}{n!}\mathbf{E}\left[\left(\Phi\left(-\infty,x\right]\right)^{(n)}\right]$ which is finite by assumption (4.3.24). (ii) **Density of $X_{(k)}$**. The announced result follows from (i) and the fact that

$$\frac{\partial}{\partial x}\int_{(-\infty,x]^k}\rho_k\left(y_1\ldots,y_k\right)\mathrm{d}y_1\ldots\mathrm{d}y_k=k\int_{(-\infty,x]^{k-1}}\rho_k\left(x,y_1\ldots,y_k\right)\mathrm{d}y_1\ldots\mathrm{d}y_{k-1}.$$

(iii) **Density of $\left(X_{(1)},\ldots,X_{(k)}\right)$**. For all $x_1<\cdots<x_k\in\mathbb{R}$,

$$\begin{aligned}&\mathbf{P}\left(X_{(1)}\leq x_1<X_{(2)}\leq x_2<\ldots<X_{(k)}\leq x_k\right)\\&\quad=\mathbf{P}\left(\Phi\left((-\infty,x_1]\right)=1,\Phi\left((x_1,x_2]\right)=1,\ldots,\Phi\left((x_{k-1},x_k]\right)=1\right)\\&\quad=\mathbf{P}\left(\bigcap_{i=1}^{k}\Phi\left((x_{i-1},x_i]\right)=1\right),\end{aligned}$$

where $x_0:=-\infty$. Let J_k be the Janossy measures of the restriction of Φ to $(-\infty,x_k]$ which is a finite point process. Then by Corollary 4.3.9

$$\mathbf{P}\left(\bigcap_{i=1}^{k}\Phi\left((x_{i-1},x_i]\right)=1\right)=J_k\left(\prod_{i=1}^{k}(x_{i-1},x_i]\right).$$

Since $R_{\Phi(-\infty,x_k]} > 2$, then by Proposition 4.3.15

$$J_k\left(\prod_{i=1}^{k}(x_{i-1},x_i]\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}M_{\Phi^{(k+n)}}\left(\prod_{i=1}^{k}(x_{i-1},x_i]\times(-\infty,x_k]^n\right).$$

Since $M_{\Phi^{(n)}}$ admits a density ρ_n with respect to the n-dimensional Lebesgue measure, then

$$J_k\left(\prod_{i=1}^{k}(x_{i-1},x_i]\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{\prod_{i=1}^{k}(x_{i-1},x_i]\times(-\infty,x_k]^n}\rho_{k+n}(y_1,\ldots,y_{k+n})\,\mathrm{d}y_1\ldots\mathrm{d}y_{k+n}.$$

Then

$$\begin{aligned}&\mathbf{P}\left(\bigcap_{i=1}^{k}\mathbf{1}\left\{X_{(i)}\in(x_{i-1},x_i]\right\}\right)\\&=\int_{\prod_{i=1}^{k}(x_{i-1},x_i]}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{(-\infty,x_k]^n}\rho_{k+n}(y_1,\ldots,y_{k+n})\,\mathrm{d}y_{k+1}\ldots\mathrm{d}y_{k+n}\right)\mathrm{d}y_1\ldots\mathrm{d}y_k,\end{aligned}$$

therefore $\left(X_{(1)},\ldots,X_{(k)}\right)$ admits the probability density function given by

$$f_k(x_1,\ldots,x_k) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{(-\infty,x_k]^n}\rho_{k+n}(x_1,\ldots,x_k,y_1\ldots,y_n)\,\mathrm{d}y_1\ldots\mathrm{d}y_n.$$

□

Example 4.3.32. *Let Φ be a Poisson point process on $\mathbb{R}$ with intensity measure $M_\Phi(\mathrm{d}x) = \lambda(x)\,\mathrm{d}x$ where $\lambda:\mathbb{R}\to\mathbb{R}_+$ is integrable on $(-\infty,a]$ for all $a\in\mathbb{R}$. By Proposition 2.3.25, for all $n\in\mathbb{N}^*$,*

$$M_{\Phi^{(n)}}\left((-\infty,a]^n\right) = \left(\int_{-\infty}^{a}\lambda(y)\,\mathrm{d}y\right)^n.$$

On the other hand, by (14.E.6)

$$M_{\Phi^{(n)}}\left((-\infty,a]^n\right) = \mathbf{E}\left[\Phi((-\infty,a])^{(n)}\right].$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\mathbf{E}\left[\Phi((-\infty,a])^{(n)}\right]}{n!}z^n &= \sum_{n=0}^{\infty}\frac{\mathbf{1}}{n!}\left(z\int_{-\infty}^{a}\lambda(y)\,\mathrm{d}y\right)^n\\&=\exp\left(z\int_{-\infty}^{a}\lambda(y)\,\mathrm{d}y\right),\end{aligned}$$

thus $R'_{\mathcal{G}_{\Phi((-\infty,a])}} = \infty$. *Then, by Proposition 4.3.28, the points of* Φ *may be sorted in the increasing order* $X_{(1)} < X_{(2)} < \ldots$, *and for all* $x \in \mathbb{R}$

$$\mathbf{P}\left(X_{(k)} \geq x\right) = \sum_{n=k}^{\infty}(-1)^{n-k}\binom{n-1}{k-1}\frac{1}{n!}\left(\int_{-\infty}^{x}\lambda(y)\,\mathrm{d}y\right)^{n}.$$

Moreover, by Proposition 2.3.25, for all $n \in \mathbb{N}^*$, $M_{\Phi^{(n)}}$ *admits the following density with respect to the Lebesgue measure*

$$\rho_n(x_1,\ldots,x_n) = \prod_{i=1}^{n}\lambda(x_i), \quad x_1,\ldots,x_n \in \mathbb{R}.$$

Then $X_{(k)}$ *admits the probability density function given by*

$$\begin{aligned} f(x) &= \frac{\lambda(x)}{(k-1)!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_{-\infty}^{x}\lambda(y)\right)^{k+n-1} \\ &= \frac{\lambda(x)}{(k-1)!}\left(\int_{-\infty}^{x}\lambda(y)\,\mathrm{d}y\right)^{k-1}\exp\left(-\int_{-\infty}^{x}\lambda(y)\,\mathrm{d}y\right) \end{aligned}$$

for all $x \in \mathbb{R}$, *and* $\left(X_{(1)},\ldots,X_{(k)}\right)$ *admits the probability density function given by*

$$\begin{aligned} f_k(x_1,\ldots,x_k) &= \left(\prod_{i=1}^{n}\lambda(x_i)\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_{-\infty}^{x_k}\lambda(y)\,\mathrm{d}y\right)^{n} \\ &= \prod_{i=1}^{n}\lambda(x_i)\exp\left(-\int_{-\infty}^{x_k}\lambda(y)\,\mathrm{d}y\right), \end{aligned}$$

for all $x_1 > \ldots > x_k \in \mathbb{R}$.

4.4 Factorial moment expansion

4.4.1 Point processes on $\mathbb{R}$

Let Φ be a simple point process on $\mathbb{R}$, $\mathbb{M}_{\mathrm{s}}(\mathbb{R})$ be the *set of simple counting measures* on $\mathbb{R}$ and let $\psi : \mathbb{M}_{\mathrm{s}}(\mathbb{R}) \to \mathbb{R}$ be a measurable function. Assume that $\psi(\Phi)$ is integrable; i.e., $\mathbf{E}[|\psi(\Phi)|] < \infty$. We shall give an expansion of the expectation $\mathbf{E}[\psi(\Phi)]$ in terms of the factorial moment measures of Φ. To this aim, we need some preliminary notation and results.

For $\mu \in \mathbb{M}_{\mathrm{s}}(\mathbb{R})$, let $\mu|_x$ be its restriction to the subset $(-\infty, x)$; i.e.,

$$\mu|_x(B) = \mu(B \cap (-\infty, x)), \quad B \in \mathcal{B}(\mathbb{R}).$$

Definition 4.4.1. *A function* $\psi : \mathbb{M}_{\mathrm{s}}(\mathbb{R}) \to \mathbb{R}$ *is said to be* continuous *at* $\pm\infty$ *if for every* $\mu, \nu \in \mathbb{M}_{\mathrm{s}}(\mathbb{R})$

$$\lim_{x\to-\infty}\psi(\mu|_x + \nu) = \psi(\nu) \quad \textit{and} \quad \lim_{x\to+\infty}\psi(\mu|_x) = \psi(\mu). \tag{4.4.1}$$

Lemma 4.4.2. Telescoping formula in $\mathbb{R}$. *Let $\psi : \mathbb{M}_{\mathrm{s}}(\mathbb{R}) \to \mathbb{R}$ be continuous at $\pm\infty$. Then, for every $\mu \in \mathbb{M}_{\mathrm{s}}(\mathbb{R})$ written as $\mu = \sum_{k\in\mathbb{Z}:t_k\in\mu} \delta_{t_k}$ where the atoms t_k are enumerated increasingly with k, the following telescoping formula holds*

$$\psi(\mu) = \psi(0) + \sum_{k\in\mathbb{Z}:t_k\in\mu} [\psi(\mu|_{t_k} + \delta_{t_k}) - \psi(\mu|_{t_k})], \tag{4.4.2}$$

where 0 is the null measure.

Proof. Note that, for any $k \in \mathbb{Z}$, $\mu|_{t_k} = \sum_{j=-\infty}^{k-1} \delta_{t_j}$. Then, for any $K \in \mathbb{Z}$,

$$\sum_{k=-K}^{K} [\psi(\mu|_{t_k} + \delta_{t_k}) - \psi(\mu|_{t_k})] = \psi\left(\mu|_{t_{K+1}}\right) - \psi\left(\mu|_{t_{-K}}\right).$$

Letting $K \to +\infty$ in the above equality and invoking (4.4.1) concludes the proof. □

For any function $\psi : \mathbb{M}_{\mathrm{s}}(\mathbb{R}) \to \mathbb{R}$ and $x \in \mathbb{R}$, we define the *first order difference operator* $\psi_x^{(1)} : \mathbb{M}_{\mathrm{s}}(\mathbb{R}) \to \mathbb{R}$ by

$$\psi_x^{(1)}(\mu) = \psi(\mu|_x + \delta_x) - \psi(\mu|_x) \tag{4.4.3}$$

and, recursively, for any $n \in \mathbb{N}^*$ and $x_1, \ldots, x_n \in \mathbb{R}$, we define *$n$-th order difference operator* $\psi^{(n)}_{x_1,\ldots,x_n} : \mathbb{M}_{\mathrm{s}}(\mathbb{R}) \to \mathbb{R}$ by

$$\psi^{(n)}_{x_1,\ldots,x_n} = \left(\psi^{(n-1)}_{x_1,\ldots,x_{n-1}}\right)^{(1)}_{x_n}. \tag{4.4.4}$$

Example 4.4.3. Difference operators for linear functions. *If $\psi(\mu) = \int_{\mathbb{R}} f(x)\,\mu(\mathrm{d}x)$ for some measurable function $f : \mathbb{R} \to \mathbb{R}_+$, then*

$$\psi_x^{(1)}(\mu) = f(x), \quad \textit{and } \psi^{(n)}_{(x_1,\ldots,x_n)}(\mu) = 0, \ \textit{for } n \geq 2.$$

Lemma 4.4.4. *If a function $\psi : \mathbb{M}_{\mathrm{s}}(\mathbb{R}) \to \mathbb{R}$ is continuous at $\pm\infty$, then so is its n-th order difference operator $\psi_x^{(n)}$ for all $x \in \mathbb{R}^n$.*

Proof. If follows from (4.4.4) that its enough to make the proof for $n = 1$, the general result follows then by induction. Let $\mu, \nu \in \mathbb{M}_{\mathrm{s}}(\mathbb{R})$. We first show that, for all $x \in \mathbb{R}$,

$$\lim_{y\to+\infty} \psi_x^{(1)}(\mu|_y) = \psi_x^{(1)}(\mu).$$

This follows from the fact that, for $y > x$,

$$\begin{aligned}\psi_x^{(1)}(\mu|_y) &= \psi((\mu|_y)|_x + \delta_x) - \psi((\mu|_y)|_x)\\ &= \psi(\mu|_x + \delta_x) - \psi(\mu|_x)\\ &= \psi_x^{(1)}(\mu).\end{aligned}$$

We now show that

$$\lim_{y\to-\infty}\psi_x^{(1)}\left(\mu|_y+\nu\right)=\psi_x^{(1)}\left(\nu\right).$$

This follows from the fact that, for $y<x$,

$$\begin{aligned}\psi_x^{(1)}\left(\mu|_y+\nu\right)&=\psi\left(\left(\mu|_y+\nu\right)|_x+\delta_x\right)-\psi\left(\left(\mu|_y+\nu\right)|_x\right)\\&=\psi\left(\mu|_y+\nu|_x+\delta_x\right)-\psi\left(\mu|_y+\nu|_x\right),\end{aligned}$$

which, by the continuity of ψ, goes when $y\to-\infty$ to

$$\psi\left(\nu|_x+\delta_x\right)-\psi\left(\nu|_x\right)=\psi_x^{(1)}\left(\nu\right).$$

□

We now present an expansion of the expectation $\mathbf{E}\left[\psi\left(\Phi\right)\right]$ in terms of the factorial moment measures of Φ.

Theorem 4.4.5. *[12, Theorem 3.2]* Factorial moment expansion. *Let Φ be a simple point process on $\mathbb{R}$ and let $\psi:\mathbb{M}_{\mathrm{s}}\left(\mathbb{R}\right)\to\mathbb{R}$ be a measurable function which is continuous at $\pm\infty$. If for all $j\in\{1,\ldots,n+1\}$,*

$$\int_{\mathbb{R}^j}\mathbf{E}\left[\left|\psi_x^{(j)}\left(\Phi_x^!\right)\right|\right]M_{\Phi^{(j)}}\left(\mathrm{d}x\right)<\infty,\tag{4.4.5}$$

then

$$\mathbf{E}\left[\psi(\Phi)\right]=\psi\left(0\right)+\sum_{j=1}^{n}\int_{\mathbb{R}^j}\psi_x^{(j)}(0)M_{\Phi^{(j)}}\left(\mathrm{d}x\right)+\int_{\mathbb{R}^{n+1}}\mathbf{E}\left[\left|\psi_x^{(n+1)}\left(\Phi_x^!\right)\right|\right]M_{\Phi^{(n+1)}}\left(\mathrm{d}x\right).\tag{4.4.6}$$

Proof. (i) Cf. [7]. We prove first the announced result for $n=0$; that is under the condition

$$\int_{\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\left|\psi_x^{(1)}\left(\mu\right)\right|\mathbf{P}_\Phi^{!x}\left(\mathrm{d}\mu\right)M_\Phi\left(\mathrm{d}x\right)<\infty,\tag{4.4.7}$$

one has

$$\mathbf{E}\left[\psi(\Phi)\right]=\psi(0)+\int_{\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\psi_x^{(1)}\left(\mu\right)\mathbf{P}_\Phi^{!x}\left(\mathrm{d}\mu\right)M_\Phi\left(\mathrm{d}x\right).\tag{4.4.8}$$

Indeed, by the Campbell-Little-Mecke formula (3.3.4),

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\psi_x^{(1)}\left(\mu\right)\mathbf{P}_\Phi^{!x}\left(\mathrm{d}\mu\right)M_\Phi\left(\mathrm{d}x\right)&=\mathbf{E}\left[\int_{\mathbb{R}}\psi_x^{(1)}\left(\Phi-\delta_x\right)\Phi\left(\mathrm{d}x\right)\right]\\&=\mathbf{E}\left[\int_{\mathbb{R}}\psi_x^{(1)}\left(\Phi\right)\Phi\left(\mathrm{d}x\right)\right]\\&=\mathbf{E}\left[\int_{\mathbb{R}}\left[\psi\left(\Phi|_x+\delta_x\right)-\psi\left(\Phi|_x\right)\right]\Phi\left(\mathrm{d}x\right)\right]\\&=\mathbf{E}\left[\sum_{k\in\mathbb{Z}:X_k\in\Phi}\left[\psi\left(\Phi|_{X_k}+\delta_{X_k}\right)-\psi\left(\Phi|_{X_k}\right)\right]\right],\end{aligned}$$

where we assume that the atoms of Φ are enumerated in the increasing order. Similarly, the condition (4.4.7) is equivalent to

$$\mathbf{E}\left[\sum_{k\in\mathbb{Z}:X_k\in\Phi}\left|\psi\left(\Phi|_{X_k}+\delta_{X_k}\right)-\psi\left(\Phi|_{X_k}\right)\right|\right]<\infty.$$

Using the dominated convergence theorem and the telescoping formula (4.4.2), we deduce that

$$\mathbf{E}\left[\sum_{k\in\mathbb{Z}:X_k\in\Phi}\left[\psi\left(\Phi|_{X_k}+\delta_{X_k}\right)-\psi\left(\Phi|_{X_k}\right)\right]\right]=\mathbf{E}\left[\psi(\Phi)\right]-\psi(0),$$

which proves (4.4.8). (ii) Let $\Phi_x^!$ be a point process with distribution $\mathbf{P}_\Phi^{!x}$. For $j\in\{1,\ldots,n\}$, isolating for all $x\in\mathbb{R}^{j+1}$ its last coordinate and invoking Fubini-Tonelli theorem, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{j+1}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}|\psi_x^{(j+1)}(\mu)|\mathbf{P}_{\Phi_x^!}(\mathrm{d}\mu)\,M_{\Phi^{(j+1)}}(\mathrm{d}x)\\
&=\int_{\mathbb{R}^j\times\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}|\psi_{x,y}^{(j+1)}(\mu)|\mathbf{P}_{\Phi_{x,y}^!}(\mathrm{d}\mu)\,M_{\Phi^{(j+1)}}(\mathrm{d}x\times\mathrm{d}y)\\
&=\int_{\mathbb{R}^j}\left(\int_{\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}|\left(\psi_x^{(j)}\right)_y^{(1)}(\mu)|\mathbf{P}_{\Phi_x^!}^{!y}(\mathrm{d}\mu)\,M_{\Phi_x^!}(\mathrm{d}y)\right)M_{\Phi^{(j)}}(\mathrm{d}x),
\end{aligned}$$

where the second equality is due to (4.4.4), and Proposition 3.3.9. Then Condition (4.4.5) implies

$$\int_{\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}|\left(\psi_x^{(j)}\right)_y^{(1)}(\mu)|\mathbf{P}_{\Phi_x^!}^{!y}(\mathrm{d}\mu)\,M_{\Phi_x^!}(\mathrm{d}y)<\infty,$$

for $M_{\Phi^{(j)}}$-almost all $x\in\mathbb{R}^j$. Applying Item (i) to the function $\psi_x^{(j)}:\mathbb{M}_{\mathrm{s}}(\mathbb{R})\to\mathbb{R}$ and to the point process $\Phi_x^!$, we get

$$\mathbf{E}\left[\psi_x^{(j)}(\Phi_x^!)\right]=\psi_x^{(j)}(0)+\int_{\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\left(\psi_x^{(j)}\right)_y^{(1)}(\mu)\,\mathbf{P}_{\Phi_x^!}^{!y}(\mathrm{d}\mu)\,M_{\Phi_x^!}(\mathrm{d}y).$$

Integrating the above equality with respect to $M_{\Phi^{(j)}}$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^j}\mathbf{E}\left[\psi_x^{(j)}(\Phi_x^!)\right]M_{\Phi^{(j)}}(\mathrm{d}x)\\
&=\int_{\mathbb{R}^j}\psi_x^{(j)}(0)M_{\Phi^{(j)}}(\mathrm{d}x)+\int_{\mathbb{R}^j}\left(\int_{\mathbb{R}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\left(\psi_x^{(j)}\right)_y^{(1)}(\mu)\,\mathbf{P}_{\Phi_x^!}^{!y}(\mathrm{d}\mu)\,M_{\Phi_x^!}(\mathrm{d}y)\right)M_{\Phi^{(j)}}(\mathrm{d}x),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\int_{\mathbb{R}^j}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\psi_x^{(j)}(\mu)\mathbf{P}_{\Phi_x^!}(\mathrm{d}\mu)\,M_{\Phi^{(j)}}(\mathrm{d}x)\\
&=\int_{\mathbb{R}^j}\psi_x^{(j)}(0)M_{\Phi^{(j)}}(\mathrm{d}x)+\int_{\mathbb{R}^{j+1}}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\psi_x^{(j+1)}(\mu)\,\mathbf{P}_{\Phi_x^!}(\mathrm{d}\mu)\,M_{\Phi^{(j+1)}}(\mathrm{d}x),
\end{aligned}$$

where twe use (4.4.4), and invoke Proposition 3.3.9. Adding Equation (4.4.8) with the above equations for $j = 1, \ldots, n$ and we get (4.4.6). $\square$

The following lemma gives a useful explicit expression of n-th order difference operator $\psi_x^{(n)}$ in function of ψ for all $x \in \mathbb{R}^n$.

Lemma 4.4.6. *For any function $\psi : \mathbb{M}_s(\mathbb{R}) \to \mathbb{R}$, $n \in \mathbb{N}^*$, and $x_1, \ldots, x_n \in \mathbb{R}$, the function $\psi^{(n)}_{x_1,\ldots,x_n}$ has the following explicit expression*

$$\psi^{(n)}_{x_1,\ldots,x_n}(\mu) = \begin{cases} \sum_{j=0}^{n} (-1)^{n-j} \sum_{J \subset \binom{[n]}{j}} \psi\left(\mu|_{x_n} + \sum_{i \in J} \delta_{x_i}\right), & \text{if } x_n < \ldots < x_1, \\ 0, & \text{otherwise,} \end{cases} \tag{4.4.9}$$

where $\binom{[n]}{j}$ denotes the collection of all subsets of $\{1, \ldots, n\}$ of cardinality j.

Proof. The proof goes by induction with respect to n. The result holds for $n = 1$ by definition (4.4.3). Assume that it holds for $n - 1$, then by (4.4.4)

$$\begin{aligned} \psi^{(n)}_{x_1,\ldots,x_n}(\mu) &= \psi^{(n-1)}_{x_1,\ldots,x_{n-1}}(\mu|_{x_n} + \delta_{x_n}) - \psi^{(n-1)}_{x_1,\ldots,x_{n-1}}(\mu|_{x_n}) \\ &= \sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J \subset \binom{[n-1]}{j}} \psi\left((\mu|_{x_n} + \delta_{x_n})|_{x_{n-1}} + \sum_{i \in J} \delta_{x_i} \right) \\ &\quad - \sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J \subset \binom{[n-1]}{j}} \psi\left((\mu|_{x_n})|_{x_{n-1}} + \sum_{i \in J} \delta_{x_i} \right). \end{aligned}$$

If $x_n \geq x_{n-1}$ then $(\mu|_{x_n})|_{x_{n-1}} = \mu|_{x_{n-1}}$ and $(\mu|_{x_n} + \delta_{x_n})|_{x_{n-1}} = \mu|_{x_{n-1}}$, thus right-hand side of the above equation vanishes. Assume now that $x_n < x_{n-1}$ then

$$(\mu|_{x_n})|_{x_{n-1}} = \mu|_{x_n}, \quad \text{and } (\mu|_{x_n} + \delta_{x_n})|_{x_{n-1}} = \mu|_{x_n} + \delta_{x_n},$$

thus

$$\begin{aligned} \psi^{(n)}_{x_1,\ldots,x_n}(\mu) = &\sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J \subset \binom{[n-1]}{j}} \psi\left(\mu|_{x_n} + \delta_{x_n} + \sum_{i \in J} \delta_{x_i} \right) \\ &- \sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J \subset \binom{[n-1]}{j}} \psi\left(\mu|_{x_n} + \sum_{i \in J} \delta_{x_i} \right), \end{aligned}$$

which is the announced result where the first sum corresponds to all subsets of $\{1, \ldots, n\}$ of cardinality $j + 1$ containing n and the second sum corresponds to all subsets of $\{1, \ldots, n - 1\}$ of cardinality j (not containing n). $\square$

Example 4.4.7. Generating function expansion. *We aim to find an expansion of the generation function defined in Definition 1.6.18. Let Φ be a simple point* *process on $\mathbb{R}$ and let $v : \mathbb{R} \to [0,1]$ be a measurable function such that the support of $1-v$ is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ and let*

$$\psi(\mu) = \prod_{x\in\mu} v(x) = \exp\left(\int_{\mathbb{R}} \log[v(x)]\,\mu(\mathrm{d}x)\right), \quad \mu \in \mathbb{M}_{\mathrm{s}}(\mathbb{R}).$$

Then by (4.4.3), for $x \in \mathbb{R}$,

$$\begin{aligned}
\psi_x^{(1)}(\mu) &= \psi(\mu|_x + \delta_x) - \psi(\mu|_x) \\
&= \prod_{y\in\mu|_x+\delta_x} v(y) - \prod_{y\in\mu|_x} v(y) \\
&= \left(\prod_{y\in\mu|_x} v(y)\right)(v(x)-1)
\end{aligned}$$

and by (4.4.9), for $x_n < \ldots < x_1 \in \mathbb{R}$,

$$\begin{aligned}
\psi^{(n)}_{x_1,\ldots,x_n}(\mu) &= \sum_{j=0}^{n}(-1)^{n-j}\sum_{J\subset\binom{[n]}{j}} \psi\left(\mu|_{x_n} + \sum_{i\in J}\delta_{x_i}\right) \\
&= \sum_{j=0}^{n}(-1)^{n-j}\sum_{J\subset\binom{[n]}{j}} \left(\prod_{y\in\mu|_{x_n}} v(y)\right)\left(\prod_{i\in J} v(x_i)\right) \\
&= \left(\prod_{y\in\mu|_{x_n}} v(y)\right)\sum_{j=0}^{n}(-1)^{n-j}\sum_{J\subset\binom{[n]}{j}}\left(\prod_{i\in J} v(x_i)\right) \\
&= (-1)^n\left(\prod_{y\in\mu|_{x_n}} v(y)\right)\prod_{k=1}^{n}(1-v(x_k)).
\end{aligned}$$

If $M_{\Phi^{(n+1)}}$ is locally finite, then condition (4.4.5) is fulfilled and by Theorem 4.4.5 the generating function of Φ admits the following expansion

$$\begin{aligned}
&\mathcal{G}_\Phi(v) \\
&= \mathbf{E}[\psi(\Phi)] \\
&= 1 + \sum_{i=1}^{n}(-1)^i \int_{x_i<\ldots<x_1}\prod_{k=1}^{i}(1-v(x_k))M_{\Phi^{(i)}}(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_i) \\
&+ (-1)^{n+1}\int_{x_{n+1}<\ldots<x_1}\int_{\mathbb{M}_{\mathrm{s}}(\mathbb{R})}\left(\prod_{y\in\mu|_{x_{n+1}}} v(y)\right)\prod_{k=1}^{n+1}(1-v(x_k))\mathbf{P}^{!x}_\Phi(\mathrm{d}\mu) \\
&\qquad M_{\Phi^{(n+1)}}(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_{n+1}).
\end{aligned}$$

Note the analogy of the above expansion with (4.3.14).

4.4.2 General marked point processes

We will now extend Theorem 4.4.5 to general marked simple point process; this extension is due to [16].

Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{G})$ and $\mathcal{B}(\mathbb{K})$, respectively. Recall that we denote by $\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})$ the space of measures $\tilde{\mu}$ on $(\mathbb{G}\times\mathbb{K},\mathcal{B}(\mathbb{G})\otimes\mathcal{B}(\mathbb{K}))$ such that $\tilde{\mu}(B\times\mathbb{K})<\infty$ for all $B\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$; and by $\tilde{\mathcal{M}}(\mathbb{G}\times\mathbb{K})$ the σ-algebra on $\tilde{\mathbb{M}}(\mathbb{G}\times\mathbb{K})$ generated by the mappings $\tilde{\mu}\mapsto\tilde{\mu}(B\times K), B\in\mathcal{B}(\mathbb{G}), K\in\mathcal{B}(\mathbb{K})$ (cf. Section 2.2.6).

Let $\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})$ be the set of counting measures $\tilde{\mu}$ on $(\mathbb{G}\times\mathbb{K},\mathcal{B}(\mathbb{G})\otimes\mathcal{B}(\mathbb{K}))$ such that the projection $\mu(\cdot)=\tilde{\mu}(\cdot\times\mathbb{K})$ is simple. (Observe that $\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})\in\tilde{\mathcal{M}}(\mathbb{G}\times\mathbb{K})$; indeed, this follows from the fact that the projection operation $\tilde{\mu}\mapsto\mu$ is measurable and from Corollary 1.6.5).

Measurable order

We assume given a (total) *order* on $\mathbb{G}$; that is, a relation denoted by $\prec$ satisfying for all $x,y,z\in\mathbb{G}$: (i) if $x\neq y$, then either $x\prec y$ or $y\prec x$; and (ii) if $x\prec y\prec z$,then $x\prec z$. Moreover, we assume that the order $\prec$ is *measurable*; i.e., the following conditions are fulfilled:

(C1) For all $a\in\mathbb{G}$, the set $\{\prec a\}:=\{x\in\mathbb{G}:x\prec a\}$ is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$.

(C2) $\{(x,y)\in\mathbb{G}^2:x\prec y\}\in\mathcal{B}(\mathbb{G})\otimes\mathcal{B}(\mathbb{G})$.

(C3) The ordered decomposition of every $\mu\in\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})$

$$\mu=\sum_{k=1}^{J}\delta_{(t_k,z_k)},\quad J\in\bar{\mathbb{N}},t_1\prec t_2\prec\ldots \tag{4.4.10}$$

is measurable; i.e., for every $k\in\mathbb{N}^*$, the mapping $\mu\mapsto(t_k(\mu),z_k(\mu))$ defined on $\left\{\mu\in\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K}):J(\mu)\geq k\right\}$ is measurable.

Lemma 4.4.8. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h spaces. The mapping $(x,\mu)\mapsto\mu(\{\prec x\}\times\mathbb{K})$ defined on $\mathbb{G}\times\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})$ equipped with the algebra $\mathcal{B}(\mathbb{G})\otimes\tilde{\mathcal{M}}(\mathbb{G}\times\mathbb{K})$ is measurable.*

Proof. It is enough to show that for all $k\in\mathbb{N}^*$,

$$\{(x,\mu)\in\mathbb{G}\times\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K}):\mu(\{\prec x\}\times\mathbb{K})<k\}\in\mathcal{B}(\mathbb{G})\otimes\tilde{\mathcal{M}}(\mathbb{G}\times\mathbb{K}).$$

To this end, for all $k\in\mathbb{N}^*$, define the mappings T_k on $\mathbb{G}\times\{\mu\in\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K}):J(\mu)\geq k\}$ by

$$T_k(x,\mu)=(x,t_k(\mu))\in\mathbb{G}^2.$$

By C3, these mappings are $(\mathcal{B}(\mathbb{G})\otimes\tilde{\mathcal{M}}(\mathbb{G}\times\mathbb{K}),\mathcal{B}(\mathbb{G})\otimes\mathcal{B}(\mathbb{G}))$-measurable. Moreover, $\{(x,\mu)\in\mathbb{G}\times\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K}):\mu(\{\prec x\}\times\mathbb{K})<k\}$ equals $(T_k)^{-1}(\{(x,y):x\prec y\text{ or }x=y\})$. Thus, the announced result follows from C2 and the fact that the diagonal is a Borel subset of $\mathbb{G}^2$. □

We give now an example of measurable order in an arbitrary l.c.s.h. space $\mathbb{G}$.

Example 4.4.9. Measurable order in an arbitrary l.c.s.h space. *Recall the order $\prec$ constructed in the proof of Proposition 1.6.11. This order satisfies C1 obviously. It satisfies also C3 as shown in the proof of Proposition 1.6.11. It remains to prove C2. This follows from the fact that*

$$\{(x,y) \in \mathbb{G}^2 : x \prec y\} = \bigcup_{n\in\mathbb{N}} \bigcup_{i<j} K_{n,i} \times K_{n,j} .$$

The order described in Example 4.4.9 for general l.c.s.h. space $\mathbb{G}$ can be replaced by more explicit ones for particular cases of $\mathbb{G}$ as shown in the following example.

Example 4.4.10. Measurable orders in $\mathbb{R}^d$. *In $\mathbb{R}$, the strict inequality $<$ is a measurable order. In $\mathbb{R}^d$ with $d \geq 2$, there is no similar natural way of ordering the points. Here are some orders in $\mathbb{R}^d$:*

(i) Lexicographic order of polar coordinates. That is, order points in the increasing order of their distances to the origin and in case of equality, use the lexicographic order of the angular coordinates.

(ii) Sort points in the order that a growing d-cube hits them, and break the ties with the lexicographic order of Cartesian coordinates.

The above orders lead to different ways of enumerating points of counting measures as already said in Example 1.6.15.

Telescoping formula

For a given point $x \in \mathbb{G}$ and each measure $\mu \in \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K})$, we define the measure $\mu|_x$ as the restriction of μ to the set $\{\prec x\} \times \mathbb{K}$, i.e.,

$$\mu|_x(B) = \mu(B \cap (\{\prec x\} \times \mathbb{K})), \quad B \in \mathcal{B}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{K}) .$$

Lemma 4.4.11. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h spaces. The mapping defined on $\mathbb{G}\times\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K})$ by $(x,\mu) \mapsto \mu|_x$ is $(\mathcal{B}(\mathbb{G})\otimes\tilde{\mathcal{M}}(\mathbb{G} \times \mathbb{K}), \tilde{\mathcal{M}}(\mathbb{G} \times \mathbb{K}))$-measurable.*

Proof. The proof follows the same lines as that of Lemma 4.4.8. □

A mapping $\psi : \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K}) \to \mathbb{R}$ is said to be $\prec$-*continuous* at ∞ if

$$\lim_{x\uparrow\infty} \psi(\mu|_x) = \psi(\mu) \tag{4.4.11}$$

for every $\mu \in \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K})$, where $x \uparrow \infty$ denotes an *unbounded increase* with respect to $\prec$ (i.e., for every sequence $x_1 \prec x_2 \prec \ldots$ such that $\bigcup_{k=1}^{\infty}\{\prec x_k\} = \mathbb{G}$, we have $\lim_{k\to\infty} \psi(\mu|_{x_k}) = \psi(\mu)$).

Lemma 4.4.12. Telescoping formula in general space. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces and $\psi : \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K}) \to \mathbb{R}$ be $\prec$-continuous at ∞. Then, for every $\mu \in \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K})$ written as in (4.4.10), the following telescoping formula holds*

$$\psi(\mu) = \psi(0) + \sum_{k=1}^{J} \left[\psi(\mu|_{t_k} + \delta_{(t_k,z_k)}) - \psi(\mu|_{t_k})\right], \tag{4.4.12}$$

where 0 is the null measure.

Proof. It follows from (4.4.10) that, for any $k \leq J(\mu)$,

$$\mu|_{t_k} = \sum_{i=1}^{k-1} \delta_{(t_i,z_i)}.$$

Then, for any $K \leq J(\mu)$,

$$\sum_{k=1}^{K} \left[\psi(\mu|_{t_k} + \delta_{(t_k,z_k)}) - \psi(\mu|_{t_k})\right] = \psi\left(\sum_{i=1}^{K} \delta_{(t_i,z_i)}\right) - \psi(0).$$

If $J(\mu) < \infty$, applying the above equality with $K = J(\mu)$ gives the announced result. Assume now that $J(\mu) = \infty$. It follows that $\bigcup_{k=1}^{\infty}\{\prec t_k\} = \mathbb{G}$ (otherwise, by condition C1, the sequence $\{t_k\}_{k\in\mathbb{N}^*}$ would have an accumulation point which contradicts the fact that μ is locally finite). Then letting $K \to \infty$ in the above equality and invoking (4.4.11) concludes the proof. $\square$

Factorial moment expansion for marked point processes

For any function $\psi : \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K}) \to \mathbb{R}$ and $(x, z) \in \mathbb{G} \times \mathbb{K}$, we define the *first order difference operator* $\psi^{(1)}_{(x,z)} : \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K}) \to \mathbb{R}$ by

$$\psi^{(1)}_{(x,z)}(\mu) = \psi\left(\mu|_x + \delta_{(x,z)}\right) - \psi\left(\mu|_x\right) \tag{4.4.13}$$

and, recursively, for $n \in \mathbb{N}^*$ and $(x_1, z_1), \ldots, (x_n, z_n) \in \mathbb{G} \times \mathbb{K}$, we define the *$n$-th order difference operator* $\psi^{(n)}_{(x_1,z_1),\ldots,(x_n,z_n)} : \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K}) \to \mathbb{R}$ by

$$\psi^{(n)}_{(x_1,z_1),\ldots,(x_n,z_n)} = \left(\psi^{(n-1)}_{(x_1,z_1),\ldots,(x_{n-1},z_{n-1})}\right)_{(x_n,z_n)}. \tag{4.4.14}$$

In order to simplify the notation, we denote the $(\mathbb{G} \times \mathbb{K})^n$-valued vectors

$$((x_1, z_1), \ldots, (x_n, z_n))$$

by (x, z).

Lemma 4.4.13. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces. For any function $\psi : \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K}) \to \mathbb{R}$, any $(x, z) \in (\mathbb{G} \times \mathbb{K})^n$ and any $n \in \mathbb{N}^*$, the n-th order difference operator $\psi^{(n)}_{(x,z)}$ is $\prec$-continuous at ∞.*

Proof. If follows from (4.4.14) that its enough to make the proof for $n=1$, the general result follows then by induction. Let $\mu \in \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})$. We have to show that, for all $(x,z)\in\mathbb{G}\times\mathbb{K}$,

$$\lim_{y\uparrow\infty}\psi^{(1)}_{(x,z)}(\mu|_y)=\psi^{(1)}_{(x,z)}(\mu).$$

Using (4.4.13) it follows that, for all $y\in\mathbb{G}$ such that $x\prec y$,

$$\begin{aligned}\psi^{(1)}_{(x,z)}(\mu|_y)&=\psi\left((\mu|_y)|_x+\delta_{(x,z)}\right)-\psi\left((\mu|_y)|_x\right)\\&=\psi\left(\mu|_x+\delta_{(x,z)}\right)-\psi\left(\mu|_x\right)\\&=\psi^{(1)}_{(x,z)}\left(\mu\right).\end{aligned}$$

Then letting $y\uparrow\infty$ completes the proof. □

Theorem 4.4.14. *[16]* Factorial moment expansion for marked point processes. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces and Φ be a marked point process on $\mathbb{G}$ with marks in $\mathbb{K}$ such that that, $\mathbf{P}$-almost surely, $\Phi(\omega)\in\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})$. Let $\psi:\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})\to\mathbb{R}$ be a measurable function which is $\prec$-continuous at ∞. If*

$$\int_{(\mathbb{G}\times\mathbb{K})^j}\mathbf{E}\left[\left|\psi^{(j)}_{(x,z)}\left(\Phi^!_{(x,z)}\right)\right|\right]M_{\Phi^{(j)}}\left(\mathrm{d}x\times\mathrm{d}z\right)<\infty,\quad j\in\{1,\ldots,n+1\}\,,\tag{4.4.15}$$

then

$$\begin{aligned}\mathbf{E}\left[\psi(\Phi)\right]=\psi\left(0\right)&+\sum_{j=1}^{n}\int_{(\mathbb{G}\times\mathbb{K})^j}\psi^{(j)}_{(x,z)}(0)\ M_{\Phi^{(j)}}\left(\mathrm{d}x\times\mathrm{d}z\right)\\&+\int_{(\mathbb{G}\times\mathbb{K})^{n+1}}\mathbf{E}\left[\left|\psi^{(n+1)}_{(x,z)}\left(\Phi^!_{(x,z)}\right)\right|\right]M_{\Phi^{(n+1)}}\left(\mathrm{d}x\times\mathrm{d}z\right).\end{aligned}\tag{4.4.16}$$

Proof. (i) We prove first the announced result for $n=0$; that is under the condition

$$\int_{\mathbb{G}\times\mathbb{K}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}\left|\psi^{(1)}_{(x,z)}\left(\mu\right)\right|\mathbf{P}^{!(x,z)}_{\Phi}\left(\mathrm{d}\mu\right)M_{\Phi}\left(\mathrm{d}x\times\mathrm{d}z\right)<\infty,\tag{4.4.17}$$

one has

$$\mathbf{E}\left[\psi(\Phi)\right]=\psi(0)+\int_{\mathbb{G}\times\mathbb{K}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}\psi^{(1)}_{(x,z)}\left(\mu\right)\mathbf{P}^{!(x,z)}_{\Phi}\left(\mathrm{d}\mu\right)M_{\Phi}\left(\mathrm{d}x\times\mathrm{d}z\right).\tag{4.4.18}$$

Invoking Corollary 3.3.7, the integral in the right-hand side of the above equation equals

$$\begin{aligned}
&\mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} \psi^{(1)}_{(x,z)}\left(\Phi-\delta_{(x,z)}\right)\Phi\left(\mathrm{d}x\times\mathrm{d}z\right)\right]\\
&\qquad=\mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} \psi^{(1)}_{(x,z)}\left(\Phi\right)\Phi\left(\mathrm{d}x\times\mathrm{d}z\right)\right]\\
&\qquad=\mathbf{E}\left[\int_{\mathbb{G}\times\mathbb{K}} \left(\psi\left(\Phi|_x+\delta_{(x,z)}\right)-\psi\left(\Phi|_x\right)\right)\Phi\left(\mathrm{d}x\times\mathrm{d}z\right)\right]\\
&\qquad=\mathbf{E}\left[\sum_{k=1}^{J(\Phi)}\left[\psi(\Phi|_{t_k}+\delta_{(t_k,z_k)})-\psi(\Phi|_{t_k})\right]\right],
\end{aligned}$$

where the first equality is due to (4.4.13). Similarly, the condition (4.4.17) is equivalent to

$$\mathbf{E}\left[\sum_{k=1}^{J(\Phi)}\left|\psi(\Phi|_{t_k}+\delta_{(t_k,z_k)})-\psi(\Phi|_{t_k})\right|\right]<\infty.$$

Using the dominated convergence theorem and the telescoping formula (4.4.12), we deduce that

$$\mathbf{E}\left[\sum_{k=1}^{J(\Phi)}\left[\psi(\Phi|_{t_k}+\delta_{(t_k,z_k)})-\psi(\Phi|_{t_k})\right]\right]=\mathbf{E}\left[\psi(\Phi)\right]-\psi(0),$$

which proves (4.4.18). (ii) Let $\Phi^{!}_{(x,z)}$ be a point process with distribution $\mathbf{P}^{!(x,z)}_{\Phi}$. For $j\in\{1,\ldots,n\}$, isolating for all $(x,z)\in(\mathbb{G}\times\mathbb{K})^{j+1}$ its last coordinate, say (y,t), and invoking Fubini-Tonelli theorem, we get

$$\begin{aligned}
&\int_{(\mathbb{G}\times\mathbb{K})^{j+1}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}|\psi^{(j+1)}_{(x,z)}\left(\mu\right)|\mathbf{P}_{\Phi^{!}_{(x,z)}}\left(\mathrm{d}\mu\right)M_{\Phi^{(j+1)}}\left(\mathrm{d}x\times\mathrm{d}z\right)\\
&=\int_{(\mathbb{G}\times\mathbb{K})^{j}\times(\mathbb{G}\times\mathbb{K})}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}|\psi^{(j+1)}_{(x,z),(y,t)}\left(\mu\right)|\mathbf{P}_{\Phi^{!}_{(x,z),(y,t)}}\left(\mathrm{d}\mu\right)\\
&\qquad\qquad M_{\Phi^{(j+1)}}\left(\mathrm{d}x\times\mathrm{d}z\times\mathrm{d}y\times\mathrm{d}t\right)\\
&=\int_{(\mathbb{G}\times\mathbb{K})^{j}}\left(\int_{\mathbb{G}\times\mathbb{K}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}|\left(\psi^{(j)}_{(x,z)}\right)^{(1)}_{(y,t)}\left(\mu\right)|\mathbf{P}^{!(y,t)}_{\Phi^{!}_{(x,z)}}\left(\mathrm{d}\mu\right)M_{\Phi^{!}_{(x,z)}}\left(\mathrm{d}y\times\mathrm{d}t\right)\right)\\
&\qquad\qquad M_{\Phi^{(j)}}\left(\mathrm{d}x\times\mathrm{d}z\right),
\end{aligned}$$

where the second equality is due to (4.4.14) and to Proposition 3.3.9. Then Condition (4.4.15) implies

$$\int_{\mathbb{G}\times\mathbb{K}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}|\left(\psi^{(j)}_{(x,z)}\right)^{(1)}_{(y,t)}\left(\mu\right)|\mathbf{P}^{!(y,t)}_{\Phi^{!}_{(x,z)}}\left(\mathrm{d}\mu\right)M_{\Phi^{!}_{(x,z)}}\left(\mathrm{d}y\times\mathrm{d}t\right)<\infty,$$

for $M_{\Phi^{(j)}}$-almost all $(x,z) \in (\mathbb{G}\times\mathbb{K})^j$. Since ψ is $\prec$-continuous at ∞ then so is $\psi^{(j)}_{(x,z)}$ by Lemma 4.4.13. Applying Item (i) to the function $\psi^{(j)}_{(x,z)}$ and to the point process $\Phi^!_{(x,z)}$, we get

$$\mathbf{E}\left[\psi^{(j)}_{(x,z)}(\Phi^!_{(x,z)})\right] = \psi_{(x,z)}(0) + \int_{\mathbb{G}\times\mathbb{K}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}\left(\psi^{(j)}_{(x,z)}\right)_{(y,t)}(\mu)\,\mathbf{P}^{!(y,t)}_{\Phi^!_{(x,z)}}(\mathrm{d}\mu)\; M_{\Phi^!_{(x,z)}}(\mathrm{d}y\times\mathrm{d}t)\,.$$

Integrating the above equality with respect to $M_{\Phi^{(j)}}$ we obtain

$$\begin{aligned}&\int_{(\mathbb{G}\times\mathbb{K})^j}\mathbf{E}\left[\psi^{(j)}_{(x,z)}(\Phi^!_{(x,z)})\right]M_{\Phi^{(j)}}(\mathrm{d}x\times\mathrm{d}z)\\ &=\int_{(\mathbb{G}\times\mathbb{K})^j}\psi^{(j)}_{(x,z)}(0)M_{\Phi^{(j)}}(\mathrm{d}x\times\mathrm{d}z)\\ &+\int_{(\mathbb{G}\times\mathbb{K})^j}\left(\int_{\mathbb{G}\times\mathbb{K}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}\left(\psi^{(j)}_{(x,z)}\right)_{(y,t)}(\mu)\,\mathbf{P}^{!(y,t)}_{\Phi^!_{(x,z)}}(\mathrm{d}\mu)\,M_{\Phi^!_{(x,z)}}(\mathrm{d}y\times\mathrm{d}t)\right)\\ &\qquad M_{\Phi^{(j)}}(\mathrm{d}x\times\mathrm{d}z)\,,\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\int_{(\mathbb{G}\times\mathbb{K})^j}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}\psi^{(j)}_{(x,z)}(\mu)\mathbf{P}_{\Phi^!_{(x,z)}}(\mathrm{d}\mu)\,M_{\Phi^{(j)}}(\mathrm{d}x\times\mathrm{d}z)\\ &=\int_{(\mathbb{G}\times\mathbb{K})^j}\psi^{(j)}_{(x,z)}(0)M_{\Phi^{(j)}}(\mathrm{d}x\times\mathrm{d}z)+\int_{(\mathbb{G}\times\mathbb{K})^{j+1}}\int_{\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})}\psi^{(j+1)}_{(x,z)}(\mu)\,\mathbf{P}_{\Phi^!_{(x,z)}}(\mathrm{d}\mu)\\ &\qquad M_{\Phi^{(j+1)}}(\mathrm{d}x\times\mathrm{d}z)\,,\end{aligned}$$

where we use (4.4.14) and invoke Proposition 3.3.9. Adding Equation (4.4.18) to the above equations for $j=1,\ldots,n$, we get (4.4.16). $\square$

Explicit expression of the difference operators

The following lemma gives a useful explicit expression of n-th order difference operator $\psi^{(n)}_{(x,z)}$ in function of ψ for all $(x,z)\in(\mathbb{G}\times\mathbb{K})^n$.

Lemma 4.4.15. *For any function $\psi:\tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G}\times\mathbb{K})\to\mathbb{R}$, $n\in\mathbb{N}^*$, and $(x,z)\in(\mathbb{G}\times\mathbb{K})^n$, the function $\psi^{(n)}_{(x,z)}$ has the following explicit expression*

$$\psi^{(n)}_{(x,z)}(\mu)=\begin{cases}\sum_{j=0}^{n}(-1)^{n-j}\sum_{J\subset\binom{[n]}{j}}\psi\left(\mu|_{x_n}+\sum_{i\in J}\delta_{(x_i,z_i)}\right) & \text{if } x_n\prec\ldots\prec x_1\\ 0 & \text{otherwise}\end{cases}\tag{4.4.19}$$

where $\binom{[n]}{j}$ denotes the collection of all subsets of $\{1,\ldots,n\}$ of cardinality j.

Proof. The proof goes by induction with respect to n. The result holds for $n = 1$ by definition (4.4.13). Assume that it holds for $n - 1$, then by (4.4.14)

$$\begin{aligned}
&\psi^{(n)}_{(x_1,z_1),\ldots,(x_n,z_n)}(\mu) \\
&\quad = \psi^{(n-1)}_{(x_1,z_1),\ldots,(x_{n-1},z_{n-1})}\left(\mu|_{x_n} + \delta_{(x_n,z_n)}\right) - \psi^{(n-1)}_{(x_1,z_1),\ldots,(x_{n-1},z_n-1)}\left(\mu|_{x_n}\right) \\
&\quad = \sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J\subset\binom{[n-1]}{j}} \psi\left(\left(\mu|_{x_n} + \delta_{(x_n,z_n)}\right)|_{x_{n-1}} + \sum_{i\in J}\delta_{(x_i,z_i)}\right) \\
&\quad - \sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J\subset\binom{[n-1]}{j}} \psi\left(\left(\mu|_{x_n}\right)|_{x_{n-1}} + \sum_{i\in J}\delta_{(x_i,z_i)}\right).
\end{aligned}$$

If $x_{n-1} \prec x_n$ or $x_{n-1} = x_n$, then

$$\left(\mu|_{x_n}\right)|_{x_{n-1}} = \mu|_{x_{n-1}} \quad \text{and} \quad \left(\mu|_{x_n} + \delta_{(x_n,z_n)}\right)||_{x_{n-1}} = \mu|_{x_{n-1}}.$$

Thus the right-hand side of the above equation vanishes. Assume now that $x_n \prec x_{n-1}$ then

$$\left(\mu|_{x_n}\right)|_{x_{n-1}} = \mu|_{x_n}, \quad \text{and} \quad \left(\mu|_{x_n} + \delta_{(x_n,z_n)}\right)|_{x_{n-1}} = \mu|_{x_n} + \delta_{(x_n,z_n)}.$$

Thus

$$\begin{aligned}
\psi^{(n)}_{(x_1,z_1),\ldots,(x_n,z_n)}(\mu) &= \sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J\subset\binom{[n-1]}{j}} \psi\left(\mu|_{x_n} + \delta_{(x_n,z_n)} + \sum_{i\in J}\delta_{(x_i,z_i)}\right) \\
&\quad - \sum_{j=0}^{n-1} (-1)^{n-1-j} \sum_{J\subset\binom{[n-1]}{j}} \psi\left(\mu|_{x_n} + \sum_{i\in J}\delta_{(x_i,z_i)}\right),
\end{aligned}$$

which is the announced result where the first sum corresponds to all subsets of $\{1,\ldots,n\}$ of cardinality $j+1$ containing n and the second sum corresponds to all subsets of $\{1,\ldots,n-1\}$ of cardinality j (not containing n). $\square$

Expansion kernels

We will now rewrite the factorial moment expansion (4.4.16) in a more usual form. To this end, we introduce the following *expansion kernels*

$$D^{(n)}_{(x,z)}\psi(\mu) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{J\subset\binom{[n]}{j}} \psi\left(\mu|_{x_*} + \sum_{i\in J}\delta_{(x_i,z_i)}\right) \tag{4.4.20}$$

$$= \sum_{J\subset\{1,\ldots,n\}} (-1)^{n-|J|}\psi\left(\mu|_{x_*} + \sum_{i\in J}\delta_{(x_i,z_i)}\right), \tag{4.4.21}$$

where $\binom{[n]}{j}$ denotes the collection of all subsets of $\{1,\ldots,n\}$ of cardinality j and $x_* := \min\{x_1,\ldots,x_n\}$ with the minimum taken with respect to the order $\prec$. Note that $D^{(n)}_{(x,z)}\psi(\mu)$ is a *symmetric* function of (x,z); i.e., invariant with respect to a permutation of the components (x_i,z_i) of $(x,z) = ((x_1,z_1),\ldots,(x_n,z_n))$. Moreover, if $x_n \prec \ldots \prec x_1$, then

$$D^{(n)}_{(x,z)}\psi(\mu) = \psi^{(n)}_{(x,z)}(\mu).$$

Thus the expansion kernels are symmetric forms of the difference operators.

Example 4.4.16. First two expansion kernels. *Applying (4.4.20) for $n = 1,2$ respectively, we get*

$$D^{(1)}_{(x,z)}\psi(\mu) = \psi\left(\mu|_x + \delta_{(x,z)}\right) - \psi(\mu|_x), \tag{4.4.22}$$

$$\begin{aligned} D^{(2)}_{(x_1,z_1),(x_2,z_2)}\psi(\mu) &= \psi\left(\mu|_{x_*} + \delta_{(x_1,z_1)} + \delta_{(x_2,z_2)}\right) \\ &\quad - \psi\left(\mu|_{x_*} + \delta_{(x_1,z_1)}\right) - \psi\left(\mu|_{x_*} + \delta_{(x_2,z_2)}\right) + \psi(\mu|_{x_*}), \end{aligned}$$

where $x_* := \min\{x_1,x_2\}$.

Example 4.4.17. Expansion kernels at the null measure. *Applying (4.4.20) for $\mu = 0$ (the null measure), we get*

$$\begin{aligned} D^{(n)}_{(x,z)}\psi(0) &= \sum_{j=0}^{n}(-1)^{n-j}\sum_{J\subset\binom{[n]}{j}}\psi\left(\sum_{i\in J}\delta_{(x_i,z_i)}\right) \\ &= \sum_{J\subset\{1,\ldots,n\}}(-1)^{n-|J|}\psi\left(\sum_{i\in J}\delta_{(x_i,z_i)}\right). \end{aligned} \tag{4.4.23}$$

In particular,

$$D^{(1)}_{(x,z)}\psi(0) = \psi\left(\delta_{(x,z)}\right) - \psi(0), \tag{4.4.24}$$

$$D^{(2)}_{(x_1,z_1),(x_2,z_2)}\psi(0) = \psi\left(\delta_{(x_1,z_1)} + \delta_{(x_2,z_2)}\right) - \psi\left(\delta_{(x_1,z_1)}\right) - \psi\left(\delta_{(x_2,z_2)}\right) + \psi(0). \tag{4.4.25}$$

Example 4.4.18. Expansion kernels for linear functions. *Assume that*

$$\psi(\mu) = \int_{\mathbb{G}\times\mathbb{K}} f(x,z)\,\mu(\mathrm{d}x\times\mathrm{d}z)$$

for some measurable function $f : \mathbb{G}\times\mathbb{K}\to\mathbb{R}_+$. Then ψ is $\prec$-continuous at ∞ since

$$\lim_{x\uparrow\infty}\psi(\mu|_x) = \lim_{x\uparrow\infty}\int_{\{\prec x\}\times\mathbb{K}} f(t,z)\,\mu(\mathrm{d}t\times\mathrm{d}z) = \psi(\mu),$$

where the last equality follows from the continuity from below of measures. Moreover

$$D^{(1)}_{(x,z)}\psi(\mu) = f(x,z), \quad \text{and } D^{(n)}_{(x,z)}\psi(\mu) = 0, \text{ for } n\geq 2.$$

Factorial moment expansion over kernels

Corollary 4.4.19. Factorial moment expansion over kernels. *Under the conditions of Theorem 4.4.14,*

$$\mathbf{E}\left[\psi(\Phi)\right] = \psi\left(0\right) + \sum_{j=1}^{n} \frac{1}{j!} \int_{(\mathbb{G}\times\mathbb{K})^j} D_{(x,z)}^{(j)}\psi(0)\ M_{\Phi^{(j)}}\left(\mathrm{d}x \times \mathrm{d}z\right)$$
$$+ \frac{1}{(n+1)!} \int_{(\mathbb{G}\times\mathbb{K})^{n+1}} \mathbf{E}\left[D_{(x,z)}^{(n+1)}\psi\left(\Phi^!_{(x,z)}\right)\right] M_{\Phi^{(n+1)}}\left(\mathrm{d}x \times \mathrm{d}z\right).$$

Moreover, if Condition (4.4.15) holds for any $j \in \mathbb{N}^*$ and

$$\lim_{j\to\infty} \left(\frac{1}{j!} \int_{(\mathbb{G}\times\mathbb{K})^j} \mathbf{E}\left[D_{(x,z)}^{(j)}\psi\left(\Phi^!_{(x,z)}\right)\right] M_{\Phi^{(j)}}\left(\mathrm{d}x \times \mathrm{d}z\right)\right) = 0,$$

then

$$\mathbf{E}\left[\psi(\Phi)\right] = \psi\left(0\right) + \sum_{j=1}^{\infty} \frac{1}{j!} \int_{(\mathbb{G}\times\mathbb{K})^j} D_{(x,z)}^{(j)}\psi(0)\ M_{\Phi^{(j)}}\left(\mathrm{d}x \times \mathrm{d}z\right).$$

Proof. For any $j \in \mathbb{N}^*$, we will say that a function or a measure on $(\mathbb{G}\times\mathbb{K})^j$ is symmetric if it is invariant with respect to a permutation of the components (x_i, z_i) of $(x,z) = ((x_1,z_1),\ldots,(x_n,z_n))$. We have already observed that $D_{(x,z)}^{(j)}\psi\left(\mu\right)$ is a symmetric function of (x,z) which coincides with $\psi_{(x,z)}^{(j)}\left(\mu\right)$ when $x_j \prec \ldots \prec x_1$. Moreover, the function $(x,z) \mapsto \mathbf{P}_\Phi^{!(x,z)}$ and the measure $M_{\Phi^{(j)}}\left(\mathrm{d}x \times \mathrm{d}z\right)$ are also symmetric. Then

$$\int_{(\mathbb{G}\times\mathbb{K})^j} D_{(x,z)}^{(j)}\psi(\mu)\ \mathbf{P}_\Phi^{!(x,z)}\left(\mathrm{d}\mu\right) M_{\Phi^{(j)}}\left(\mathrm{d}x \times \mathrm{d}z\right)$$
$$= j! \int_{(\mathbb{G}\times\mathbb{K})^j} \psi_{(x,z)}^{(j)}(\mu)\ \mathbf{P}_\Phi^{!(x,z)}\left(\mathrm{d}\mu\right) M_{\Phi^{(j)}}\left(\mathrm{d}x \times \mathrm{d}z\right).$$

Invoking Theorem 4.4.14 concludes the proof. □

Corollary 4.4.20. Factorial moment expansion for Poisson. *In the conditions of Theorem 4.4.14, if moreover Φ is a Poisson point process on $\mathbb{G}\times\mathbb{K}$ with intensity measure Λ, then Condition (4.4.15) writes*

$$\int_{(\mathbb{G}\times\mathbb{K})^j} \mathbf{E}\left[\left|D_{(x,z)}^{(j)}\psi\left(\Phi\right)\right|\right] \Lambda^j\left(\mathrm{d}x \times \mathrm{d}z\right) < \infty, \quad j \in \{1,\ldots,n+1\}, \quad (4.4.26)$$

and (4.4.16) reads

$$\mathbf{E}\left[\psi(\Phi)\right] = \psi\left(0\right) + \sum_{j=1}^{n} \frac{1}{j!} \int_{(\mathbb{G}\times\mathbb{K})^j} D_{(x,z)}^{(j)}\psi(0)\ \Lambda^j\left(\mathrm{d}x \times \mathrm{d}z\right)$$
$$+ \frac{1}{(n+1)!} \int_{(\mathbb{G}\times\mathbb{K})^{n+1}} \mathbf{E}\left[D_{(x,z)}^{(n+1)}\psi\left(\Phi\right)\right] \Lambda^{n+1}\left(\mathrm{d}x \times \mathrm{d}z\right).$$

Proof. Since Φ is Poisson, it follows from Proposition 2.3.25 that for any $j \in \mathbb{N}^*$, $M_{\Phi^{(j)}} = \Lambda^j$. Moreover, by Slivnyak's theorem 3.2.4, $\Phi^!_{(x,z)} \stackrel{\text{dist.}}{=} \Phi$ for Λ^j-almost all $(x, z) \in (\mathbb{G} \times \mathbb{K})^j$. Then applying Corollary 4.4.19 concludes the proof. □

Example 4.4.21. Factorial moment expansion for homogeneous Poisson. *In the conditions of Theorem 4.4.14, if moreover Φ is a homogeneous Poisson point process on $\mathbb{G} \times \mathbb{K}$ with intensity λ, then Condition (4.4.15) writes*

$$\int_{(\mathbb{G}\times\mathbb{K})^j} \mathbf{E}\left[\left|D^{(j)}_{(x,z)}\psi(\Phi)\right|\right] \mathrm{d}x \times \mathrm{d}z < \infty, \quad j \in \{1, \ldots, n+1\}, \tag{4.4.27}$$

and (4.4.16) reads

$$\begin{aligned}\mathbf{E}[\psi(\Phi)] = \psi(0) &+ \sum_{j=1}^{n} \frac{\lambda^j}{j!} \int_{(\mathbb{G}\times\mathbb{K})^j} D^{(j)}_{(x,z)}\psi(0)\, \mathrm{d}x \times \mathrm{d}z \\ &+ \frac{\lambda^{n+1}}{(n+1)!} \int_{(\mathbb{G}\times\mathbb{K})^{n+1}} \mathbf{E}\left[D^{(n+1)}_{(x,z)}\psi(\Phi)\right] \mathrm{d}x \times \mathrm{d}z.\end{aligned} \tag{4.4.28}$$

In this case, for all $j \in \{1, \ldots, n\}$,

$$\left.\frac{\mathrm{d}^j \mathbf{E}[\psi(\Phi)]}{\mathrm{d}\lambda^j}\right|_{\lambda=0} = \int_{(\mathbb{G}\times\mathbb{K})^j} D^{(j)}_{(x,z)}\psi(0)\, \mathrm{d}x \times \mathrm{d}z.$$

That is, considering $\mathbf{E}[\psi(\Phi)]$ as a function of the intensity λ, the above formula gives its successive derivatives at $\lambda = 0$.

4.4.3 Shot-noise functions

We will study in the present section the following particular function

$$\psi(\mu) = g(\mu(f)), \quad \mu \in \tilde{\mathbb{M}}_{\mathrm{s}}(\mathbb{G} \times \mathbb{K}), \tag{4.4.29}$$

for some given functions $f : \mathbb{G} \times \mathbb{K} \to \mathbb{R}_+$ assumed measurable and $g : \mathbb{R}_+ \to \mathbb{R}$.

Lemma 4.4.22. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h spaces and ψ be defined by (4.4.29). If the function g is continuous, then ψ is $\prec$-continuous at ∞.*

Proof. We have shown in Example 4.4.18 that the mapping $\mu \mapsto \mu(f)$ is $\prec$-continuous at ∞, then so is ψ. □

We will give sufficient conditions for (4.4.15) to hold for the above function ψ (by extending the results of [39]).

Proposition 4.4.23. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h spaces and ψ be defined by (4.4.29). Then for any $n \in \mathbb{N}^*$,*

$$\left|D^{(n)}_{(x,z)}\psi(\mu)\right| \le 2^n \|g\|_\infty, \tag{4.4.30}$$

where $\|g\|_\infty := \sup_{x\in\mathbb{R}_+}|g(x)|$. *Moreover, if the function* g *is* k *times differentiable for some* $k \in \mathbb{N}^*$, *then for any* $n \geq k$ *and any injective function* $\sigma : \{1,\ldots,k\} \to \{1,\ldots,n\}$,

$$\left|D^{(n)}_{(x,z)}\psi(\mu)\right| \leq 2^{n-k}\left\|g^{(k)}\right\|_\infty \prod_{i=1}^{k} f\left(x_{\sigma(i)}, z_{\sigma(i)}\right), \tag{4.4.31}$$

where $g^{(k)}$ *is the* k*-th order derivative of* g.

Proof. In order to give the basic idea, we first prove (4.4.31) for $n = k = 1$. By (4.4.22)

$$\begin{aligned} D^{(1)}_{(x,z)}\psi(\mu) &= \psi\left(\mu|_x + \delta_{(x,z)}\right) - \psi(\mu|_x) \\ &= g\left(\mu|_x(f) + f((x,z))\right) - g\left(\mu|_x(f)\right) \\ &= f(x,z)\int_0^1 g'\left(\mu|_x(f) + \tau f((x,z))\right)\mathrm{d}\tau. \end{aligned}$$

Then

$$\left|D^{(1)}_{(x,z)}\psi(\mu)\right| \leq \|g'\|_\infty\ f(x,z).$$

Consider now the general case $n \geq 1$. The expression (4.4.21) of the expansion kernel may be written as follows

$$D^{(n)}_{(x,z)}\psi(\mu) = \sum_{(b_1,\ldots,b_n)\in\mathcal{P}_n}(-1)^{n-\sum_{i=1}^n b_i}g\left(\mu|_{x_*}(f) + \sum_{i=1}^n b_i f(x_i,z_i)\right),$$

where $\mathcal{P}_n$ denotes the set of all n-tuples $(b_1,\ldots,b_n) \in \{0,1\}^n$. Since $\mathcal{P}_n$ has cardinality 2^n, the inequality (4.4.30) follows immediately. Let $\beta := \mu|_{x_*}(f)$ and $\gamma_i := f(x_i,z_i)$, then

$$D^{(n)}_{(x,z)}\psi(\mu) = \sum_{(b_1,\ldots,b_n)\in\mathcal{P}_n}(-1)^{\sum_{i=1}^n(1-b_i)}g\left(\beta + \sum_{i=1}^n b_i\gamma_i\right).$$

We now partition the set $\mathcal{P}_n$ into 2^{n-k} subsets where each subset A corresponds to some fixed $(b_{k+1},\ldots,b_n) \in \mathcal{P}_{n-k}$; that is

$$A(b_{k+1},\ldots,b_n) = \{(b_1,\ldots,b_k,b_{k+1},\ldots,b_n) : (b_1,\ldots,b_k) \in \mathcal{P}_k\}.$$

So we have

$$D^{(n)}_{(x,z)}\psi(\mu) = \sum_{(v_1,\ldots,v_{n-k})\in\mathcal{P}_{n-k}}(-1)^{\sum_{i=1}^{n-k}(1-v_i)}H(A(v_1,\ldots,v_{n-k})), \tag{4.4.32}$$

where

$$H(A(v_1,\ldots,v_{n-k})) = \sum_{(b_1,\ldots,b_k)\in\mathcal{P}_k}(-1)^{\sum_{i=1}^k(1-b_i)}g\left(\beta + \sum_{i=1}^{n-k}v_i\gamma_{i+k} + \sum_{i=1}^k b_i\gamma_i\right).$$

Taking the absolute value of (4.4.32), we obtain

$$|D^{(n)}_{(x,z)}\psi(\mu)| \leq \sum_{(v_1,\dots,v_{n-k})\in\mathcal{P}_{n-k}} |H(v_1,\dots,v_{n-k})|. \tag{4.4.33}$$

Let

$$G(\theta_1,\dots,\theta_k) := g^{(k)}\left(\beta + \sum_{i=1}^{n-k} v_i z_{i+k} + \sum_{i=1}^{k} \theta_i\gamma_i\right)\prod_{i=1}^{k}\gamma_i.$$

Integrating the above function respectively with respect to $\theta_1,\dots,\theta_k \in [0,1]$ gives

$$\int_0^1\cdots\int_0^1 G(\theta_1,\dots,\theta_k)\mathrm{d}\theta_1\dots\mathrm{d}\theta_k = H(A(v_1,\dots,v_{n-k})),$$

and hence

$$|H(A(v_1,\dots,v_{n-k}))| \leq \int_0^1\cdots\int_0^1 |G(\theta_1,\dots,\theta_k)|\mathrm{d}\theta_1\dots\mathrm{d}\theta_k \leq \left\|g^{(k)}\right\|_\infty \prod_{i=1}^{k}\gamma_i.$$

Substituting the above inequality into (4.4.33), we get

$$|D^{(n)}_{(x,z)}\psi(\mu)| \leq \sum_{(v_1,\dots,v_{n-k})\in\mathcal{P}_{n-k}} \left\|g^{(k)}\right\|_\infty \prod_{i=1}^{k}\gamma_i = 2^{n-k}\left\|g^{(k)}\right\|_\infty \prod_{i=1}^{k}\gamma_i,$$

which proves (4.4.31) for $\sigma(i) = i$, $1 \leq i \leq k$. Since $D^{(n)}_{(x,z)}\psi(\mu)$ is a symmetric function of (x,z); the inequality (4.4.31) holds for any injective function $\sigma : \{1,\dots,k\} \to \{1,\dots,n\}$. □

Corollary 4.4.24. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces and ψ be defined by (4.4.29). If the function g is n times differentiable for some $n \in \mathbb{N}^*$, then*

$$\left|D^{(n)}_{(x,z)}\psi(\mu)\right| \leq a_n \prod_{i=1}^{n}\min\left(1, f\left(x_i, z_i\right)\right),$$

where

$$a_n = \max\left\{2^{n-k}\left\|g^{(k)}\right\|_\infty : k = 0,\dots,n\right\}.$$

Proof. This follows from Proposition 4.4.23 and the fact that $\min(1,a)\min(1,b) = \min(1,a,b,ab)$. □

Corollary 4.4.25. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h spaces and ψ be defined by (4.4.29) where the function $g : \mathbb{R}_+ \to \mathbb{R}$ is assumed to be $n+1$ times differentiable with bounded derivatives up to order $n+1$ (for some $n \in \mathbb{N}$). Then a sufficient condition for (4.4.15) to holds is*

$$\int_{(\mathbb{G}\times\mathbb{K})^j} \prod_{i=1}^{j}\min\left(1, f\left(x_i, z_i\right)\right) M_{\Phi^{(j)}}(\mathrm{d}x \times \mathrm{d}z) < \infty, \quad j \in \{1,\dots,n+1\}.$$

When Φ is a Poisson point process, the above inequality simplifies to

$$\int_{\mathbb{G}\times\mathbb{K}} \min\left(1, f(x,z)\right) M_\Phi(\mathrm{d}x \times \mathrm{d}z) < \infty. \tag{4.4.34}$$

Proof. The result for a general point process follows from Corollary 4.4.24. In the particular case of a Poisson point process, the factorial moment measure $M_{\Phi^{(j)}}$ has the product form (2.3.18) which concludes the proof. $\square$

Example 4.4.26. Shannon capacity expansion in Poisson wireless networks. *We model the base stations (BS) locations in a wireless network with a homogeneous Poisson point process $\Phi = \sum_{n \in \mathbb{N}} \delta_{X_n}$ on $\mathbb{R}^2$ with intensity λ. We define the* interference *by*

$$I = \sum_{n \in \mathbb{N}} f(X_n) = f(\Phi),$$

where $f(x) = |x|^{-\beta}$ for some constant $\beta > 2$; cf. Examples 2.6.8 and 2.6.11. Consider the Shannon capacity defined by

$$g(I) = \log_2 \left(1 + \frac{S}{N + I}\right),$$

where S and N are two positive constants (representing respectively the received power and noise power). Let ψ be defined by (4.4.29); then the above Shannon capacity equals

$$\psi(\Phi) = g(f(\Phi)).$$

We aim to find an expansion of $\mathbf{E}[\psi(\Phi)]$ in function of λ.

The function g is continuous on $\mathbb{R}_+$,then ψ is $\prec$-continuous at ∞ by Lemma 4.4.22. Note that

$$|g(x)| \leq g(0) < \infty, \quad x \in \mathbb{R}_+.$$

Moreover,

$$g'(x) = \frac{-S}{(N+x)^2 \log\left(1 + \frac{S}{N+x}\right) \log 2}, \quad x \in \mathbb{R}_+,$$

is bounded since $\lim_{x \to \infty} g'(x) = 0$. Similarly, g is infinitely differentiable with bounded derivatives on $\mathbb{R}_+$. The left-hand side of (4.4.34) equals

$$\begin{aligned} 2\pi\lambda \int_0^\infty \min(1, f(r))\, r\mathrm{d}r &= 2\pi\lambda \int_0^\infty \min\left(1, r^{-\beta}\right) r\mathrm{d}r \\ &= 2\pi\lambda \left(1 + \int_1^\infty r^{1-\beta}\mathrm{d}r\right) \\ &= 2\pi\lambda \frac{\beta - 1}{\beta - 2} < \infty. \end{aligned}$$

Then by Corollary 4.4.25, the condition (4.4.15) holds for and $n \in \mathbb{N}$. Moreover, by (4.4.24)

$$D_x^{(1)} \psi(0) = g(f(x)) - g(0),$$

and by (4.4.25)

$$D_{x,y}^{(2)} \psi(0) = g(f(x) + f(y)) - g(f(x)) - g(f(y)) + g(0).$$

Therefore, by (4.4.28)

$$\mathbf{E}\left[\psi\left(\Phi\right)\right]=\log_2\left(1+\frac{S}{N}\right)+2\pi\lambda\int_0^\infty\left[g\left(f\left(r\right)\right)-g\left(0\right)\right]\; r\mathrm{d}r$$
$$+\pi\lambda^2\int_0^\infty\int_0^\infty\left[g\left(f\left(r\right)+f\left(s\right)\right)-g\left(f\left(r\right)\right)-g\left(f\left(s\right)\right)+g\left(0\right)\right]\; rs\mathrm{d}r\mathrm{d}s+o\left(\lambda^2\right).$$

In particular,

$$\left.\frac{\mathrm{d}\mathbf{E}\left[\psi(\Phi)\right]}{\mathrm{d}\lambda}\right|_{\lambda=0}=2\pi\int_0^\infty\left[g\left(f\left(r\right)\right)-g\left(0\right)\right]\; r\mathrm{d}r$$
$$=2\pi\int_0^\infty\left[\log_2\left(1+\frac{S}{N+f\left(r\right)}\right)-\log_2\left(1+\frac{S}{N}\right)\right]\; r\mathrm{d}r, \tag{4.4.35}$$

which integrates the effect of a single interferer relatively to the case where there is no interference.

4.5 Further examples

4.5.1 For Section 4.2

Example 4.5.1. Gamma random measure. *Let Φ be a random measure on a l.c.s.h. space $\mathbb{G}$ such that for any pairwise disjoint sets $B_1,\ldots,B_k\in\mathcal{B}\left(\mathbb{G}\right)$, the random variables $\Phi\left(B_1\right),\ldots,\Phi\left(B_k\right)$ are independent Gamma random variables with respective Laplace transforms*

$$\mathcal{L}_{\Phi(B_j)}\left(t\right)=\mathbf{E}\left[e^{-t\Phi(B_j)}\right]=\left(1+\lambda t\right)^{-\alpha(B_j)}, \tag{4.5.1}$$

where $\lambda\in\mathbb{R}_+^*$ and α is a given locally finite measure on $\mathbb{G}$.

Consider first a simple function $f=\sum_{j=1}^n a_j\mathbf{1}_{B_j}$ where $a_1,\ldots,a_n\geq 0$ and $B_1,\ldots,B_n\in\mathcal{B}\left(\mathbb{G}\right)$ are pairwise disjoint. Then

$$\begin{aligned}\mathbf{E}\left[\exp\left(-\int_{\mathbb{G}}f\mathrm{d}\Phi\right)\right]&=\mathbf{E}\left[\exp\left(-\sum_{j=1}^n a_j\Phi(B_j)\right)\right]\\&=\prod_{j=1}^n\mathbf{E}\left[e^{-a_j\Phi(B_j)}\right]\\&=\prod_{j=1}^n\left(1+\lambda a_j\right)^{-\alpha(B_j)}\\&=\exp\left[-\sum_{j=1}^n\log\left(1+\lambda a_j\right)\alpha\left(B_j\right)\right]\\&=\exp\left(-\int_{\mathbb{G}}\log\left(1+\lambda f\left(x\right)\right)\alpha\left(\mathrm{d}x\right)\right).\end{aligned}$$

If f is a general measurable nonnegative function on $\mathbb{G}$, there exists an increasing sequence of simple functions converging to it; and therefore the monotone convergence theorem shows that the Laplace transform of Φ equals

$$\mathcal{L}_\Phi(f) = \exp\left(-\int_{\mathbb{G}} \log(1+\lambda f(x))\,\alpha(\mathrm{d}x)\right),$$

for all measurable $f : \mathbb{G} \to \mathbb{R}_+$. Using the expansion

$$\log(1+u) = -\sum_{r=1}^{n} \frac{(-u)^r}{r} + o(u^n)$$

and the dominated convergence theorem, we deduce that for any measurable $f : \mathbb{G} \to \mathbb{R}_+$ which is bounded with support in $\mathcal{B}_c(\mathbb{G})$

$$\mathcal{L}_\Phi(tf) = \sum_{r=1}^{n} \frac{(-t)^r}{r} \lambda^r \int_{\mathbb{G}} f(x)^r \alpha(\mathrm{d}x) + o(t^n).$$

Comparing the above equation with (4.2.4) shows that

$$\int_{\mathbb{G}^r} f(x_1)\ldots f(x_r)\, C_r(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r) = (r-1)!\lambda^r \int_{\mathbb{G}} f(x)^r \alpha(\mathrm{d}x).$$

Thus the cumulant measures of a Gamma random measure are given by

$$C_r(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r) = (r-1)!\lambda^r \alpha(\mathrm{d}x_1)\,\delta_{x_1}(\mathrm{d}x_2)\ldots\delta_{x_1}(\mathrm{d}x_r).$$

Therefore, the mean measure of Φ equals

$$M_\Phi(\mathrm{d}x) = C_\Phi(\mathrm{d}x) = \lambda\alpha(\mathrm{d}x)$$

and its second moment measure follows from (4.1.4)

$$\begin{aligned} M_{\Phi^2}(\mathrm{d}x \times \mathrm{d}y) &= C_2(\mathrm{d}x \times \mathrm{d}y) + M_\Phi(\mathrm{d}x)\, M_\Phi(\mathrm{d}y) \\ &= \lambda^2 \alpha(\mathrm{d}x)\,\delta_x(\mathrm{d}y) + \lambda^2 \alpha(\mathrm{d}x)\,\alpha(\mathrm{d}y). \end{aligned}$$

(In particular, for all $B \in \mathcal{B}(\mathbb{G})$, $\mathbf{E}[\Phi(B)] = \lambda\alpha(B)$ and $\mathrm{var}[\Phi(B)] = \lambda^2\alpha(B)$ as expected for a Gamma random variable.)

Example 4.5.2. Dirichlet random measure. *Let Φ be a Gamma random measure as in Example 4.5.1 where the measure α is assumed non-null and finite. Since $\mathbf{E}[\Phi(\mathbb{G})] = \lambda\alpha(\mathbb{G}) < \infty$, then $\Phi(\mathbb{G})$ is almost surely finite. On the other hand,*

$$\begin{aligned} \mathbf{P}(\Phi(\mathbb{G}) = 0) &= \lim_{t\uparrow\infty} \mathbf{E}[e^{-t\Phi(\mathbb{G})}] \\ &= \lim_{t\uparrow\infty} (1+\lambda t)^{-\alpha(\mathbb{G})} = 0, \end{aligned}$$

where the first equality is a general property of nonnegative random variables and the second one follows from (4.5.1). Then, $\tilde{\Phi}$ *defined by*

$$\tilde{\Phi}(B) = \Phi(B)/\Phi(\mathbb{G}), \quad B \in \mathcal{B}(\mathbb{G})$$

is a random measure by Proposition 1.1.7(iii), called a Dirichlet random measure.

Example 4.5.3. Compound Poisson point process. *Let* $\bar{\Phi}$ *be a compound Poisson point process as in Example 2.3.21 with Poisson parent process* Φ *and descendant processes* $\{\Phi_x\}_{x\in\mathbb{G}}$ *with* $\Phi_x = Z_x\delta_0$*. Its generating function is given by Equation (2.3.17)*

$$\mathcal{G}_{\bar{\Phi}}(v) = \exp\left[-\int_{\mathbb{G}} (1 - \mathcal{G}_{Z_x}(v(x))) M_\Phi(\mathrm{d}x)\right].$$

Therefore, for $1 - h \in \mathcal{V}(\mathbb{G})$ *and* $\rho \in (0,1)$*,*

$$\log \mathcal{G}_{\bar{\Phi}}(1-\rho h) = \int_{\mathbb{G}} (\mathcal{G}_{Z_x}(1 - \rho h(x)) - 1) M_\Phi(\mathrm{d}x)$$

Assume that $K = \sup_{x\in\mathbb{G}} \mathbf{E}[Z_x^n] < \infty$*, then it follows from Lemma 13.A.21 that*

$$\mathcal{G}_{Z_x}(y) = 1 + \sum_{r=1}^{n} \mathbf{E}\left[Z_x^{(r)}\right] \frac{(y-1)^r}{r!} + \frac{(y-1)^n}{n!}\varepsilon_{x,n}(y), \quad \forall y \in \mathbb{R}, |y| \leq 1,$$

where $|\varepsilon_{x,n}(y)| \leq 2K$ *and* $\lim_{y\uparrow 1} \varepsilon_{x,n}(y) = 0$*. Combining the above two equalities, we get*

$$\begin{aligned}\log \mathcal{G}_{\bar{\Phi}}(1-\rho h) = &\sum_{r=1}^{n} \frac{(-\rho)^r}{r!} \int_{\mathbb{G}} \mathbf{E}\left[Z_x^{(r)}\right] h(x)^r M_\Phi(\mathrm{d}x) \\ &+ \frac{(-\rho)^n}{n!} \int_{\mathbb{G}} h(x)^n \varepsilon_{x,n}(1 - \rho h(x)) M_\Phi(\mathrm{d}x).\end{aligned}$$

Assuming that $\int_{\mathbb{G}} h(x)^n M_\Phi(\mathrm{d}x) < \infty$*, then it follows from the dominated convergence theorem that*

$$\lim_{\rho\downarrow 0} \int_{\mathbb{G}} h(x)^n \varepsilon_{x,n}(1 - \rho h(x)) M_\Phi(\mathrm{d}x) = 0,$$

thus

$$\log \mathcal{G}_{\bar{\Phi}}(1-\rho h) = \sum_{r=1}^{n} \frac{(-\rho)^r}{r!} \int_{\mathbb{G}} \mathbf{E}\left[Z_x^{(r)}\right] h(x)^r M_\Phi(\mathrm{d}x) + o(\rho^n),$$

which compared to (4.2.7) implies that the factorial cumulant measures of $\bar{\Phi}$ *are given by*

$$C_{(r)}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r) = \mathbf{E}\left[Z_{x_1}^{(r)}\right] M_\Phi(\mathrm{d}x_1)\,\delta_{x_1}(\mathrm{d}x_2)\ldots\delta_{x_1}(\mathrm{d}x_r). \quad (4.5.2)$$

Therefore the r-th factorial cumulant measure of a compound Poisson point process is concentrated on the diagonal where it reduces to the intensity measure of the parent process Φ multiplied by the r-th factorial moment of the cluster size.

Example 4.5.4. Negative binomial distributions. *Let Φ be a mixed Poisson point process as in Example 2.3.7; that is a Cox point process Φ directed by $\Lambda = X\mu$ where X has the gamma probability distribution with shape α and scale λ. Its generating function is given by (2.3.5); that is*

$$\mathcal{G}_\Phi(v) = \left(1 + \lambda \int_{\mathbb{G}} (1-v)\,\mathrm{d}\mu\right)^{-\alpha}, \quad v \in \mathcal{V}(\mathbb{G}).$$

Thus for $1 - h \in \mathcal{V}(\mathbb{G})$ such that $\int_{\mathbb{G}} h(x)\,\mu(\mathrm{d}x) < \infty$ and $\rho \in (0,1)$,

$$\begin{aligned}\log \mathcal{G}_\Phi(1-\rho h) &= -\alpha \log\left(1 + \lambda\rho \int_{\mathbb{G}} h(x)\,\mu(\mathrm{d}x)\right)\\ &= \alpha \sum_{r=1}^{n} \frac{\left[-\lambda\rho \int_{\mathbb{G}} h(x)\,\mu(\mathrm{d}x)\right]^r}{r} + o(\rho^n),\end{aligned}$$

which compared to (4.2.7) implies that the factorial cumulant measures of Φ are given by

$$C_{(r)}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_r) = \alpha(r-1)!\lambda^r \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_r).$$

4.5.2 For Section 4.4

Example 4.5.5. Palm-Khinchin equations. *Consider $\psi(\mu) = \mathbf{1}\{\mu((0,t)) \geq k\}$ for some $t > 0$, $k \geq 0$. Then by (4.4.3)*

$$\begin{aligned}\psi_x^{(1)}(\mu) &= \psi(\mu|_x + \delta_x) - \psi(\mu|_x)\\ &= [\mathbf{1}\{\mu((0,x)) \geq k-1\} - \mathbf{1}\{\mu((0,x)) \geq k\}]\,\mathbf{1}\{0 < x < t\}\end{aligned}$$

and by (4.4.9)

$$\begin{aligned}\psi_{x_1,x_2}^{(2)}(\mu) &= \psi(\mu|_{x_2} + \delta_{x_1} + \delta_{x_2}) - \psi(\mu|_{x_2} + \delta_{x_1}) - \psi(\mu|_{x_2} + \delta_{x_2}) + \psi(\mu|_{x_2})\\ &= [\mathbf{1}\{\mu((0,x_2)) \geq k-2\} - 2 \times \mathbf{1}\{\mu((0,x_2)) \geq k-1\}\\ &\quad + \mathbf{1}\{\mu((0,x_2)) \geq k\}]\mathbf{1}\{0 < x_2 < x_1 < t\}.\end{aligned}$$

Now Theorem 4.4.5 with $n = 0$ for a simple stationary point process Φ with intensity λ gives

$$\mathbf{P}(\Phi((0,t)) \geq k) = \mathbf{1}\{k=0\} + \int_0^t \left[\mathbf{P}\left(\Phi_x^!((0,x)) \geq k-1\right) - \mathbf{P}\left(\Phi_x^!((0,x)) \geq k\right)\right]\lambda \mathrm{d}x,$$

or with $n = 1$,

$$
\begin{aligned}
&\mathbf{P}\left(\Phi\left((0,t)\right) \geq k\right) \\
&= \mathbf{1}\left\{k = 0\right\} + \lambda t[\mathbf{1}\left\{k \leq 1\right\} - 1(k = 0)] \\
&+ \int_0^t \int_{x_2}^t \left[\mathbf{P}\left(\Phi^!_{x_1,x_2}\left((0,x_2)\right) \geq k-2\right) - 2\mathbf{P}\left(\Phi^!_{x_1,x_2}\left((0,x_2)\right) \geq k-1\right)\right. \\
&\qquad \left. + \mathbf{P}\left(\Phi^!_{x_1,x_2}\left((0,x_2)\right) \geq k\right)\right] M_{\Phi^{(2)}}\left(\mathrm{d}x_1 \times \mathrm{d}x_2\right).
\end{aligned}
$$

The above two equations are called the Palm-Khinchin equations.

Example 4.5.6. Expansion kernels for shot-noise exponential. *Let* $\mathbb{G}$ *and* $\mathbb{K}$ *be two l.c.s.h. spaces. Consider a measurable function* $f : \mathbb{G} \times \mathbb{K} \to \mathbb{R}_+$ *and let*

$$
\psi\left(\mu\right) = e^{-\mu(f)}, \quad \mu \in \tilde{\mathbb{M}}_{\mathrm{s}}\left(\mathbb{G} \times \mathbb{K}\right), \tag{4.5.3}
$$

where $\mu\left(f\right) = \int_{\mathbb{G}\times\mathbb{K}} f\left(x,z\right) \; \mu\left(\mathrm{d}x \times \mathrm{d}z\right)$. *We have shown in Example 4.4.18 that the mapping* $\mu \mapsto \mu\left(f\right)$ *is* $\prec$*-continuous at* ∞, *then so is* ψ.

Moreover, applying (4.4.23), we get

$$
\begin{aligned}
D^{(n)}_{(x,z)}\psi\left(0\right) &= \sum_{J\subset\{1,\ldots,n\}} \left(-1\right)^{n-|J|} e^{-\sum_{i\in J} f(x_i,z_i)} \\
&= \sum_{J\subset\{1,\ldots,n\}} \left(-1\right)^{n-|J|} \prod_{i\in J} e^{-f(x_i,z_i)} \\
&= \prod_{k=1}^n \left(e^{-f(x_k,z_k)} - 1\right).
\end{aligned} \tag{4.5.4}
$$

Moreover, since $\psi\left(\mu+\nu\right) = \psi\left(\mu\right)\psi\left(\nu\right)$, *it follows from (4.4.20) that*

$$
D^{(n)}_{(x,z)}\psi\left(\mu\right) = \psi\left(\mu|_{x_*}\right) \times D^{(n)}_{(x,z)}\psi\left(0\right).
$$

In particular,

$$
\left|D^{(n)}_{(x,z)}\psi\left(\mu\right)\right| \leq \prod_{k=1}^n \left(1 - e^{-f(x_k,z_k)}\right). \tag{4.5.5}
$$

Example 4.5.7. Poisson Laplace transform expansion. *Let* $\mathbb{G}$ *and* $\mathbb{K}$ *be two l.c.s.h. spaces,* Φ *be a Poisson point process on* $\mathbb{G} \times \mathbb{K}$ *with intensity measure* Λ, *and let* ψ *be given by (4.5.3) for some measurable function* $f : \mathbb{G} \times \mathbb{K} \to \mathbb{R}_+$. *Assume that*

$$
a := \int_{\mathbb{G}\times\mathbb{K}} \left(1 - e^{-f(x,z)}\right) \Lambda\left(\mathrm{d}x \times \mathrm{d}z\right) < \infty.
$$

For any $j \in \mathbb{N}^*$., *the left-hand side of Condition (4.4.26) writes*

$$
\begin{aligned}
\int_{(\mathbb{G}\times\mathbb{K})^j} \mathbf{E}\left[\left|D^{(j)}_{(x,z)}\psi\left(\Phi\right)\right|\right] \Lambda^j\left(\mathrm{d}x \times \mathrm{d}z\right) &\leq \int_{(\mathbb{G}\times\mathbb{K})^j} \prod_{k=1}^j \left(1 - e^{-f(x_k,z_k)}\right) \Lambda^j\left(\mathrm{d}x \times \mathrm{d}z\right) \\
&= a^j < \infty,
\end{aligned}
$$

where the first inequality is due to (4.5.5). The above inequality shows also that

$$\frac{1}{j!}\int_{(\mathbb{G}\times\mathbb{K})^j}\mathbf{E}\left[\left|D^{(j)}_{(x,z)}\psi\left(\Phi\right)\right|\right]\Lambda^j\left(\mathrm{d}x\times\mathrm{d}z\right)\leq\frac{a^j}{j!}\to 0,\quad \text{as } j\to\infty.$$

Moreover, invoking (4.5.4) we get

$$\int_{(\mathbb{G}\times\mathbb{K})^j}D^{(j)}_{(x,z)}\psi(0)\,\Lambda^j\left(\mathrm{d}x\times\mathrm{d}z\right)=\int_{(\mathbb{G}\times\mathbb{K})^j}\left[\prod_{k=1}^{j}\left(e^{-f(x_k,z_k)}-1\right)\right]\Lambda^j\left(\mathrm{d}x\times\mathrm{d}z\right)=\left(-1\right)^j a^j.$$

It follows from Corollary 4.4.20 that

$$\mathbf{E}\left[\psi(\Phi)\right]=1+\sum_{j=1}^{\infty}\frac{\left(-1\right)^j}{j!}a^j=e^{-a}.$$

As expected, we retrieve the expression (2.1.1) of the Laplace transform of a Poisson point process.

Example 4.5.8. Shannon capacity derivative for isolated points point process. *The context is the same as Example 4.4.26, expect that the the base stations (BS) locations are modelled by the following Matérn I hard-core point process (cf. Example 3.4.1)*

$$\Phi_1=\sum_{k\in\mathbb{Z}}\delta_{X_k}\mathbf{1}\left\{\Phi\left(B\left(X_k,h\right)\right)=1\right\},$$

where $\Phi=\sum_{k\in\mathbb{Z}}\delta_{X_k}$ is a hmogeneous Poisson point process on $\mathbb{R}^2$ with intensity λ and $h>0$ is a given constant. Since

$$\Phi_1^{(j)}\left(B\right)\leq\Phi^{(j)}\left(B\right),\quad \text{for all } j\in\mathbb{N}, B\in\mathcal{B}\left(\mathbb{G}\right)^{\otimes j},$$

then for all $j\in\mathbb{N}$,

$$\begin{aligned}
&\int_{(\mathbb{R}^2\times\mathbb{K})^j}\prod_{i=1}^{j}\min\left(1,f\left(x_i,z_i\right)\right)M_{\Phi_1^{(j)}}\left(\mathrm{d}x\times\mathrm{d}z\right)\\
&\quad\leq\int_{(\mathbb{R}^2\times\mathbb{K})^j}\prod_{i=1}^{j}\min\left(1,f\left(x_i,z_i\right)\right)M_{\Phi^{(j)}}\left(\mathrm{d}x\times\mathrm{d}z\right)\\
&\quad=\left(2\pi\lambda\int_0^\infty\min\left(1,f\left(r\right)\right)r\mathrm{d}r\right)^j<\infty.
\end{aligned}$$

It follows from Corollary 4.4.25 that the condition (4.4.15) holds for and $n\in\mathbb{N}$. Recall that the mean measure of Φ_1 is given by (3.4.1)

$$\mathbf{E}\left[\Phi_1\left(\mathrm{d}x\right)\right]=\lambda e^{-\lambda\pi h^2}\mathrm{d}x.$$

Then the first order derivative of $\mathbf{E}\left[\psi(\Phi_1)\right]$ at $\lambda=0$ is the same as in the Poisson case (4.4.35). For the second order derivative, we need the second order factorial moment measure which will be given in Example 7.4.4 below.

4.6 Exercises

4.6.1 For Section 4.1

Exercise 4.6.1. No aligned points in homogeneous Poisson point processes. *Consider a homogeneous Poisson point process Φ on $\mathbb{R}^2$ with intensity $\lambda \in \mathbb{R}_+^*$. Show that* $\mathbf{P}$*-almost surely:*

1. *Φ has not three aligned points.*

2. *Φ has not four points on the same circle.*

Solution 4.6.1. *Recall that the n-th factorial moment measure of Φ is given by (2.3.18).*

1. *Observe that the probability that there exists a 3-tuple of aligned points is smaller than the expectation of the number of such 3-tuples. A 3-tuple $X, Y, Z \in \Phi$ is aligned iff Z is on the straight line containing X and Y, which we write $Z \in (X, Y)$. Thus*

$$\begin{aligned}
&\mathbf{P}\left(\{\exists 3\text{-tuple of aligned points in } \Phi\}\right)\\
&\quad \leq \mathbf{E}[\operatorname{card}(\{3\text{-tuples of aligned points in } \Phi\})]\\
&\quad = \mathbf{E}\left[\sum_{(X,Y,Z)\in\Phi^{(3)}} \mathbf{1}\left\{Z \in (X,Y)\right\}\right]\\
&\quad = \int_{(\mathbb{R}^2)^3} \mathbf{1}\left\{z \in (x,y)\right\} M_{\Phi^{(3)}}\left(\mathrm{d}x \times \mathrm{d}y \times \mathrm{d}z\right)\\
&\quad = \lambda^3 \int_{(\mathbb{R}^2)^2} \left(\int_{\mathbb{R}^2} \mathbf{1}\left\{z \in (x,y)\right\} \mathrm{d}z\right) \mathrm{d}x\mathrm{d}y = 0,
\end{aligned}$$

   where the third line is due to Campbell's averaging formula (1.2.2) and the fourth one is due to (2.3.18).

2. *Analogously, denoting the circumscribed circle of (X, Y, Z) by $\mathcal{C}(X, Y, Z)$,*

$$\begin{aligned}
&\mathbf{P}\left(\{\exists 4 \text{ points of } \Phi \text{ on the same circle}\}\right)\\
&\quad \leq \mathbf{E}[\operatorname{card}(\{4\text{-tuple in } \Phi \text{ on the same circle}\})]\\
&\quad = \mathbf{E}\left[\sum_{(X,Y,Z,T)\in\Phi^{(3)}} \mathbf{1}\left\{T \in \mathcal{C}(X,Y,Z)\right\}\right]\\
&\quad = \lambda^4 \int_{(\mathbb{R}^2)^4} \mathbf{1}\left\{t \in \mathcal{C}(X,Y,Z)\right\} \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t = 0.
\end{aligned}$$

Exercise 4.6.2. Moment measures of independently marked point processes. *Let $\mathbb{G}$ and $\mathbb{K}$ be two l.c.s.h. spaces equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{G})$ and $\mathcal{B}(\mathbb{K})$ respectively. Let $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z_k)}$ be an independently marked point* *process on $\mathbb{G} \times \mathbb{K}$ associated to the ground process $\Phi = \sum_{k \in \mathbb{Z}} \delta_{X_k}$, on $\mathbb{G}$ through the probability kernel $\tilde{p}(\cdot, \cdot)$. Show that, for all $n \in \mathbb{N}^*$, the n-th moment measure of $\tilde{\Phi}$ is given by*

$$M_{\tilde{\Phi}^n}(\mathrm{d}x_1 \times \mathrm{d}z_1 \times \ldots \times \mathrm{d}x_n \times \mathrm{d}z_n) = M_{\Phi^n}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n) \prod_{i=1}^n \tilde{p}(x_i, \mathrm{d}z_i).$$

Solution 4.6.2. *For all $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{G}), K_1, \ldots, K_n \in \mathcal{B}(\mathbb{K})$,*

$$\begin{aligned}
&M_{\tilde{\Phi}^n}(B_1 \times K_1 \times \cdots \times B_n \times K_n) \\
&\quad = \mathbf{E}\left[\tilde{\Phi}^n (B_1 \times K_1 \times \cdots \times B_n \times K_n)\right] \\
&\quad = \mathbf{E}\left[\sum_{k_1, \ldots, k_n \in \mathbb{Z}} \prod_{i=1}^n \mathbf{1}\{X_{k_i} \in B_i, Z_{k_i} \in K_i\}\right] \\
&\quad = \mathbf{E}\left[\mathbf{E}\left[\sum_{k_1, \ldots, k_n \in \mathbb{Z}} \prod_{i=1}^n \mathbf{1}\{X_{k_i} \in B_i, Z_{k_i} \in K_i\} \,\middle|\, \Phi\right]\right] \\
&\quad = \mathbf{E}\left[\sum_{k_1, \ldots, k_n \in \mathbb{Z}} \prod_{i=1}^n \mathbf{1}\{X_{k_i} \in B_i\} \tilde{p}(X_{k_i}, K_i)\right] \\
&\quad = \mathbf{E}\left[\int_{\mathbb{G}^n} \left(\prod_{i=1}^n \mathbf{1}\{x_i \in B_i\} \tilde{p}(x_i, K_i)\right) \Phi^n (\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n)\right] \\
&\quad = \int_{\mathbb{G}^n} \left(\prod_{i=1}^n \mathbf{1}\{x_i \in B_i\} \tilde{p}(x_i, K_i)\right) M_{\Phi^n}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_n),
\end{aligned}$$

where the last equality follows from Campbell's averaging formula 1.2.2 for the point process Φ^n.

Exercise 4.6.3. Poisson shot-noise third moment. *Let $f_i : \mathbb{G} \to \mathbb{R}_+$ be measurable ($i \in \{1, 2, 3\}$) and Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$. Express the expectation $\mathbf{E}[\Phi(f_1)\Phi(f_2)\Phi(f_3)]$ as sum of integrals with respect to M_Φ of products of the functions f_i. Deduce the expression of the shot-noise third moment $\mathbf{E}\left[\Phi(f)^3\right]$ for all measurable $f : \mathbb{G} \to \mathbb{R}_+$.*

Solution 4.6.3. *Applying Campbell averaging formula (1.2.2) to the point process Φ^3 we get*

$$\mathbf{E}[\Phi(f_1)\Phi(f_2)\Phi(f_3)] = \int_{\mathbb{G}^3} f_1(x_1) f_2(x_2) f_3(x_3) M_{\Phi^3}(\mathrm{d}x_1 \times \mathrm{d}x_2 \times \mathrm{d}x_3).$$

On the other hand,

$$\begin{aligned}M_{\Phi^3}(\mathrm{d}x_1 \times \mathrm{d}x_2 \times \mathrm{d}x_3) &= C_3(\mathrm{d}x_1 \times \mathrm{d}x_2 \times \mathrm{d}x_3) \\ &+ \sum\nolimits^* C_2(\mathrm{d}x_1 \times \mathrm{d}x_2) \times C_\Phi(\mathrm{d}x_3) \\ &+ C_\Phi(\mathrm{d}x_1) C_\Phi(\mathrm{d}x_2) C_\Phi(\mathrm{d}x_3) \\ &= M_\Phi(\mathrm{d}x_1)\delta_{x_1}(\mathrm{d}x_2)\delta_{x_1}(\mathrm{d}x_3) \\ &+ \sum\nolimits^* M_\Phi(\mathrm{d}x_1)\delta_{x_1}(\mathrm{d}x_2) M_\Phi(\mathrm{d}x_3) \\ &+ M_\Phi(\mathrm{d}x_1) M_\Phi(\mathrm{d}x_2) M_\Phi(\mathrm{d}x_3),\end{aligned}$$

where the first equality follows from (4.1.3) with $\sum^*$ denotes the sum of the three terms of the same type, and the second equality follows from (4.2.5). Therefore,

$$\begin{aligned}\mathbf{E}[\Phi(f_1)\Phi(f_2)\Phi(f_3)] &= M_\Phi(f_1 f_2 f_3) \\ &+ M_\Phi(f_1 f_2) M_\Phi(f_3) + M_\Phi(f_1 f_3) M_\Phi(f_2) + M_\Phi(f_2 f_3) M_\Phi(f_1) \\ &+ M_\Phi(f_1) M_\Phi(f_2) M_\Phi(f_3).\end{aligned}$$

In particular,

$$\mathbf{E}\left[\Phi(f)^3\right] = M_\Phi\left(f^3\right) + 3M_\Phi\left(f^2\right) M_\Phi(f) + M_\Phi(f)^3.$$

4.6.2 For Section 4.2

Exercise 4.6.4. Factorial moment and cumulant measures of Poisson point processes. *Consider a Poisson point process Φ on a l.c.s.h. space $\mathbb{G}$.*

1. *Use the generating function expansion (4.2.6) to prove Proposition 2.3.25; that is the factorial moment measures are $M_{\Phi^{(r)}} = (M_\Phi)^r$.*
2. *Use the expansion (4.2.7) to prove that the factorial cumulant measures are $C_{(1)} = M_\Phi$ and $C_{(r)} = 0, \forall r \geq 2$. (This result was already obtained in Example 4.1.13.)*

Solution 4.6.4. *It follows from Example 4.2.3 that all the moment measures of Φ are locally finite.*

1. The generating function of Φ is given by (2.1.3); thus for a measurable function $h : \mathbb{G} \to [0,1]$ whose support is in $\mathcal{B}_c(\mathbb{G})$ and $\rho \in (0,1)$,

$$\mathcal{G}_\Phi(1-\rho h) = \exp\left[-\rho \int_{\mathbb{G}} h \mathrm{d}M_\Phi\right] = \sum_{r=0}^{\infty} \frac{(-\rho)^r}{r!}\left(\int_{\mathbb{G}} h \mathrm{d}M_\Phi\right)^r,$$

which compared to (4.2.6) implies $M_{\Phi^{(r)}} = (M_\Phi)^r$.

2. Moreover,

$$\log \mathcal{G}_\Phi(1-\rho h) = -\rho \int_{\mathbb{G}} h \mathrm{d}M_\Phi,$$

thus, by the expansion (4.2.7), $C_{(1)} = M_\Phi$ and $C_{(r)} = 0, \forall r \geq 2$.

Exercise 4.6.5. Second moment measure of the mixed Binomial point process. *Let $\Phi = \sum_{j=1}^{N} \delta_{X_j}$ be a mixed Binomial point process as in Example 2.2.28. Let λ be the probability distribution of the atom X_1. Assume that $\mathbf{E}\left[N^2\right] < \infty$.*

1. *Using the expansion (4.2.3) of the Laplace transform (2.2.13) of Φ, check that the mean measure of Φ is*

$$M_\Phi(\mathrm{d}x) = \mathbf{E}[N]\,\lambda(\mathrm{d}x)$$

which was already proved in (2.2.12); and show that the second moment measures of Φ is

$$M_{\Phi^2}(\mathrm{d}x \times \mathrm{d}y) = \mathbf{E}[N]\,\lambda(\mathrm{d}x)\,\delta_x(\mathrm{d}y) + \mathbf{E}\left[N^{(2)}\right]\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y).$$

2. *Deduce that the second moment measure, the second cumulant measure and the second factorial cumulant measure are given respectively by*

$$\begin{aligned} M_{\Phi^{(2)}}(\mathrm{d}x \times \mathrm{d}y) &= \mathbf{E}\left[N^{(2)}\right]\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y),\\ C_2(\mathrm{d}x \times \mathrm{d}y) &= \mathbf{E}[N]\,\lambda(\mathrm{d}x)\,\delta_x(\mathrm{d}y) + c_{(2)}\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y),\\ C_{(2)}(\mathrm{d}x \times \mathrm{d}y) &= c_{(2)}\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y), \end{aligned}$$

where $\mathbf{E}\left[N^{(2)}\right]$ is the second factorial cumulant of N. (The expression of $M_{\Phi^{(2)}}$ is a particular case of (4.3.10).)

3. *Prove that for any $A, B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,*

$$\operatorname{cov}(\Phi(A), \Phi(B)) = \mathbf{E}[N]\,\lambda(A \cap B) + c_{(2)}\lambda(A)\,\lambda(B).$$

4. *Check the above result in the particular case when N is Poisson.*

Solution 4.6.5. *1. Observe that for any $A, B \in \mathcal{B}(\mathbb{G})$,*

$$M_{\Phi^2}(A \times B) = \mathbf{E}[\Phi(A)\,\Phi(B)] \le \mathbf{E}\left[\Phi(\mathbb{G})^2\right] = \mathbf{E}\left[N^2\right] < \infty.$$

Then M_{Φ^2} is a finite measure. Let $f \in \mathfrak{F}_+(\mathbb{G})$ be bounded. Then

$$\int_{\mathbb{G}^2} f(x)\,f(y)\,M_{\Phi^2}(\mathrm{d}x \times \mathrm{d}y) < \infty.$$

Then the expansion (4.2.3) applies with $n = 2$.

By the expression (2.2.13) of the Laplace transform of Φ, we have for any $t \in \mathbb{R}_+$,

$$\mathcal{L}_\Phi(tf) = \mathcal{G}_N\left(\mathcal{L}_{f(X_1)}(t)\right), \quad f \in \mathfrak{F}_+(\mathbb{G}),$$

By Lemma 13.A.21

$$\mathcal{G}_N(x) = 1 + \sum_{k=1}^{2} \mathbf{E}\left[N^{(k)}\right]\frac{(x-1)^k}{k!} + o\left((x-1)^2\right), \quad \forall x \in \mathbb{R}, |x| \le 1.$$

Since f is bounded, $\mathbf{E}\left[f(X_1)^2\right]<\infty$, then by Lemma 13.B.1,

$$\mathcal{L}_{f(X_1)}(t)=1+\sum_{r=1}^{2}(-1)^r\,\mathbf{E}\left[f(X_1)^r\right]\frac{t^r}{r!}+o\left(t^2\right),\quad t\in\mathbb{R}_+,$$

Combining the above two expansions, we get

$$\begin{aligned}\mathcal{G}_N\left(\mathcal{L}_{f(X_1)}(t)\right)&=1+\mathbf{E}[N]\sum_{r=1}^{2}(-1)^r\,\mathbf{E}\left[f(X_1)^r\right]\frac{t^r}{r!}\\&\quad+\frac{\mathbf{E}\left[N^{(2)}\right]}{2!}\left(\sum_{r=1}^{2}(-1)^r\,\mathbf{E}\left[f(X_1)^r\right]\frac{t^r}{r!}\right)^2+o\left(t^2\right)\\&=1-\mathbf{E}[N]\,\mathbf{E}[f(X_1)]\,t\\&\quad+\frac{t^2}{2}\left(\mathbf{E}[N]\,\mathbf{E}\left[f(X_1)^2\right]+\mathbf{E}\left[N^{(2)}\right]\mathbf{E}[f(X_1)]^2\right)+o\left(t^2\right)\end{aligned}$$

Comparing the above expansion with (4.2.3) shows that

$$\int_{\mathbb{G}}f(x)\,M_\Phi(\mathrm{d}x)=\mathbf{E}[N]\,\mathbf{E}[f(X_1)]$$

and

$$\int_{\mathbb{G}^2}f(x)\,f(y)\,M_{\Phi^2}(\mathrm{d}x\times\mathrm{d}y)=\mathbf{E}[N]\,\mathbf{E}\left[f(X_1)^2\right]+\mathbf{E}\left[N^{(2)}\right]\mathbf{E}[f(X_1)]^2.$$

The announced expressions of the first and second moment measures then follow respectively from the above two equations.

2. Applying (14.E.5) we deduce that for any $A,B\in\mathcal{B}(\mathbb{G})$,

$$\begin{aligned}M_{\Phi^{(2)}}(A\times B)&=M_{\Phi^2}(A\times B)-M_\Phi(A\cap B)\\&=\mathbf{E}[N]\,F(A\cap B)+\mathbf{E}\left[N^{(2)}\right]F(A)\,F(B)-\mathbf{E}[N]\,F(A\cap B)\\&=\mathbf{E}\left[N^{(2)}\right]F(A)\,F(B).\end{aligned}$$

Moreover, by (4.1.4)

$$\begin{aligned}C_2(\mathrm{d}x\times\mathrm{d}y)&=M_{\Phi^2}(\mathrm{d}x\times\mathrm{d}y)-M_\Phi(\mathrm{d}x)\,M_\Phi(\mathrm{d}y)\\&=\mathbf{E}[N]\,F(\mathrm{d}x)\,\delta_x(\mathrm{d}y)+\left(\mathbf{E}\left[N^{(2)}\right]-\mathbf{E}[N]^2\right)F(\mathrm{d}x)\,F(\mathrm{d}y)\\&=\mathbf{E}[N]\,F(\mathrm{d}x)\,\delta_x(\mathrm{d}y)+c_{(2)}F(\mathrm{d}x)\,F(\mathrm{d}y)\end{aligned}$$

where the last equality is due to (13.A.36). Finally, by (4.1.4)

$$\begin{aligned}C_{(2)}(\mathrm{d}x\times\mathrm{d}y)&=M_{\Phi^{(2)}}(\mathrm{d}x\times\mathrm{d}y)-M_\Phi(\mathrm{d}x)\,M_\Phi(\mathrm{d}y)\\&=\left(\mathbf{E}\left[N^{(2)}\right]-\mathbf{E}[N]^2\right)F(\mathrm{d}x)\,F(\mathrm{d}y)=c_{(2)}F(\mathrm{d}x)\,F(\mathrm{d}y).\end{aligned}$$

3. By (4.1.5), for any $A, B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,

$$\operatorname{cov}(\Phi(A), \Phi(B)) = C_2(A \times B) = \mathbf{E}[N]\, F(A \cap B) + \mathrm{c}_{(2)} F(A)\, F(B).$$

4. When N *is Poisson, it follows from (13.A.28) that* $\mathrm{c}_{(2)} = 0$. *Then the above equality gives*

$$\operatorname{cov}(\Phi(A), \Phi(B)) = \mathbf{E}[N]\, F(A \cap B).$$

On the other hand, when N *is Poisson,* Φ *is a Poisson point process. Then, the above result may be deduced immediately from (2.4.3.*

Chapter 5

Determinantal and permanental point processes

Determinantal point processes allow one to model some spatial correlations between the atoms of the point process. They were first introduced by O. Macchi [65] and are of great interest in Mathematical Physics.

5.1 Determinantal point process basics

Recall that the context is that described in Section 1.1; in particular $\mathbb{G}$ is a l.c.s.h. space, $\mathcal{B}(\mathbb{G})$ is the associated Borel σ-algebra, and $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ is the set of relatively compact measurable subsets of $\mathbb{G}$.

5.1.1 Definition and basic properties

Definition 5.1.1. Determinantal point process. *Let μ be a locally finite measure on the l.c.s.h. space $\mathbb{G}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function. A point process Φ on $\mathbb{G}$ is said to be* determinantal *with* background measure μ *and* kernel K *if for all $k \in \mathbb{N}^*$, the k-th factorial moment measure $M_{\Phi^{(k)}}$ admits a density with respect to the product measure μ^k which equals*

$$\rho^{(k)}(x_1, \ldots, x_k) = \det\left(K(x_i, x_j)\right)_{1 \le i,j \le k}, \quad \text{for } \mu^k\text{-almost all } (x_1, \ldots, x_k) \in \mathbb{G}^k, \tag{5.1.1}$$

where $(a_{ij})_{1 \le i,j \le k}$ denotes the matrix with entries a_{ij} and $\det(\cdot)$ denotes the determinant. The function $\rho^{(k)}$ is called the k-th factorial moment density *with respect to μ^k.*

Observe that the mean measure of a determinantal point process Φ with background measure μ and kernel K is given by

$$M_{\Phi}(B)=\int_{B}K(x,x)\,\mu(\mathrm{d}x)\,,\quad B\in\mathcal{B}(\mathbb{G})\,. \tag{5.1.2}$$

Remark 5.1.2. *Let Φ be a determinantal point process on a l.c.s.h. space $\mathbb{G}$ with background measure μ and kernel $K:\mathbb{G}^2\to\mathbb{C}$ having the form*

$$K(x,y)=\sqrt{f(x)}\tilde{K}(x,y)\sqrt{f(y)},\quad (x,y)\in\mathbb{G}^2,$$

for some given measurable functions $f:\mathbb{G}\to\mathbb{R}_+$ and $\tilde{K}:\mathbb{G}^2\to\mathbb{C}$. Then Φ is also a determinantal point process on $\mathbb{G}$ with background measure $\tilde{\mu}(\mathrm{d}x)=f(x)\,\mu(\mathrm{d}x)$ and kernel $\tilde{K}$. Indeed, for any $(x_1,\ldots,x_k)\in\mathbb{G}^k$,

$$\det\left(K(x_i,x_j)\right)_{1\le i,j\le k}=\left[\prod_{i=1}^{k}f(x_i)\right]\det\left(\tilde{K}(x_i,x_j)\right)_{1\le i,j\le k}.$$

Thus

$$\begin{aligned}&\det\left(K(x_i,x_j)\right)_{1\le i,j\le k}\ \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k)\\&\quad=\det\left(\tilde{K}(x_i,x_j)\right)_{1\le i,j\le k}\ \tilde{\mu}(\mathrm{d}x_1)\ldots\tilde{\mu}(\mathrm{d}x_k)\,.\end{aligned}$$

Lemma 5.1.3. Thinning of determinantal point process. *Let Φ be a determinantal point process on a l.c.s.h. space $\mathbb{G}$ with background measure μ and kernel K, $p:\mathbb{G}\to[0,1]$ some measurable function, and let $\tilde{\Phi}$ the thinning of Φ with retention function p. Then $\tilde{\Phi}$ is a determinantal point process on $\mathbb{G}$ with background measure μ and kernel*

$$\tilde{K}(x,y)=\sqrt{p(x)}K(x,y)\sqrt{p(y)},\quad x,y\in\mathbb{G}.$$

Proof. By Proposition 2.3.24, for any $k\in\mathbb{N}^*$ and any $B\in\mathcal{B}(\mathbb{G})^{\otimes k}$,

$$\begin{aligned}M_{\tilde{\Phi}^{(k)}}(B)&=\int_{B}p(x_1)\ldots p(x_n)\det\left(K(x_i,x_j)\right)_{1\le i,j\le k}\ \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k)\\&=\int_{B}\det\left(\sqrt{p(x_i)}K(x_i,x_j)\sqrt{p(x_j)}\right)_{1\le i,j\le k}\ \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k)\,.\end{aligned}$$

□

Lemma 5.1.4. *A determinantal point process on a l.c.s.h. space $\mathbb{G}$ is simple.*

Proof. Observe that

$$\begin{aligned}&M_{\Phi^{(2)}}(\{(x,x):x\in\mathbb{G}\})\\&\quad=\int_{\mathbb{G}^2}\mathbf{1}\{x_1=x_2\}\,\rho^{(2)}(x_1,x_2)\ \mu(\mathrm{d}x_1)\,\mu(\mathrm{d}x_2)\\&\quad=\int_{\mathbb{G}^2}\mathbf{1}\{x_1=x_2\}\det\left(K(x_1,x_2)\right)_{1\le i,j\le 2}\ \mu(\mathrm{d}x_1)\,\mu(\mathrm{d}x_2)=0.\end{aligned}$$

Then Φ is simple by Lemma 2.3.23(ii). □

The following lemma shows that the restriction of a determinantal point process on $\mathbb{G}$ to any measurable subset of $\mathbb{G}$ is also determinantal.

Lemma 5.1.5. Determinantal point process restriction. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$, μ some locally finite measure on $\mathbb{G}$, and $K : \mathbb{G}^2 \to \mathbb{C}$ a measurable function. For any $D \in \mathcal{B}(\mathbb{G})$, let μ_D be the restriction of μ to D and K_D the restriction of K to $D \times D$.*

(i) If Φ is a determinantal point process on $\mathbb{G}$ with background measure μ and kernel K, then, for any $D \in \mathcal{B}(\mathbb{G})$, the restriction of Φ to D is a determinantal point process on D with background measure μ_D and kernel K_D.

(ii) Conversely, if for any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, the restriction of Φ to D is a determinantal point process on D with background measure μ_D and kernel K_D, then Φ is a determinantal point process on $\mathbb{G}$ with background measure μ and kernel K.

Proof. (i) Indeed, letting Φ_D be the restriction of Φ to D, then for all $B \in \mathcal{B}(D)^k$,

$$\begin{aligned}
\mathbf{E}\left[\Phi_D^{(k)}(B)\right] &= \mathbf{E}\left[\Phi^{(k)}(B)\right] \\
&= \int_B \det\left(K(x_i, x_j)\right)_{1\leq i,j\leq k} \; \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k) \\
&= \int_B \det\left(K_D(x_i, x_j)\right)_{1\leq i,j\leq k} \; \mu_D(\mathrm{d}x_1)\ldots\mu_D(\mathrm{d}x_k).
\end{aligned}$$

(ii) For any $B_1, \ldots, B_k \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, let $D = B_1 \cup \ldots \cup B_k$ which is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$. Then

$$\begin{aligned}
M_{\Phi^{(k)}}(B_1 \times \cdots \times B_k) &= \mathbf{E}\left[\Phi^{(k)}(B_1 \times \cdots \times B_k)\right] \\
&= \mathbf{E}\left[\Phi_D^{(k)}(B_1 \times \cdots \times B_k)\right] \\
&= \int_{B_1\times\cdots\times B_k} \det\left(K_D(x_i, x_j)\right)_{1\leq i,j\leq k} \; \mu_D(\mathrm{d}x_1)\cdots\mu_D(\mathrm{d}x_k) \\
&= \int_{B_1\times\cdots\times B_k} \det\left(K(x_i, x_j)\right)_{1\leq i,j\leq k} \; \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k),
\end{aligned}$$

and thus, by [11, Theorem 10.3 p.163], $M_{\Phi^{(k)}}$ admits $\det\left(K(x_i, x_j)\right)_{1\leq i,j\leq k}$ as a density with respect to μ^k. □

Example 5.1.6. Poisson is determinantal. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with diffuse locally finite intensity measure μ. Then by Proposition 2.3.25, the k-th factorial moment measure is $M_{\Phi^{(k)}} = \mu^k$ for all $k \in \mathbb{N}^*$. Then Φ is a determinantal point process with background measure μ and kernel*

$$K(x, y) = \mathbf{1}_{\{x=y\}}, \quad x, y \in \mathbb{G}. \tag{5.1.3}$$

Indeed, obviously $\det(K(x,x)) = 1$ *for all* $x \in \mathbb{G}$ *and for any density* $\rho^{(1)}$ *of* $M_{\Phi^{(1)}}$ *with respect to* μ*, necessarily* $\rho^{(1)}(x) = 1$*, for* μ*-almost all* $x \in \mathbb{G}$*. Consider now some* $k \geq 2$ *and let*

$$\mathbb{G}_2 = \left\{(x,y) \in \mathbb{G}^2 : x \neq y\right\}.$$

Since μ *is diffuse, then* $\mu^2\left(\mathbb{G}^2 \backslash \mathbb{G}_2\right) = 0$*. Then by Lemma 14.A.1(ii),* $\mu^k(A_k) = 0$ *where*

$$A_k = \left\{(x_1,\ldots,x_k) \in \mathbb{G}^k : x_i = x_j \text{ for some } i \neq j\right\}.$$

Observe that for any $(x_1,\ldots,x_k) \in \mathbb{G}^k \backslash A_k$*,the matrix* $(K(x_i,x_j))_{1\leq i,j\leq k}$ *is the identity matrix denoted by* I_k*. Consequently,*

$$(K(x_i,x_j))_{1\leq i,j\leq k} = I_k, \quad \text{for } \mu^k\text{-almost all } x \in (x_1,\ldots,x_k) \in \mathbb{G}^k. \tag{5.1.4}$$

Therefore, for any density $\rho^{(k)}$ *of* $M_{\Phi^{(k)}}$ *with respect to* μ^k*,*

$$\rho^{(k)}(x_1,\ldots,x_k) = \det(K(x_i,x_j))_{1\leq i,j\leq k} = 1,$$

for μ^k*-almost all* $x_1,\ldots,x_k \in \mathbb{G}$*.*

Remark 5.1.7. Non-uniqueness of kernel*. The kernel of a determinantal point process is not unique. Let* Φ *be a determinantal point process on a l.c.s.h. space* $\mathbb{G}$ *with background measure* μ *and kernel* K*, and let* $h : \mathbb{G} \to \mathbb{R}^*$ *be a measurable function. Define* $\hat{K} : \mathbb{G}^2 \to \mathbb{C}$ *by*

$$\hat{K}(x,y) = h(x)\,K(x,y)\,h(y)^{-1}, \quad (x,y) \in \mathbb{G}^2.$$

Then Φ *admits also* $\hat{K}$ *as kernel with respect to the background measure* μ*.*

Indeed, for any $(x_1,\ldots,x_k) \in \mathbb{G}^k$*, let* $A = (K(x_i,x_j))_{1\leq i,j\leq k}$ *and let* H *be a diagonal matrix of dimension* k *with* $H_{ii} = h(x_i)$ $(i = 1,\ldots,k)$*. Then*

$$\begin{aligned}\det\left(\hat{K}(x_i,x_j)\right)_{1\leq i,j\leq k} &= \det\left(HAH^{-1}\right) \\ &= \det(A) = \det(K(x_i,x_j))_{1\leq i,j\leq k}.\end{aligned}$$

5.1.2 Indistinguishable kernels

We will now introduce a notion of *indistinguishability* of kernels of determinantal point processes.

Definition 5.1.8. Indistinguishable kernels*. Let* μ *be a locally finite measure on a l.c.s.h. space* $\mathbb{G}$*. Two measurable functions* K *and* $\tilde{K}$ *from* $\mathbb{G}^2$ *to* $\mathbb{C}$ *are called* μ-indistinguishable *if the two following conditions hold true:*

$$\begin{cases} \tilde{K}(x,x) = K(x,x), & \text{for } \mu\text{-almost all } x \in \mathbb{G}, \\ \tilde{K}(x,y) = K(x,y), & \text{for } \mu^2\text{-almost all } (x,y) \in \mathbb{G}^2. \end{cases} \tag{5.1.5}$$

Remark 5.1.9. *Note that the second condition in (5.1.5) is not enough to ensure μ-indistinguishability as shown by the example $K(x,y) = \mathbf{1}_{\{x=y\}}$ and $\tilde{K}(x,y) = 0$ for all $x, y \in \mathbb{G}$, and μ any diffuse (non null) measure on $\mathbb{G}$.*

Lemma 5.1.10. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let K and $\tilde{K}$ be two measurable functions from $\mathbb{G}^2$ to $\mathbb{C}$ such that there exists some $\mathbb{G}_1 \in \mathcal{B}(\mathbb{G})$ such that $\mu(\mathbb{G} \backslash \mathbb{G}_1) = 0$ and*

$$\tilde{K}(x,y) = K(x,y), \quad \textit{for all } x, y \in \mathbb{G}_1.$$

Then K and $\tilde{K}$ are μ-indistinguishable.

Proof. This follows from the fact that $\mu^2\left(\mathbb{G}^2 \backslash \mathbb{G}_1^2\right) = 0$ by Lemma 14.A.1(i). □

Proposition 5.1.11. *Let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$ and let K and $\tilde{K}$ be two μ-indistinguishable measurable functions from $\mathbb{G}^2$ to $\mathbb{C}$. Then the following results hold true.*

(i) For all $k \in \mathbb{N}^*$ and for μ^k-almost all $(x_1, \ldots, x_k) \in \mathbb{G}^k$,

$$\det\left(\tilde{K}(x_i, x_j)\right)_{1 \le i,j \le k} = \det\left(K(x_i, x_j)\right)_{1 \le i,j \le k}. \tag{5.1.6}$$

(ii) If Φ and $\tilde{\Phi}$ are determinantal point processes with background measure μ and kernels K and $\tilde{K}$ respectively, then Φ and $\tilde{\Phi}$ have the same factorial moment measures.

(iii) If Φ is a determinantal point process with background measure μ and kernel K, then Φ admits also kernel $\tilde{K}$ (with respect to the background measure μ).

Proof. (i) Equality (5.1.6) is obvious for $k = 1$. Consider now some $k \ge 2$. Let

$$\mathbb{G}_1 = \left\{x \in \mathbb{G} : \tilde{K}(x,x) = K(x,x)\right\}$$

and

$$\mathbb{G}_2 = \left\{(x,y) \in \mathbb{G}^2 : \tilde{K}(x,y) = K(x,y)\right\}.$$

Since K and $\tilde{K}$ be two μ-indistinguishable, then $\mu(\mathbb{G} \backslash \mathbb{G}_1) = 0$ and $\mu^2\left(\mathbb{G}^2 \backslash \mathbb{G}_2\right) = 0$. Let

$$B_k = \mathbb{G}^k \backslash \mathbb{G}_1^k$$

and

$$A_k = \left\{(x_1, \ldots, x_k) \in \mathbb{G}^k : (x_i, x_j) \in \mathbb{G}^2 \backslash \mathbb{G}_2 \text{ for some } i \neq j\right\}.$$

It follows from Lemma 14.A.1 that

$$\mu^k(B_k) = 0, \quad \text{and } \mu^k(A_k) = 0.$$

Then $\mu^k(A_k \cup B_k) = 0$. Observing that the equality in Equation (5.1.6) holds for any $(x_1, \ldots, x_k) \in \mathbb{G}^k \backslash (A_k \cup B_k)$ concludes the proof. (ii) This follows from (i) and the very definition of a determinantal point process. (iii) Same argument as (ii). □

Some additional assumptions are needed for the existence and uniqueness of a determinantal point process with given background measure μ and kernel K and this is what we explore next.

5.1.3 Uniqueness of the distribution

We begin with proving uniqueness for *finite* determinantal point process.

Lemma 5.1.12. Uniqueness of finite determinantal point process. *Let Φ be a determinantal point process on a l.c.s.h. space $\mathbb{G}$ with background measure μ and kernel K where μ is a locally finite measure on $\mathbb{G}$. Assume moreover that $\int_{\mathbb{G}} K(x,x)\,\mu(\mathrm{d}x) < \infty$ (equivalently, by (5.1.2), $\mathbf{E}[\Phi(\mathbb{G})] < \infty$) and that for any $k \geq 2$, the matrix $(K(x_i,x_j))_{1\leq i,j\leq k}$ is Hermitian nonnegative-definite for μ^k-almost all $(x_1,\ldots,x_k) \in \mathbb{G}^k$. Then the following results hold true:*

(i)
$$\mathbf{E}\left[\Phi(\mathbb{G})^{(k)}\right] \leq \mathbf{E}[\Phi(\mathbb{G})]^k, \quad \text{for all } k \in \mathbb{N}^*, \tag{5.1.7}$$
and
$$\mathbf{E}\left[(1+s)^{\Phi(\mathbb{G})}\right] \leq e^{s\mathbf{E}[\Phi(\mathbb{G})]}, \quad \text{for all } s \in \mathbb{R}_+^*. \tag{5.1.8}$$

(ii) The radius of convergence $R_{\mathcal{G}_{\Phi(\mathbb{G})}}$ of the generating function $\mathcal{G}_{\Phi(\mathbb{G})}$ (cf. Definition 13.A.11) is infinite.

(iii) The distribution of Φ is uniquely determined by μ and K.

Proof. (i) By the very definition of a determinantal point process, for any $k \geq 2$,

$$\begin{aligned}\mathbf{E}\left[\Phi(\mathbb{G})^{(k)}\right] &= \int_{\mathbb{G}^k} \det(K(x_i,x_j))_{1\leq i,j\leq k}\,\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k)\\ &\leq \int_{\mathbb{G}^k} \prod_{i=1}^k K(x_i,x_i)\,\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k) = \mathbf{E}[\Phi(\mathbb{G})]^k,\end{aligned}$$

where the second inequality follows from the Hadamard's inequality (15.A.1). (ii) It follows from the above inequality that, for any $s \in \mathbb{R}_+^*$,

$$\mathbf{E}\left[(1+s)^{\Phi(\mathbb{G})}\right] = \sum_{k\in\mathbb{N}} \frac{1}{k!}\mathbf{E}\left[\Phi(\mathbb{G})^{(k)}\right] s^k \leq \sum_{k\in\mathbb{N}} \frac{(s\mathbf{E}[\Phi(\mathbb{G})])^k}{k!} = e^{s\mathbf{E}[\Phi(\mathbb{G})]},$$

where the first equality is due to Lemma 13.A.15. Therefore, the radius of convergence $R_{\mathcal{G}_{\Phi(\mathbb{G})}}$ of the generating function $\mathcal{G}_{\Phi(\mathbb{G})}$ is infinite. (iii) Since $R_{\mathcal{G}_{\Phi(\mathbb{G})}} = \infty$, then Proposition 4.3.20(ii) implies that the distribution of Φ is characterized by its factorial moment measures which are uniquely determined by μ and K. $\square$

Remark 5.1.13. *There exist determinantal point processes with non-Hermitian kernels; see e.g. [91, §2.2, §2.5].*

We deduce now uniqueness for *general* (i.e., not necessarily finite) determinantal point processes.

Corollary 5.1.14. Uniqueness of general determinantal point processes. *Let Φ be a determinantal point process on a l.c.s.h. space $\mathbb{G}$ with background measure μ and kernel K where μ is a locally finite measure on $\mathbb{G}$. Assume moreover that, for any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,*

$$\int_D K(x,x)\,\mu(\mathrm{d}x) < \infty$$

and for any $k \geq 2$, the matrix $(K(x_i,x_j))_{1\leq i,j\leq k}$ is Hermitian nonnegative-definite for μ^k-almost all $(x_1,\ldots,x_k) \in \mathbb{G}^k$. Then, for any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, $\mathbf{E}[\Phi(D)] < \infty$,

$$\mathbf{E}\left[\Phi(D)^{(k)}\right] \leq \mathbf{E}[\Phi(D)]^k, \quad \text{for all } k \in \mathbb{N}^*,$$

and

$$\mathbf{E}\left[(1+s)^{\Phi(D)}\right] \leq e^{s\mathbf{E}[\Phi(D)]}, \quad \text{for all } s \in \mathbb{R}_+^*.$$

Moreover, the distribution of Φ is uniquely determined by μ and K.

Proof. By Lemma 5.1.5(i), for all $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, the restriction of Φ to D is a determinantal point process which we denote by Φ_D. Applying Lemma 5.1.12 to Φ_D gives the announced inequalities and shows that the distribution of Φ_D is uniquely determined by μ restricted to D and K restricted to D^2. This being true for any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, Corollary 1.3.4 allows one to conclude. □

Example 5.1.15. *Following Example 5.1.6, note that the Poisson point process having a diffuse locally finite intensity measure μ is the unique (in distribution) determinantal point process with background measure μ and kernel $K(x,y) = \mathbf{1}_{\{x=y\}}$ for all $x,y \in \mathbb{G}$ since by (5.1.4), $(K(x_i,x_j))_{1\leq i,j\leq k}$ is the identity matrix (and, therefore, Hermitian nonnegative-definite) for μ^k-almost all $(x_1,\ldots,x_k) \in \mathbb{G}^k$.*

Corollary 5.1.16. *Let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$ and let K and $\tilde{K}$ be two μ-indistinguishable measurable functions from $\mathbb{G}^2$ to $\mathbb{C}$. Assume moreover that K satisfies the conditions of Corollary 5.1.14. Then two determinantal point processes with background measure μ and respective kernels K and $\tilde{K}$ have the same distribution.*

Proof. Let Φ and $\tilde{\Phi}$ be determinantal point processes with background measure μ and kernels K and $\tilde{K}$ respectively. By Proposition 5.1.11(iii), $\tilde{\Phi}$ admits also kernel K (with respect to the background measure μ). Then by Corollary 5.1.14, Φ and $\tilde{\Phi}$ have the same distribution. □

Remark 5.1.17. *Note that it is not enough to assume the second condition in (5.1.5) instead of μ-indistinguishablility in Corollary 5.1.16. Indeed, assume* *as in Remark 5.1.9 a diffuse (non-null) measure μ, $K(x,y) = \mathbf{1}_{\{x=y\}}$, and $\tilde{K}(x,y) = 0$ for all $x, y \in \mathbb{G}$. Then by Example 5.1.15, the determinantal point process with background measure μ and kernel K is a Poisson point process with intensity measure μ; whereas the determinantal point process with background measure μ and kernel $\tilde{K}$ is the null point process.*

5.1.4 Generating function and Laplace transform

We give first expansions of the generating function and Laplace transform of a finite determinantal point process. Recall Definition 4.3.12 of the generating function and Laplace transform of finite point processes.

Proposition 5.1.18. Generating function of finite determinantal point process. *Under the conditions of Lemma 5.1.12, the determinantal point process Φ is almost surely finite and we have the following expansions of its generating function and Laplace transform.*

(i) For all bounded measurable functions $v : \mathbb{G} \to \mathbb{C}$,

$$\mathcal{G}_\Phi(v) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left[\prod_{i=1}^{k} (1 - v(x_i))\right] \det(K(x_i, x_j))_{1 \le i,j \le k} \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_k). \tag{5.1.9}$$

(ii) For any measurable function $f : \mathbb{G} \to \mathbb{R}$ such that $\inf_{x \in \mathbb{G}} f(x) > -\infty$,

$$\mathcal{L}_\Phi(f) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left(1 - e^{-f(x_1)}\right) \ldots \left(1 - e^{-f(x_k)}\right) \det(K(x_i, x_j))_{1 \le i,j \le k} \, \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_k). \tag{5.1.10}$$

(iii) The void probability of Φ equals

$$\mathbf{P}(\Phi(B) = 0) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{B^k} \det(K(x_i, x_j))_{1 \le i,j \le k} \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_k), \tag{5.1.11}$$

for any $B \in \mathcal{B}(\mathbb{G})$.

Moreover, the above three series are absolutely convergent.

Proof. The fact that Φ is almost surely finite follows from the following assumption in Lemma 5.1.12

$$\mathbf{E}[\Phi(\mathbb{G})] = M_\Phi(\mathbb{G}) = \int_{\mathbb{G}} K(x,x) \, \mu(\mathrm{d}x) < \infty,$$

where the second equality is due to (5.1.2). (i) Observe that, by (5.1.8), $R_{\mathcal{G}_{\Phi(\mathbb{G})}} = \infty$ (where $R_{\mathcal{G}_{\Phi(\mathbb{G})}}$ is the radius of convergence of the generating function $\mathcal{G}_{\Phi(\mathbb{G})}$). Then Proposition 4.3.15(i) gives the announced expansion. (ii) Applying (i) for $v := e^{-f}$ and using (4.3.12) gives the announced result. (iii) This follows from Proposition 4.3.1 and the fact that $R_{\mathcal{G}_{\Phi(\mathbb{G})}} = \infty$. $\square$

We now deduce the generating function and Laplace transform of a general determinantal point process.

Corollary 5.1.19. Generating function of general determinantal point processes. *Under the conditions of Corollary 5.1.14, the following results hold true.*

(i) The expansion (5.1.9) of the generation function holds for any bounded measurable function $v : \mathbb{G} \to \mathbb{C}$ such that the support of $1 - v$ is in $\mathcal{B}_c(\mathbb{G})$.

(ii) The expansion (5.1.10) of the Laplace transform holds for any measurable function $f : \mathbb{G} \to \mathbb{R}$ whose support is in $\mathcal{B}_c(\mathbb{G})$ and such that $\inf_{x\in\mathbb{G}} f(x) > -\infty$.

(iii) The expansion (5.1.11) of the void probability holds for any $B \in \mathcal{B}_c(\mathbb{G})$.

Moreover, the series in the above three expansions are absolutely convergent.

Proof. (i) Let D be the support of $1 - v$. By lemma 5.1.5(i), the restriction of Φ to D is a determinantal point process which we denote by Φ_D. Applying Proposition 5.1.18(i) to Φ_D gives the announced expansion. (ii) This follows from Proposition 5.1.18(ii) with the same argument as above for D being the support of f. (iii) This is immediate from Proposition 5.1.18(iii). □

5.1.5 Inequalities for moment measures

We will now give bounds on the moment measures of a determinantal point process.

Proposition 5.1.20. Inequalities for the moment measures of determinantal point processes. *Under the conditions of Corollary 5.1.14, the following results hold true.*

(i) All the moment measures and factorial moment measures of the determinantal point process Φ are locally finite.

(ii) For any $k \in \mathbb{N}^*$,

$$M_{\Phi^{(k)}}(B_1 \times \cdots \times B_k) \le \prod_{i=1}^{k} M_\Phi(B_i), \quad B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G}). \quad (5.1.12)$$

(iii) Le $\rho^{(k)}$ be the k-th factorial moment density with respect to μ^k. Then for any $n, m, l \in \mathbb{N}$,

$$\begin{aligned}&\rho^{(n+m+l)}(x_1, \ldots, x_{n+m+l})\,\rho^{(l)}(x_{n+m+1}, \ldots, x_{n+m+l})\\&\le \rho^{(n+l)}(x_1, \ldots, x_n, x_{n+m+1}, \ldots, x_{n+m+l})\,\rho^{(m+l)}(x_{n+1}, \ldots, x_{n+m+l}),\end{aligned}$$

with the convention that $\rho^{(k)}$ equals 1 when $k = 0$.

Proof. (i) The fact that the factorial moment measures are locally finite follows from (5.1.7). The moment measures are also locally finite by Equation (13.A.23). (ii) By the definition 5.1.1 of determinantal point processes, for any $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G})$,

$$\begin{aligned} M_{\Phi^{(k)}}(B_1 \times \cdots \times B_k) &= \int_{B_1 \times \ldots \times B_k} \det\left(K(x_i, x_j)\right)_{1 \le i,j \le k} \ \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_k) \\ &\le \int_{B_1 \times \ldots \times B_k} \left(\prod_{i=1}^{k} K(x_i, x_i) \right) \mu^k(\mathrm{d}x) \\ &= \prod_{i=1}^{k} \int_{B_i} K(x_i, x_i)\, \mu(\mathrm{d}x_i) = \prod_{i=1}^{k} M_\Phi(B_i), \end{aligned}$$

where the second line is due to Hadamard's inequality (15.A.1) and the last equality follows from (5.1.2). (iii) This follows from (15.A.2) when $l = 0$ and from (15.A.3) when $l \in \mathbb{N}^*$. □

Corollary 5.1.21. *Under the conditions of Corollary 5.1.14, the moment measures and the factorial moment measures of the determinantal point process Φ are not larger than the respective measures of the Poisson point process with the same mean measure.*

Proof. For factorial moment measures, the announced inequality follows from Proposition 5.1.20(ii) and Proposition 2.3.25. Invoking Equation (13.A.23) allows one to conclude. □

5.2 Existence of determinantal point processes with regular kernels

In this section, we will give sufficient conditions for the existence of determinantal point processes with given kernel. We begin by considering *canonical kernels* as in Definition 16.A.24; which we remind below. The existence of more general determinantal point processes will be proved in Section 5.2.4.

5.2.1 Canonical determinantal point processes

We denote by $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ the space of measurable functions $f : \mathbb{G} \to \mathbb{C}$ which are square-integrable with respect to μ; cf. Definition 16.A.1.

Definition 5.2.1. Canonical kernels; reminder. *Let μ be a σ-finite measure on a l.c.s.h. space $\mathbb{G}$, $\{\varphi_n\}_{n \in \mathbb{N}^*}$ be a sequence in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, and $\{\lambda_n\}_{n \in \mathbb{N}^*}$ be nonnegative real numbers such that $\sum_{n \in \mathbb{N}^*} \lambda_n \|\varphi_n\|^2 < \infty$. Let $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function such that*

$$K(x, y) = \begin{cases} \sum_{n \in \mathbb{N}^*} \lambda_n \varphi_n(x) \varphi_n(y)^* & \text{if } x, y \in \mathbb{G}_1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2.1}$$

where

$$\mathbb{G}_1 = \left\{ x \in \mathbb{G} : \sum_{n \in \mathbb{N}^*} \lambda_n \left| \varphi_n(x) \right|^2 < \infty \right\}. \tag{5.2.2}$$

Then K is called a pre-canonical kernel *associated to $\{\varphi_n\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$. If $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is orthonormal, then K is called* canonical kernel*.*

Definition 5.2.2. *A determinantal point process Φ with canonical kernel as in Definition 5.2.1 is called a* canonical determinantal point process*. In the particular case when the coefficients λ_n are in $\{0,1\}$ for all $n \in \mathbb{N}^*$, we say that the process Φ is an* elementary determinantal point process*.*

In what follows we aim to show existence of canonical determinantal point processes. We begin by showing the existence of the elementary determinantal point processes.

Lemma 5.2.3. Construction of elementary determinantal point processes *[49, Lemma 4.5.1]. Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let $(\varphi_1, \ldots, \varphi_N)$ be an orthonormal set in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ for some given $N \in \mathbb{N}^*$. Then there exists a determinantal point process Φ on $\mathbb{G}$ with kernel $K(x,y) = \sum_{n=1}^N \varphi_n(x)\varphi_n(y)^*$, $x, y \in \mathbb{G}$. Moreover, Φ has N points almost surely and the corresponding Janossy measure (4.3.6) is given by*

$$J_N(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_N) = \det\left(K(x_i, x_j)\right)_{1 \le i,j \le N} \, \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_N).$$

Proof. Observe that K is Hermitian and that, for any $x, y, z \in \mathbb{G}$,

$$\begin{aligned}
\int_{\mathbb{G}} K(x,y)K(y,z)\mu(\mathrm{d}y) &= \int_{\mathbb{G}} \sum_{n,m=1}^N \varphi_n(x)\varphi_n(y)^*\varphi_m(y)\varphi_m(z)^*\mu(\mathrm{d}y) \\
&= \sum_{n,m=1}^N \varphi_n(x)\varphi_m(z)^* \mathbf{1}_{\{n=m\}} = K(x,z),
\end{aligned}$$

where the second equality is due to the fact that $(\varphi_1, \ldots, \varphi_N)$ is orthonormal. Then it follows from [70, Theorem 5.1.4] that for any $l \in \{1, \ldots, N\}$,

$$\int_{\mathbb{G}} \det\left(K(x_i, x_j)\right)_{1 \le i,j \le l} \, \mu(\mathrm{d}x_l) = (N - l + 1) \det\left(K(x_i, x_j)\right)_{1 \le i,j \le l-1}. \tag{5.2.3}$$

Applying the identity (5.2.3) recursively, it follows that

$$\int_{\mathbb{G}^N} \frac{1}{N!} \det\left(K(x_i, x_j)\right)_{1 \le i,j \le N} \, \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_N) = 1. \tag{5.2.4}$$

On the other hand, for any $k \in \mathbb{N}^*$ and any $(x_1, \ldots, x_k) \in \mathbb{G}^k$, the matrix $\left(K(x_i, x_j)\right)_{1 \le i,j \le k}$ is Hermitian nonnegative-definite since its is the sum of the Hermitian nonnegative-definite matrices $\left(\varphi_n(x_i)\varphi_n(x_j)^*\right)_{1 \le i,j \le k}$ $(n =$

$1, \ldots, N)$. Then $\det \left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq k} \geq 0$. Let Π_N be the probability measure on $\mathbb{G}^N$ defined by

$$\Pi_N \left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_N\right) := \frac{1}{N!} \det \left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq N} \ \mu\left(\mathrm{d}x_1\right) \ldots \mu\left(\mathrm{d}x_N\right),$$

and let Φ be a point process on $\mathbb{G}$ with N points which are generated according to the above distribution. We will show that Φ is a determinantal process with kernel K. (i) Consider first some integer $k > N$. Since Φ has N points, the k-th factorial moment measure of Φ is the null measure. On the other hand, the matrix $\left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq k}$ equals BB^*, where B is a matrix defined by

$$B_{ij} = \left(\varphi_j\left(x_i\right)\right), \quad 1 \leq i \leq k, 1 \leq j \leq N.$$

Then $\left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq k}$ has rank N and therefore its determinant is zero. Therefore, Equation (5.1.1) holds for $k > N$. (ii) Consider now some $k \in \{1, \ldots, N\}$. By (4.3.6), the only non-zero Janossy measure of Φ is

$$J_N \left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_N\right) := \det \left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq N} \ \mu\left(\mathrm{d}x_1\right) \ldots \mu\left(\mathrm{d}x_N\right).$$

Then the k-th factorial moment measure of Φ may be deduced from (4.3.9) as follows

$$\begin{aligned}
&M_{\Phi^{(k)}} \left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k\right) \\
&= \frac{J_N \left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k \times \mathbb{G}^{N-k}\right)}{(N-k)!} \\
&= \frac{1}{(N-k)!} \left[\int_{\mathbb{G}^{N-k}} \det \left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq N} \ \mu\left(\mathrm{d}x_{k+1}\right) \ldots \mu\left(\mathrm{d}x_N\right)\right] \mu\left(\mathrm{d}x_1\right) \ldots \mu\left(\mathrm{d}x_k\right) \\
&= \det \left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq k} \ \mu\left(\mathrm{d}x_1\right) \ldots \mu\left(\mathrm{d}x_k\right),
\end{aligned}$$

where the last equality is due to (5.2.3). Thus Equation (5.1.1) holds also for $k \in \{1, \ldots, N\}$ which concludes the proof. □

Remark 5.2.4. *Here is an alternative proof of (5.2.4). Since*

$$K(x,y) = \sum_{n=1}^{N} \varphi_n(x) \varphi_n(y)^*,$$

it follows from Lemma 16.A.25(vi) that

$$\begin{aligned}
\det \left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq N} &= \sum_{1 \leq n_1 < \ldots < n_N \leq N} \left|\det \left(\varphi_{n_j}\left(x_i\right)\right)_{1 \leq i,j \leq N}\right|^2 \\
&= \left|\det \left(\varphi_j\left(x_i\right)\right)_{1 \leq i,j \leq N}\right|^2 \\
&= \left|\sum_{\pi \in S_N} \mathrm{sgn}(\pi) \prod_{k=1}^{N} \varphi_{\pi(k)}(x_k)\right|^2 \\
&= \sum_{\pi, \tau \in S_N} \mathrm{sgn}(\pi\tau) \prod_{k=1}^{N} \varphi_{\pi(k)}(x_k) \varphi_{\tau(k)}(x_k)^*,
\end{aligned}$$

where S_N is the set of permutations of $\{1,\ldots,N\}$ and $\mathrm{sgn}(\pi)$ *denotes the signature of the permutation π. Integrating the above equality over $\mathbb{G}^N$ and using the fact that $(\varphi_1,\ldots,\varphi_N)$ is an orthonormal set gives (5.2.4).*

We show now the existence of a determinantal point process with a given canonical kernel K and give an explicit construction of the point process.

Theorem 5.2.5. Construction of canonical determinantal point processes *[49, Theorem 4.5.3]. Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal set in $L^2_{\mathbb{C}}(\mu,\mathbb{G})$ and let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be such that each λ_n is in $[0,1]$ and $\sum_{n\in\mathbb{N}^*}\lambda_n<\infty$. Let $Z=\{Z_n\}_{n\in\mathbb{N}^*}$ be a sequence of independent Bernoulli random variables with respective means $\{\lambda_n\}_{n\in\mathbb{N}^*}$. Given a realization $z=\{z_n\}_{n\in\mathbb{N}^*}$ of Z such that $\sum_{n\in\mathbb{N}^*}z_n<\infty$, let*

$$K_z(x,y)=\sum_{n\in\mathbb{N}^*}z_n\varphi_n(x)\varphi_n(y)^*,\quad x,y\in\mathbb{G},$$

and let Φ_z be an elementary determinantal point process with background measure μ and kernel K_z. Then the mixture Φ_Z in the sense of Definition 2.2.25 is a well defined point process on $\mathbb{G}$ which is determinantal with background measure μ and kernel K given by (5.2.1) (that is the canonical kernel associated to $\{\varphi_n\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$).

Proof. Let $\mathbb{G}_1$ be given by (5.2.2). By Lemma 16.A.25(i), $\mu(\mathbb{G}\backslash\mathbb{G}_1)=0$. It follows from Lemma 5.1.10 and Proposition 5.1.11 that it is enough to specify the kernel K on $\mathbb{G}_1\times\mathbb{G}_1$ as in (5.2.1). Moreover, K is in $L^2_{\mathbb{C}}(\mu^2,\mathbb{G}^2)$ by Lemma 16.A.25(iv). (i) Observe that since $\mathbf{E}\left[\sum_{n\in\mathbb{N}^*}Z_n\right]=\sum_{n\in\mathbb{N}^*}\lambda_n<\infty$, then $\sum_{n\in\mathbb{N}^*}Z_n<\infty$ almost surely. Let $z=\{z_n\}_{n\in\mathbb{N}^*}$ be a realization of Z such that $\sum_{n\in\mathbb{N}^*}z_n<\infty$. Then only a finite number of the z_n are non-zero, thus there exists a determinantal process Φ_z with kernel K_z by Lemma 5.2.3. Namely Φ_z has $N=\sum_{n\in\mathbb{N}^*}z_n$ points almost surely and the corresponding Janossy measure is

$$J^z_N(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_N):=\det\left(K_z(x_i,x_j)\right)_{1\le i,j\le N}\ \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_N).\quad(5.2.5)$$

The finite dimensional distributions of Φ_z are related to its Janossy measures by Corollary 4.3.9 as follows. For all disjoint sets $A_1,\ldots,A_k\in\mathcal{B}(\mathbb{G})$ and all $n_1,\ldots,n_k\in\mathbb{N}$ such that $n_1+\ldots+n_k=n\le N$,

$$\mathbf{P}\left(\Phi_z(A_1)=n_1,\ldots,\Phi_z(A_k)=n_k\right)=\frac{1}{n_1!\ldots n_k!}\frac{J^z_N\left(A_1^{n_1}\times\cdots\times A_k^{n_k}\times B^r\right)}{(N-n)!},$$

where $B=(A_1\cup\ldots\cup A_k)^{\mathrm{c}}$ and $r=N-n$. The above quantity is measurable with respect to z. Then, by Lemma 2.2.26, the mixture Φ_Z in the sense of Definition 2.2.25 is a well defined point process on $\mathbb{G}$. We will show now that Φ_Z is determinantal. Indeed, for any $k\in\mathbb{N}^*$, the k-th factorial moment measure of Φ_Z equals

$$\begin{aligned} M_{\Phi_Z^{(k)}}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k\right) &= \mathbf{E}\left[\Phi_Z^{(k)}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k\right)\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\Phi_Z^{(k)}\left(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k\right) \mid Z\right]\right] \\ &= \mathbf{E}\left[\det\left(K_Z\left(x_i, x_j\right)\right)_{1\le i,j\le k}\right] \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right). \end{aligned}$$

It remains to show that, for μ^k-almost all $(x_1, \ldots, x_k) \in \mathbb{G}^k$,

$$\mathbf{E}\left[\det\left(K_Z\left(x_i, x_j\right)\right)_{1\le i,j\le k}\right] = \det\left(K\left(x_i, x_j\right)\right)_{1\le i,j\le k}, \tag{5.2.6}$$

which we do next. (ii) We will first prove (5.2.6) for some truncation of the kernels. More precisely, for any $N \in \mathbb{N}$, we will prove that

$$\mathbf{E}\left[\det\left(K_Z^{(N)}\left(x_i, x_j\right)\right)_{1\le i,j\le k}\right] = \det\left(K^{(N)}\left(x_i, x_j\right)\right)_{1\le i,j\le k}, \tag{5.2.7}$$

where

$$K_Z^{(N)}(x,y) = \sum_{n=1}^{N} Z_n\varphi_n(x)\varphi_n(y)^*, \quad x,y \in \mathbb{G},$$

and

$$K^{(N)}(x,y) = \sum_{n=1}^{N} \lambda_n\varphi_n(x)\varphi_n(y)^*, \quad x,y \in \mathbb{G}.$$

Indeed, observe that $\left(K_Z^{(N)}\left(x_i, x_j\right)\right)_{1\le i,j\le k} = AB$ where A is the matrix in $\mathbb{C}^{k\times N}$ defined by

$$A_{in} = Z_n\varphi_n\left(x_i\right), \quad 1 \le i \le k, 1 \le n \le N,$$

and B is the matrix in $\mathbb{C}^{N\times k}$ defined by

$$B_{nj} = \varphi_n\left(x_j\right)^*, \quad 1 \le n \le N, 1 \le j \le k.$$

By the Cauchy-Binet formula [48, §0.8.7 p.22], we get

$$\det(AB) = \sum_{1\le n_1,\ldots,n_k\le N} \det(A[n_1,\ldots,n_k])\det(B\left\{n_1,\ldots,n_k\right\}),$$

where $A[n_1, \ldots, n_k]$ is the submatrix of A composed by the columns numbered $n_1, \ldots, n_k$ and $B\{n_1, \ldots, n_k\}$ is the submatrix of B composed by the rows numbered $n_1, \ldots, n_k$. Moreover, by the very definition of the determinant of a matrix

$$\det\left(A[n_1,\ldots,n_k]\right) = \sum_{\pi\in S_k} \mathrm{sgn}(\pi)\prod_{l=1}^{k} Z_{\pi(n_l)}\varphi_{\pi(n_l)}(x_l),$$

where S_k is the set of permutations of $\{n_1, \ldots, n_k\}$ and $\mathrm{sgn}(\pi)$ denotes the signature of the permutation π. Then

$$\mathbf{E}\left[\det(A[n_1, \ldots, n_k])\right] = \det(C[n_1, \ldots, n_k]),$$

where C is a matrix in $\mathbb{C}^{k\times N}$ defined by

$$C_{in} = \lambda_n \ \varphi_n\left(x_i\right), \quad 1 \leq i \leq k, 1 \leq n \leq N,$$

Therefore,

$$\begin{aligned}\mathbf{E}\left[\det(AB)\right] &= \sum_{1\leq n_1,\ldots,n_k\leq N} \mathbf{E}\left[\det(A[n_1, \ldots, n_k])\right] \det(B\left\{n_1, \ldots, n_k\right\}) \\ &= \sum_{1\leq n_1,\ldots,n_k\leq N} \det(C[n_1, \ldots, n_k]) \det(B\left\{n_1, \ldots, n_k\right\}) \\ &= \det\left(K^{(N)}\left(x_i, x_j\right)\right)_{1\leq i,j\leq k},\end{aligned}$$

where the last equality is due the Cauchy-Binet formula and the fact that $\left(K^{(N)}\left(x_i, x_j\right)\right)_{1\leq i,j\leq k} = CB$. This proves (5.2.7). (iii) We take now the limit of (5.2.7) when $N \to \infty$. The right-hand side of (5.2.7) converges to $\det\left(K\left(x_i, x_j\right)\right)_{1\leq i,j\leq k}$ for μ^k-almost all $(x_1, \ldots, x_k) \in \mathbb{G}^k$. Then it remains to show that, for any $(x_1, \ldots, x_k) \in \mathbb{G}^k$,

$$\lim_{N\to\infty} \mathbf{E}\left[\det\left(K_Z^{(N)}\left(x_i, x_j\right)\right)_{1\leq i,j\leq k}\right] = \mathbf{E}\left[\det\left(K_Z\left(x_i, x_j\right)\right)_{1\leq i,j\leq k}\right], \quad (5.2.8)$$

to finish the proof of (5.2.6). Recall that, almost surely $\sum_{n\in\mathbb{N}^*} Z_n < \infty$. Then only a finite number of the Z_n are non-zero, thus for any $x, y \in \mathbb{G}$,

$$\lim_{N\to\infty} K_Z^{(N)}(x, y) = \lim_{N\to\infty} \sum_{n=1}^{N} Z_n \varphi_n(x)\varphi_n(y)^* = K_Z(x, y).$$

Therefore, since the determinant of a matrix is a multinomial of its components, for any $(x_1, \ldots, x_k) \in \mathbb{G}^k$,

$$\lim_{N\to\infty} \det\left(K_Z^{(N)}\left(x_i, x_j\right)\right)_{1\leq i,j\leq k} = \det\left(K_Z\left(x_i, x_j\right)\right)_{1\leq i,j\leq k}.$$

Taking the expectation in the above equality we get

$$\mathbf{E}\left[\lim_{N\to\infty} \det\left(K_Z^{(N)}\left(x_i, x_j\right)\right)_{1\leq i,j\leq k}\right] = \mathbf{E}\left[\det\left(K_Z\left(x_i, x_j\right)\right)_{1\leq i,j\leq k}\right].$$

Since $\det\left(K_Z^{(N)}\left(x_i, x_j\right)\right)_{1\leq i,j\leq k}$ is increasing with N by Lemma 16.A.25(v), we may exchange the expectation and the limit in the right-hand side of the above equality by the monotone convergence theorem to get (5.2.8). $\square$

Corollary 5.2.6. *Let Φ be a determinantal point process as in Theorem 5.2.5. Then $\Phi(\mathbb{G})$ is the sum of independent Bernoulli random variables with respective means $\{\lambda_n\}_{n\in\mathbb{N}^*}$. In particular, the void probability of Φ is smaller than that of Poisson point process with the same mean measure.*

Proof. Denote the point process Φ by Φ_Z as in Theorem 5.2.5. Observe that, given $Z = z$, then Φ_Z is an elementary determinantal point process with $\sum_{n\in\mathbb{N}^*} z_n$ points by Lemma 5.2.3. Then $\Phi_Z(\mathbb{G}) = \sum_{n\in\mathbb{N}^*} Z_n$, which proves the first part of the Corollary. It follows that

$$\mathbf{P}\left(\Phi_Z(\mathbb{G}) = 0\right) = \mathbf{P}\left(\sum_{n\in\mathbb{N}^*} Z_n = 0\right) = \prod_{n\in\mathbb{N}^*}(1-\lambda_n) \leq e^{-\sum_{n\in\mathbb{N}^*}\lambda_n},$$

since $1-\lambda \leq e^{-\lambda}$ for any real λ; which shows the last assertion of the Corollary. □

5.2.2 Integral operator: essentials

We will now attempt to represent a general kernel $K : \mathbb{G}^2 \to \mathbb{C}$ in its canonical form so that it can be used to construct the corresponding determinantal point process. In this regard, we shall use some results from functional analysis presented in details in Chapter 16 which we summarize now.

In the whole section, let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$ and $K \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$. Observe that since $\mathbb{G}$ is l.c.s.h, then the space $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable by Lemma 16.A.2. We introduce an *integral operator* $\mathcal{K}_{\mathbb{G}}$ associated to K defined on $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ by

$$\mathcal{K}_{\mathbb{G}} f(x) = \int_{\mathbb{G}} K(x,y) f(y) \mu(\mathrm{d}y), \quad f \in L^2_{\mathbb{C}}(\mu, \mathbb{G}), x \in \mathbb{G},$$

cf. Definition 16.A.8. The properties of this operator are stated in Section 16.A. In particular, the set of eigenvalues of $\mathcal{K}_{\mathbb{G}}$ is at most countable, and has at most one accumulation point, namely, 0. Moreover, each non-zero eigenvalue has finite multiplicity; cf. Proposition 16.A.11(ii).

The operator $\mathcal{K}_{\mathbb{G}}$ is Hermitian iff

$$K(x,y) = K(y,x)^*, \quad \text{for } \mu^2\text{-almost all } (x,y) \in \mathbb{G}^2,$$

cf. Lemma 16.A.9(vi). In this case, K has the following representation in $L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$

$$K(x,y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x) \varphi_n(y)^*,$$

where $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ with respective *real* eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$; cf. Proposition 16.A.13(i)-(iii).

A Hermitian operator $\mathcal{K}_{\mathbb{G}}$ is called *trace class* if $\sum_{n\in\mathbb{N}^*} |\lambda_n| < \infty$; cf. Corollary 16.A.18(i).

Finally, a Hermitian operator $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite (i.e., $\langle \mathcal{K}_{\mathbb{G}} f, f\rangle \geq 0$ for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$) iff $\lambda_n \geq 0$ for all $n \in \mathbb{N}^*$; cf. Proposition 16.A.13(v).

Remark 5.2.7. Gaussian covariance functions define nonnegative-definite operator. *If μ is a finite measure and $\mathbb{G}$ is a compact metric space, then an integral operator $\mathcal{K}_{\mathbb{G}}$ with continuous Hermitian kernel K is Hermitian nonnegative-definite iff there exist a Gaussian process indexed by* $\operatorname{supp}(\mu)$ *with covariance function K; cf. Proposition 16.A.31(i) and Theorem 16.A.32.*

5.2.3 Canonical version of a kernel

We introduce now the notion of canonical version of a kernel.

Definition 5.2.8. Canonical version of a kernel. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let $K \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$. Assume that the integral operator $\mathcal{K}_{\mathbb{G}}$ is Hermitian, nonnegative-definite, and trace class. Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ with respective eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$. Then the kernel $\tilde{K}$ defined by the right-hand side of (5.2.1) (with $\mathbb{G}_1$ given by (5.2.2)) is called the* canonical version *of K.*

Note that K and its canonical version $\tilde{K}$ are not necessarily μ-indistinguishable in general. Hence, a determinantal point process constructed with kernel $\tilde{K}$ does not necessarily admit K as kernel. In what follows we will develop sufficient conditions for indistinguishablility between K and $\tilde{K}$. Our first observation is that this holds for pre-canonical kernels K.

Proposition 5.2.9. Kernel versus its canonical version. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let $K \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$. Assume that the integral operator $\mathcal{K}_{\mathbb{G}}$ is Hermitian, nonnegative-definite, and trace class. Then the following results hold.*

(i) The kernel K and its canonical version coincide μ^2-almost everywhere.

(ii) The kernel K and its canonical version lead to the same integral operator.

(iii) If $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite, then for all integers $k \geq 2$, the matrix $(K(x_i, x_j))_{1\leq i,j\leq k}$ is Hermitian and nonnegative-definite for μ^k-almost all $(x_1, \ldots, x_k) \in \mathbb{G}^k$.

(iv) If K is a pre-canonical kernel (cf. Definition 5.2.1), then K is μ-indistinguishable from its canonical version.

Proof. (i) This follows from Proposition 16.A.13(iii). (ii) This follows from (i) and Lemma 16.A.9(iv). (iii) This follows from Proposition 16.A.13(vi). (iv) Assume that K is a pre-canonical kernel; i.e., it has the form (5.2.1) where $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is an arbitrary sequence in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ are nonnegative real numbers such that $\sum_{n\in\mathbb{N}^*} \lambda_n \|\varphi_n\|^2 < \infty$. Let $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, let P_D be the projection from $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ to $L^2_{\mathbb{C}}(\mu, D)$ defined by (16.A.25), let K_D be the restriction of K to $D \times D$ and let $\mathcal{K}_D$ be the integral operator associated to K_D.

Since K has the form (5.2.1), then $K_D(x,y) = \sum_{n\in\mathbb{N}^*} \lambda_n P_D\varphi_n(x) P_D\varphi_n(y)^*$, which is a pre-canonical kernel. Thus, by Proposition 16.A.26(iii),

$$\mathrm{tr}\,(\mathcal{K}_D) = \int_D K(x,x)\mu\,(\mathrm{d}x)\,. \tag{5.2.9}$$

Let $\tilde{K}$ be a canonical version of K and let $\tilde{K}_D$ be the restriction of $\tilde{K}$ to $D\times D$. By Item (i), $K\,(x,y)$ and $\tilde{K}\,(x,y)$ coincide for μ^2-almost all $(x,y)\in D^2$. Then, by Lemma 16.A.9(iv), the integral operator associated to $\tilde{K}_D$ is also $\mathcal{K}_D$. Since $\tilde{K}_D$ is a pre-canonical kernel, then, again by Proposition 16.A.26(iii),

$$\mathrm{tr}\,(\mathcal{K}_D) = \int_D \tilde{K}(x,x)\mu\,(\mathrm{d}x)\,. \tag{5.2.10}$$

Comparing (5.2.9) and (5.2.10), we get $\int_D K(x,x)\mu\,(\mathrm{d}x) = \int_D \tilde{K}(x,x)\mu\,(\mathrm{d}x)$. This being true for any $D\in\mathcal{B}_{\mathrm{c}}\,(\mathbb{G})$, it follows that $K\,(x,x) = \tilde{K}\,(x,x)$, for μ-almost all $x\in\mathbb{G}$. □

Proposition 5.2.9(iv) shows that a pre-canonical kernel K is indistinguishable from its canonical version $\tilde{K}$. Hence, a determinantal point process constructed with kernel $\tilde{K}$ in Theorem 5.2.5 (under assumption $\lambda_n\in[0,1]$ for all $n\in\mathbb{N}^*$) admits also K as its kernel; cf. Proposition 5.1.11(iii). This is not necessarily true in full generality as we will see in Example 5.2.15. This motivates the introduction of the notion of *regular* kernels.

5.2.4 Regular kernels

We shall define several regularity classes for kernels; some of them will be usefull later when studying the broader class of α-determinantal point processes in Section 5.3 which includes the determinantal point processes of Definition 5.1.1 as a particular case corresponding to $\alpha=-1$.

Definition 5.2.10. Regularizable kernel. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, let $K:\mathbb{G}^2\to\mathbb{C}$ be a measurable function, and let $\alpha\in\mathbb{R}^*_-$.*

(i) K is called regularizable *on $\mathbb{G}$ iff (i) $K\in L^2_{\mathbb{C}}\left(\mu^2,\mathbb{G}^2\right)$; and (ii) the integral operator $\mathcal{K}_{\mathbb{G}}$ defined by (16.A.1) is Hermitian, nonnegative-definite, and trace class. If moreover the eigenvalues of $\mathcal{K}_{\mathbb{G}}$ are not larger than $-1/\alpha\in\mathbb{R}^*_+$, then K is called* α-regularizable *on $\mathbb{G}$.*

(ii) K is called locally regularizable *(resp.* locally α-regularizable*) on $\mathbb{G}$ iff K is regularizable (resp. α-regularizable) on all $D\in\mathcal{B}_{\mathrm{c}}\,(\mathbb{G})$.*

Definition 5.2.11. Regular kernel. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, let $K:\mathbb{G}^2\to\mathbb{C}$ be a measurable function, and let $\alpha\in\mathbb{R}^*_-$.*

(i) K is called regular *(resp.* α-regular*) on $\mathbb{G}$ iff K is regularizable (resp. α-regularizable) on $\mathbb{G}$ and μ-indistinguisable from its canonical version.*

(ii) K *is called* locally regular *(resp.* locally α-regular*) on* $\mathbb{G}$ *iff* K *is regular (resp.* α*-regular) on all* $D \in \mathcal{B}_c(\mathbb{G})$*.*

The considered measure μ will often be clear from the context; otherwise we will say K is regularizable, regular, etc. *with respect to* μ.

Note that if K is regularizable (resp. α-regularizable) on $\mathbb{G}$, then its canonical version $\tilde{K}$ is regular (resp. α-regular) on $\mathbb{G}$ by Proposition 5.2.9(iv).

Lemma 5.2.12. Global versus local regularity. *Let* μ *be a locally finite measure on a l.c.s.h. space* $\mathbb{G}$, $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$ *and* $\mathcal{K}_{\mathbb{G}}$ *the associated integral operator. For any* $D \in \mathcal{B}_c(\mathbb{G})$*, let* $\mathcal{K}_D$ *be the integral operator associated to the restriction of* K *to* $D \times D$*. Then the following results hold.*

(i) $\mathcal{K}_{\mathbb{G}}$ *is Hermitian iff* $\mathcal{K}_D$ *is Hermitian for any* $D \in \mathcal{B}_c(\mathbb{G})$*.*

(ii) $\mathcal{K}_{\mathbb{G}}$ *is nonnegative-definite iff* $\mathcal{K}_D$ *is nonnegative-definite for any* $D \in \mathcal{B}_c(\mathbb{G})$*.*

(iii) K *is regularizable on* $\mathbb{G}$ *iff it is locally regularizable on* $\mathbb{G}$ *and* $\mathcal{K}_{\mathbb{G}}$ *is trace class.*

Proof. By Lemma 1.1.4, there exists increasing sets $D_1, D_2, \ldots \in \mathcal{B}_c(\mathbb{G})$ such that $\mathbb{G} = \bigcup_{k\in\mathbb{N}^*} D_k$. (i) Necessity is obvious. It remains to prove sufficiency. For any $k \in \mathbb{N}^*$, the integral operator $\mathcal{K}_{D_k}$ is Hermitian, then by Lemma 16.A.9(vi), $K(x,y) = K(y,x)^*$ for μ^2-almost all $(x,y) \in D_k^2$. Thus $K(x,y) = K(y,x)^*$ for μ^2-almost all $(x,y) \in \mathbb{G}^2$. Invoking again Lemma 16.A.9(vi) shows that $\mathcal{K}_{\mathbb{G}}$ is Hermitian. (ii) Only sufficiency needs to be proved. Let $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Then $\mathbf{1}_{D_k} \times f \underset{k\to\infty}{\to} f$ in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, thus $\mathcal{K}_{\mathbb{G}}\mathbf{1}_{D_k} \times f \underset{k\to\infty}{\to} \mathcal{K}_{\mathbb{G}} f$ in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ by Lemma 16.A.9(ii). By the bicontinuity of the inner product in Hilbert spaces [21, Theorem 1.3.3 p.57], $\langle \mathcal{K}_{\mathbb{G}}\mathbf{1}_{D_k} f, \mathbf{1}_{D_k} f\rangle \underset{k\to\infty}{\to} \langle \mathcal{K}_{\mathbb{G}} f, f\rangle$. Since for any $k \in \mathbb{N}^*$, $\langle \mathcal{K}_{\mathbb{G}}\mathbf{1}_{D_k} f, \mathbf{1}_{D_k} f\rangle = \langle \mathcal{K}_{D_k}\mathbf{1}_{D_k} f, \mathbf{1}_{D_k} f\rangle$ is nonnegartive, then $\langle \mathcal{K}_{\mathbb{G}} f, f\rangle \geq 0$. Therefore, $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite. (iii) *Necessity.* Assume that K is regularizable on $\mathbb{G}$. Then the integral operator $\mathcal{K}_{\mathbb{G}}$ is trace class by definition. Let $D \in \mathcal{B}_c(\mathbb{G})$ and let $\mathcal{K}_D$ be the integral operator associated to the restriction of K to $D \times D$. By Lemma 16.A.23(ii), for any $f \in L^2_{\mathbb{C}}(\mu, D)$,

$$\mathcal{K}_D f = \sum_{n\in\mathbb{N}^*} \langle f, P_D\varphi_n\rangle \lambda_n P_D \varphi_n,$$

where equality is in $L^2_{\mathbb{C}}(\mu, D)$ and P_D is the projection defined by (16.A.25). Observe moreover that $\sum_{n\in\mathbb{N}^*} \lambda_n \|P_D\varphi_n\|^2 \leq \sum_{n\in\mathbb{N}^*} \lambda_n \|\varphi_n\|^2 = \sum_{n\in\mathbb{N}^*} \lambda_n < \infty$. Then by Proposition 16.A.26(i), $\mathcal{K}_D$ is the integral operator associated to the pre-canonical kernel associated to $\{P_D\varphi_n\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$. Thus by Proposition 16.A.26(ii)-(iii), $\mathcal{K}_D$ is Hermitian, nonnegative-definite, and trace class. This being true for any $D \in \mathcal{B}_c(\mathbb{G})$, it follows that K is locally regularizable on $\mathbb{G}$. *Sufficiency.* Assume that $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$ is locally regularizable on $\mathbb{G}$ and that the integral operator $\mathcal{K}_{\mathbb{G}}$ is trace class. Then by Items (i)-(ii), $\mathcal{K}_{\mathbb{G}}$ is Hermitian and nonnegative-definite. Thus K is regularizable on $\mathbb{G}$ by definition. □

By Mercer's theorem 16.A.28, regularizable continuous kernels are regular:

Corollary 5.2.13. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be a continuous function. Then the following results hold.*

(i) If K is locally regularizable on $\mathbb{G}$, then K is locally regular on $\mathbb{G}$.

(ii) If K is regularizable on $\mathbb{G}$, then K is regular on $\mathbb{G}$.

Proof. (i) Assume that K is locally regularizable on $\mathbb{G}$. Let D be a compact subset of $\mathbb{G}$, K_D be the restriction of K to $D \times D$, μ_D be the restriction of μ to D, and $\tilde{K}_D$ be a canonical version of K_D. Mercer's theorem 16.A.28 shows that

$$K_D(x,y) = \tilde{K}_D\left(x,y\right), \quad x,y \in \operatorname{supp}\left(\mu_D\right). \tag{5.2.11}$$

By Lemma 14.B.3, $\mu_D\left(D \backslash \operatorname{supp}\left(\mu_D\right)\right) = 0$, then Lemma 5.1.10 implies that K_D is indistinguisable from $\tilde{K}_D$. Then K is locally regular on $\mathbb{G}$. (ii) Assume that K is regularizable on $\mathbb{G}$. By Lemma 5.2.12, K is locally regularizable on $\mathbb{G}$. Then by Item (i), K is locally regular on $\mathbb{G}$. Then (5.2.11) holds true for any compact subset D of $\mathbb{G}$. By Lemma 1.1.4, $\mathbb{G}$ may be covered by a countable union of compact sets $\{D_n\}_{n\in\mathbb{N}^*}$. Since for each D_n, the equality (5.2.11) holds, it follows that

$$K(x,y) = \tilde{K}\left(x,y\right), \quad x,y \in \operatorname{supp}\left(\mu\right).$$

Then K is regular on $\mathbb{G}$. □

Remark 5.2.14. Shift-invariant kernels and α-regularity. *In the particular case of shift-invariant kernels on $\mathbb{R}^d$ (i.e. $K(x,y) = K(x+t, y+t)$ for all $x,y,t \in \mathbb{R}^d$), sufficient conditions for α-*regularity *will be given in Theorem 5.6.8 in terms of the Fourier transform of $K(x,0)$.*

Example 5.2.15. The Poisson kernel is not regular. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with diffuse intensity measure μ such that $0 < \mu\left(\mathbb{G}\right) < \infty$. We have already seen in Example 5.1.6 that Φ is a determinantal point process with background measure μ and kernel $K\left(x,y\right) = \mathbf{1}_{\{x=y\}}$ for all $x,y \in \mathbb{G}$. The integral operator associated to K is*

$$\mathcal{K}_{\mathbb{G}} f(x) = \int_{\mathbb{G}} \mathbf{1}_{\{x=y\}} f\left(y\right) \mu\left(\mathrm{d}y\right) = 0.$$

Then the canonical version of K is the null kernel; that is $\tilde{K}\left(x,y\right) = 0$ for all $x,y \in \mathbb{G}$, leading to the null point process. Observe that $K\left(x,x\right) = 1 \neq \tilde{K}\left(x,x\right)$ for all $x \in \mathbb{G}$, therefore, K is not regular.

Theorem 5.2.16. Existence of determinantal point processes with regular kernels. *Let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$, let $K : \mathbb{G}^2 \to \mathbb{C}$ be (-1)-regular on $\mathbb{G}$. Then the following results hold.*

(i) There exists a determinantal point process Φ on $\mathbb{G}$ with background measure μ and kernel K. Moreover, the distribution of Φ is uniquely determined by μ and K.

(ii) *The distribution of the total number of points of* Φ *is given by*

$$\mathbf{P}\left(\Phi\left(\mathbb{G}\right)=k\right)=\sum_{1\leq n_1<\ldots<n_k}\left[\lambda_{n_1}\ldots\lambda_{n_k}\prod_{l\neq n_1,\ldots,n_k}\left(1-\lambda_l\right)\right],\quad k\in\mathbb{N}, \tag{5.2.12}$$

where $\{\lambda_n\}_{n\in\mathbb{N}^*}$ *are the eigenvalues of the integral operator* $\mathcal{K}_{\mathbb{G}}$ *associated to* K. *In particular, the void probability of* Φ *equals* $\mathbf{P}\left(\Phi\left(\mathbb{G}\right)=0\right)=\prod_{n\in\mathbb{N}^*}\left(1-\lambda_n\right)$.

(iii) *For any* $k\in\mathbb{N}^*$, *the* k*-th Janossy measure* J_k *of* Φ *admits the following density with respect to* μ^k

$$\begin{aligned}&\sigma_k\left(x_1,\ldots,x_k\right) \\ &=\sum_{1\leq n_1<\ldots<n_k}\left[\lambda_{n_1}\ldots\lambda_{n_k}\prod_{l\neq n_1,\ldots,n_k}\left(1-\lambda_l\right)\right]\left|\det(\varphi_{n_j}(x_i))_{1\leq i,j\leq k}\right|^2,\end{aligned} \tag{5.2.13}$$

for μ^k*-almost all* $(x_1,\ldots,x_k)\in\mathbb{G}^k$, *where* $\{\varphi_n\}_{n\in\mathbb{N}^*}$ *is an orthonormal basis of* $L^2_{\mathbb{C}}\left(\mu,\mathbb{G}\right)$ *composed of eigenvectors of* $\mathcal{K}_{\mathbb{G}}$ *and* $\{\lambda_n\}_{n\in\mathbb{N}^*}$ *are the corresponding eigenvalues.*

Proof. (i) Let $\tilde{K}$ be a canonical version of K. By Theorem 5.2.5, there exists a determinantal point process Φ on $\mathbb{G}$ with background measure μ and kernel $\tilde{K}$. Since K and $\tilde{K}$ are μ-indistinguishable, then Φ has also kernel K with respect to the background measure μ by Proposition 5.1.11(iii). Uniqueness follows from Corollary 5.1.14 and Equation (16.A.35). (ii) With the notation in Theorem 5.2.5, the point process $\Phi=\Phi_Z$ has $\Phi_Z\left(\mathbb{G}\right)=\sum_{n\in\mathbb{N}^*}Z_n$ points almost surely. Then

$$\begin{aligned}&\mathbf{P}\left(\Phi_Z\left(\mathbb{G}\right)=k\right) \\ &=\sum_{1\leq n_1<\cdots<n_k}\mathbf{P}\left(Z_{n_1}=\cdots=Z_{n_k}=1,Z_{n_i}=0\text{ for }i\in\mathbb{N}^*\backslash\left\{n_1,\ldots,n_k\right\}\right) \\ &=\sum_{1\leq n_1<\ldots<n_k}\left[\lambda_{n_1}\ldots\lambda_{n_k}\prod_{l\neq n_1,\ldots,n_k}\left(1-\lambda_l\right)\right],\end{aligned}$$

where the last equality follows from the fact that $Z=\{Z_n\}_{n\in\mathbb{N}^*}$ is a sequence of independent Bernoulli random variables with respective means $\{\lambda_n\}_{n\in\mathbb{N}^*}$. Applying (5.2.12) with $k=0$ gives the expression fo the void probability of Φ. (iii) With the notation in Theorem 5.2.5, given a realization $z=\{z_n\}_{n\in\mathbb{N}^*}$ of $Z=\{Z_n\}_{n\in\mathbb{N}^*}$, the Janossy measure of Φ_z is given by (5.2.5)

$$J_N^z\left(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_N\right):=\det\left(K_z\left(x_i,x_j\right)\right)_{1\leq i,j\leq N}\ \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_N\right),$$

where $N=\sum_{n\in\mathbb{N}^*}z_n$. Let $1\leq n_1<\cdots<n_N$ be the indexes n for which $z_n=1$, then, by Lemma 16.A.25(vi),

$$\det\left(K_z\left(x_i,x_j\right)\right)_{1\leq i,j\leq N}=\left|\det(\varphi_{n_j}(x_i))_{1\leq i,j\leq N}\right|^2.$$

Thus, deconditioning with respect to $Z = \{Z_n\}_{n\in\mathbb{N}^*}$, it follows that, for any $k \in \mathbb{N}^*$, the Janossy measure $J_k(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_k)$ of Φ admits the following density with respect to μ^k

$$\begin{aligned}
&\sigma_k(x_1, \ldots, x_k) \\
&= \sum_{1\leq n_1<\ldots<n_k} \mathbf{P}\left(Z_{n_1} = \ldots = Z_{n_k} = 1, Z_{n_i} = 0 \text{ for } i \in \mathbb{N}^* \setminus \{n_1, \ldots, n_N\}\right) \\
&\qquad\qquad \left|\det(\varphi_{n_j}(x_i))_{1\leq i,j\leq k}\right|^2 \\
&= \sum_{1\leq n_1<\ldots<n_k} \left[\lambda_{n_1} \ldots \lambda_{n_k} \prod_{l\neq n_1,\ldots,n_k} (1-\lambda_l)\right] \left|\det(\varphi_{n_j}(x_i))_{1\leq i,j\leq k}\right|^2 .
\end{aligned}$$

□

We aim now to extend the above result to locally regular kernels which allows to construct determinantal point processes with infinite number of atoms.

Theorem 5.2.17. Existence of determinantal point processes with locally regular kernels. *Let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be locally (-1)-regular on $\mathbb{G}$. Then the following results hold true.*

(i) There exists a determinantal point process Φ on $\mathbb{G}$ with background measure μ and kernel K. Moreover, the distribution of Φ is uniquely determined by μ and K.

(ii) Let $D \in \mathcal{B}_c(\mathbb{G})$, let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, D)$ composed of eigenvectors of $\mathcal{K}_D$ (the integral operator associated to the restriction of K to $D\times D$) and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding eigenvalues. Then the distribution of $\Phi(D)$ is given by (5.2.12) and for any $k \in \mathbb{N}^*$, the k-th Janossy measure J_k of the restriction of Φ to D admits the density σ_k given by (5.2.13) with respect to μ^k.

The proof of Theorem 5.2.17 will be given in Section 5.2.4. It relies on the construction of determinantal point processes on compact subsets of $\mathbb{G}$ and extension to the whole space by verifying *Kolmogorov consistency conditions.*

In the particular case of continuous kernels, the result of Theorem 5.2.17 may be strengthened as follows.

Corollary 5.2.18. Existence of determinantal point process with continuous kernels. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be continuous and locally (-1)-regularizable on $\mathbb{G}$. Then there exists a determinantal point process Φ on $\mathbb{G}$ with background measure μ and kernel K. Moreover, the distribution of Φ is uniquely determined by μ and K.*

Proof. By Corollary 5.2.13(i), K is locally (-1)-regular on $\mathbb{G}$. Theorem 5.2.17 allows one to conclude. □

Proof of Theorem 5.2.17

Before proving the above theorem, we need some preliminary results.

Lemma 5.2.19. Kolmogorov consistency conditions. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, $K : \mathbb{G}^2 \to \mathbb{C}$ be regularizable on $\mathbb{G}$, and $\tilde{K}$ be a canonical version of K. Then the following results hold.*

(i) For any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, let K_D be the restriction of K to $D \times D$, and $\widetilde{K_D}$ be a canonical version of K_D. Then $\widetilde{K_D}$ is μ-indistinguishable from $\tilde{K}_D$, the restriction of $\tilde{K}$ to $D \times D$.

(ii) Let $\alpha \in \mathbb{R}_-^*$. If K is α-regularizable (resp. regular, α-regular) on $\mathbb{G}$, then K is locally α-regularizable (resp. locally regular, locally α-regular) on $\mathbb{G}$.

(iii) Assume that K is (-1)-regularizable. Let Φ be a determinantal point process on $\mathbb{G}$ with kernel $\tilde{K}$ and background measure μ and let Φ^D be a determinantal point process on D with kernel $\widetilde{K_D}$ and background measure μ restricted to D. Then Φ^D has the same distribution as the restriction of Φ to D.

Proof. (i) Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding respective eigenvalues. The canonical version $\tilde{K}$ of K is defined by

$$\tilde{K}(x, y) = \mathbf{1}\{x, y \in \mathbb{G}_1\} \times \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x) \varphi_n(y)^*,$$

where $\mathbb{G}_1$ is defined by (5.2.2). Then

$$\tilde{K}_D(x, y) = \mathbf{1}\{x, y \in \mathbb{G}_1\} \times \sum_{n\in\mathbb{N}^*} \lambda_n P_D\varphi_n(x) P_D\varphi_n(y)^*,$$

which is a pre-canonical kernel. Observe that K_D and $\tilde{K}_D$ lead to the same integral operator by Lemma 16.A.23(ii) and Proposition 16.A.26(i). Then K_D and $\tilde{K}_D$ have the same canonical version; that is $\widetilde{K_D}$. Since $\tilde{K}_D$ is a pre-canonical kernel, then it is indistinguishable from its canonical version $\widetilde{K_D}$ by Proposition 5.2.9(iv). (ii) *Step 1*: α-regularizable $\Rightarrow$ locally α-regularizable. Assume that K is α-regularizable on $\mathbb{G}$. Observe that, for any $f \in L^2_{\mathbb{C}}(\mu, D)$,

$$\|\mathcal{K}_D f\|^2 \leq \|\mathcal{K}_{\mathbb{G}} f\|^2 \leq |\alpha|^{-2} \|f\|^2,$$

where the last equality is due to Proposition 16.A.13(iv). Then $\|\mathcal{K}_D\| \leq |\alpha|^{-1}$, thus the eigenvalues of $\mathcal{K}_D$ are not larger than $|\alpha|^{-1}$. Then K is locally α-regularizable on $\mathbb{G}$. *Step 2*: regular $\Rightarrow$ locally regular. Assume that K is regular on $\mathbb{G}$. By Lemma 5.2.12, K is locally regularizable on $\mathbb{G}$. It remains to show that, for any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, K_D is indistinguisable from its canonical version $\widetilde{K_D}$. Since K is regular, then K_D is indistinguisable from its canonical version $\tilde{K}_D$, the restriction of $\tilde{K}$ to $D \times D$. Item (i) permits to conclude. *Step 3*: α-regular $\Rightarrow$ locally α-regular. This follows from the first two steps. (iii) Observe that $\tilde{K}$ is regular. Moreover, by (iii), K is locally (-1)-regularizable on $\mathbb{G}$, then $\widetilde{K_D}$ is (-1)-regular. The existence of Φ and Φ^D then follows from Theorem 5.2.5. On the other hand, by Lemma 5.1.5(i), the restriction Φ_D of Φ to D is a determinantal point process on D with kernel $\tilde{K}_D$ and background measure μ restricted to D. Further, by Item (i) above, $\widetilde{K_D}$ is μ-indistinguishable from $\tilde{K}_D$. Then, by Proposition 5.1.11(iii), Φ^D has also kernel $\tilde{K}_D$. Corollary 5.1.16 implies that Φ^D and Φ_D have the same distribution. □

Lemma 5.2.20. *Let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be locally (-1)-regularizable on $\mathbb{G}$. For any $D \in \mathcal{B}_c(\mathbb{G})$, let K_D be the restriction of K to $D \times D$, and let $\widetilde{K_D}$ be a canonical version of K_D. Then there exists a determinantal point process Φ on $\mathbb{G}$ with background measure μ such that, for any $D \in \mathcal{B}_c(\mathbb{G})$, the restriction of Φ to D has for kernel $\widetilde{K_D}$.*

Proof. By Theorem 5.2.5, for each $D \in \mathcal{B}_c(\mathbb{G})$, there exists a determinantal point process Φ^D on $\mathbb{G}$ with kernel $\widetilde{K_D}$ and background measure μ. It remains to show the compatibility between the distributions of the point processes Φ^D over the different $D \in \mathcal{B}_c(\mathbb{G})$ to show that they are the distributions of the restrictions of the same determinantal point process Φ on $\mathbb{G}$. To do so consider the family of finite dimensional distributions of all the Φ^D when D ranges over $\mathcal{B}_c(\mathbb{G})$. By Lemma 5.2.19(iii), this family satisfies the Kolmogorov consistency conditions for point processes [31, Theorem 9.2.X] which concludes the proof. □

We are now ready to prove Theorem 5.2.17.

Proof of Theorem 5.2.17. (i) Since K is locally (-1)-regular, then for any $D \in \mathcal{B}_c(\mathbb{G})$, K_D is indistinguisable from its canonical version $\widetilde{K_D}$. Lemma 5.2.20 and Proposition 5.1.11(iii) show that there exists a determinantal point process Φ on $\mathbb{G}$ with background measure μ such that for any $D \in \mathcal{B}_c(\mathbb{G})$, the restriction of Φ to D has kernel K_D. Lemma 5.1.5(ii) implies that Φ has kernel K on $\mathbb{G}$. Uniqueness follows again from Corollary 5.1.14. (ii) This follows from Theorem 5.2.16(ii)-(iii) applied to the restriction of Φ to any $D \in \mathcal{B}_c(\mathbb{G})$. □

5.3 α-Determinantal point processes

5.3.1 Definition and basic properties

We now consider the class of α-determinantal point processes which includes the determinantal point processes of Definition 5.1.1 as a particular case corresponding to $\alpha = -1$. In this regard, we need the notion of α-determinant of a finite dimensional matrix; cf. Definition 15.A.6.

Definition 5.3.1. α-Determinantal point process. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, let $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function and let*

$\alpha \in \mathbb{R}$. A point process Φ on $\mathbb{G}$ is said to be an α-determinantal point process *with* background measure μ *and* kernel K *if for all $k \in \mathbb{N}^*$, the k-th factorial moment measure $M_{\Phi^{(k)}}$ admits a density with respect to the product measure μ^k which equals*

$$\rho^{(k)}(x_1, \ldots, x_k) = \det_\alpha (K(x_i, x_j))_{1 \le i,j \le k}, \quad \text{for } \mu^k\text{-almost all } x_1, \ldots, x_k \in \mathbb{G}, \tag{5.3.1}$$

where $\det_\alpha$ denotes the α-determinant (15.A.12). The function $\rho^{(k)}$ is called the k-th factorial moment density *with respect to μ^k. For $\alpha = 1$, we say that Φ on $\mathbb{G}$ is a* permanental point process. *For $\alpha = -1$, we retrieve the determinantal point processes of Definition 5.1.1.*

Remark 5.3.2. *The terminology for α-determinantal point processes is not unanimous in the literature.*

In the first paper by O. Macchi, the determinantal and permanental point processes are called fermion *and* boson *processes, respectively. This terminology is used by several other authors; e.g. in [88].*

For $\alpha > 0$, what we call an α-determinantal point process is called a permanent process *with parameter $1/\alpha$ in [68, §3] and an $1/\alpha$-permanental point processes in [62, Definition 12.2].*

For $\alpha < 0$, what we call an α-determinantal point process is called a determinant process *in [68, §6].*

Similarly, our α-determinantal point processes are called α-permanental point processes *by some authors.*

Observe that the mean measure of an α-determinantal point process Φ with background measure μ and kernel K is given by

$$M_\Phi(B) = \int_B K(x,x)\,\mu(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{G}). \tag{5.3.2}$$

In particular, $K(x,x) \in \mathbb{R}_+$ for μ-almost all $x \in \mathbb{G}$.

Example 5.3.3. Poisson is α-determinantal. *By the same arguments as in Example 5.1.6, for all $\alpha \in \mathbb{R}$, a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with a diffuse and locally finite intensity measure μ is an α-determinantal point process with background measure μ and kernel $K(x,y) = \mathbf{1}_{\{x=y\}}$ for all $x, y \in \mathbb{G}$. Indeed, by (5.1.4),*

$$\det_\alpha (K(x_i, x_j))_{1 \le i,j \le k} = 1, \quad \text{for } \mu^k\text{-almost all } x_1, \ldots, x_k \in \mathbb{G}.$$

Here is an extension of Lemma 5.1.3 for the thinning of α-determinantal point processes.

Lemma 5.3.4. Thinning of α-determinantal point process. *Let $\alpha \in \mathbb{R}$ and let Φ be an α-determinantal point process on a l.c.s.h. space $\mathbb{G}$ with background measure μ and kernel K, $p : \mathbb{G} \to [0,1]$ be some measurable function, and let $\tilde{\Phi}$ be the thinning of Φ with retention function p. Then $\tilde{\Phi}$ is an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel*

$$\tilde{K}(x,y) = \sqrt{p(x)}K(x,y)\sqrt{p(y)}, \quad x, y \in \mathbb{G}.$$

Proof. The proof follows the same lines as that of Lemma 5.1.3, with $\det(\cdot)$ replaced by $\det_\alpha(\cdot)$. □

The following proposition will be useful for the construction of α-determinantal point processes by superposition.

Proposition 5.3.5. Superposition of α-determinantal point processes. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, $K : \mathbb{G}^2 \to \mathbb{C}$ a measurable function, and $\Phi_1, \ldots, \Phi_m$ independent point processes such that Φ_l is an α_l-determinantal point process on $\mathbb{G}$ with background measure μ and kernel $\alpha_l^{-1}K$ for all $1 \le l \le m$. Assume that $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^*$ are such that $\alpha_1^{-1} + \cdots + \alpha_m^{-1} \neq 0$. Then the superposition $\Phi = \Phi_1 + \cdots + \Phi_m$ is an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel $\alpha^{-1}K$, where $\alpha = \left(\alpha_1^{-1} + \cdots + \alpha_m^{-1}\right)^{-1}$.*

Proof. It follows from [37] that for any matrix $A = (A_{ij})_{1\le i,j\le k} \in \mathbb{C}^{k\times k}$,

$$\det_\alpha(\alpha^{-1}A) = \sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^m \det_{\alpha_l}\left(\alpha_l^{-1}\left(A_{ij}\right)_{i,j\in I_l}\right), \tag{5.3.3}$$

where the summation $\sum_{\{I_1,\ldots,I_m\}}$ is over all partitions $\{I_1, \ldots, I_m\}$ of $\{1, \ldots, k\}$. On the other hand, by Lemma 14.E.5, for any $k \in \mathbb{N}^*$ and any $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{G})$,

$$\Phi^{(k)}\left(B_1 \times \cdots \times B_k\right) = \sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^m \Phi_l^{(|I_l|)}\left(\prod_{u\in I_l} B_u\right).$$

Taking expectation and using the independence of the Φ_l's, we get

$$\begin{aligned}
&M_{\Phi^{(k)}}\left(B_1 \times \cdots \times B_k\right)\\
&\quad= \sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^m M_{\Phi_l^{(|I_l|)}}\left(\prod_{u\in I_l} B_u\right)\\
&\quad= \sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^m \int_{\prod_{u\in I_l} B_u} \det_{\alpha_l}\left(\alpha_l^{-1}K\left(x_i, x_j\right)\right)_{i,j\in I_l} \prod_{u\in I_l} \mu\left(\mathrm{d}x_u\right)\\
&\quad= \sum_{\{I_1,\ldots,I_m\}} \int_{B_1\times\ldots\times B_k} \left(\prod_{l=1}^m \det_{\alpha_l}\left(\alpha_l^{-1}K\left(x_i, x_j\right)\right)_{i,j\in I_l}\right) \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right)\\
&\quad= \int_{B_1\times\ldots\times B_k} \left(\sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^m \det_{\alpha_l}\left(\alpha_l^{-1}K\left(x_i, x_j\right)\right)_{i,j\in I_l}\right) \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right)\\
&\quad= \int_{B_1\times\ldots\times B_k} \det_\alpha\left(\alpha^{-1}K\left(x_i, x_j\right)\right)_{1\le i,j\le k} \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right),
\end{aligned}$$

where the last equality is due to (5.3.3). □

Similarly to Lemma 5.1.5, we have:

Lemma 5.3.6. α-Determinantal point process restriction. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$, μ some locally finite measure on $\mathbb{G}$, $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function and $\alpha \in \mathbb{R}$. For any $D \in \mathcal{B}(\mathbb{G})$, let μ_D be the restriction of μ to D and K_D be the restriction of K to $D \times D$.*

(i) If Φ is an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel K, then, for any $D \in \mathcal{B}(\mathbb{G})$, the restriction of Φ to D is an α-determinantal point process on D with background measure μ_D and kernel K_D.

(ii) Inversely, if for any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, the restriction of Φ to D is an α-determinantal point process on D with background measure μ_D and kernel K_D, then Φ is an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel K.

Proof. The proof follows in the same lines as that of Lemma 5.1.5. □

Similarly to Proposition 5.1.11, we have:

Proposition 5.3.7. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, let K and $\tilde{K}$ be two μ-indistinguishable measurable functions from $\mathbb{G}^2$ to $\mathbb{C}$ and let $\alpha \in \mathbb{R}$. Then the following results hold true.*

(i) For all $k \in \mathbb{N}^*$,

$$\det\nolimits_\alpha \left(\tilde{K}(x_i, x_j)\right)_{1\le i,j\le k} = \det\nolimits_\alpha \left(K(x_i, x_j)\right)_{1\le i,j\le k},$$

for μ^k-almost all $(x_1, \ldots, x_k) \in \mathbb{G}^k$.

(ii) If Φ and $\tilde{\Phi}$ are α-determinantal point processes with background measure μ and kernels K and $\tilde{K}$ respectively, then Φ and $\tilde{\Phi}$ have the same factorial moment measures.

(iii) If Φ is an α-determinantal point process with background measure μ and kernel K, then Φ has also kernel $\tilde{K}$ (with respect to the background measure μ).

Proof. The proof follows in the same lines as that of Proposition 5.1.11. □

5.3.2 Uniqueness of distribution

Here is the analogue of Lemma 5.1.12 for α-determinantal point processes.

Proposition 5.3.8. *Let $\alpha \in \mathbb{R}^*$ and let Φ be an α-determinantal point process on a l.c.s.h. space $\mathbb{G}$ with background measure μ and kernel K where μ is a locally finite measure on $\mathbb{G}$. Assume moreover that $\int_{\mathbb{G}} K(x,x)\,\mu(\mathrm{d}x) < \infty$ (equivalently, by (5.3.2), $\mathbf{E}[\Phi(\mathbb{G})] < \infty$) and that for any $k \geq 2$, the* *matrix* $(K(x_i, x_j))_{1\leq i,j\leq k}$ *is Hermitian nonnegative-definite for* μ^k*-almost all* $(x_1, \ldots, x_k) \in \mathbb{G}^k$*. Then*

$$\mathbf{E}\left[\Phi(\mathbb{G})^{(k)}\right] \leq k!\left(|\alpha|\,\mathbf{E}[\Phi(\mathbb{G})]\right)^k, \quad \text{for all } k \in \mathbb{N}^*,$$

and

$$\mathbf{E}\left[(1+s)^{\Phi(\mathbb{G})}\right] \leq \frac{1}{1-s|\alpha|\,\mathbf{E}[\Phi(\mathbb{G})]}, \quad \text{for all } s \in \left[0, \frac{1}{|\alpha|\,\mathbf{E}[\Phi(\mathbb{G})]}\right).$$

Moreover, the radius of convergence $R_{\mathcal{G}_{\Phi(\mathbb{G})}}$ *of the generating function* $\mathcal{G}_{\Phi(\mathbb{G})}$ *is larger than or equal to* $1 + \frac{1}{|\alpha|\mathbf{E}[\Phi(\mathbb{G})]}$*. Further, the distribution of* Φ *is uniquely determined by* μ *and* K*.*

Proof. Let $A = (A_{ij})_{1\leq i,j\leq k} \in \mathbb{C}^{k\times k}$ be a Hermitian nonnegative-definite matrix. Observe that

$$\begin{aligned}
\det{}_\alpha(A) &\leq \sum_{\pi\in S_k} |\alpha|^{k-\mathrm{cyc}(\pi)} \prod_{i=1}^k \left|A_{i\pi(i)}\right| \\
&\leq |\alpha|^k \sum_{\pi\in S_k} \prod_{i=1}^k \left|A_{i\pi(i)}\right| \\
&\leq |\alpha|^k\, k! \prod_{i=1}^k A_{ii},
\end{aligned} \tag{5.3.4}$$

where the third equality follows from (15.A.4). Then by the very definition of an α-determinantal point process, for any $k \geq 2$,

$$\begin{aligned}
\mathbf{E}\left[\Phi(\mathbb{G})^{(k)}\right] &= \int_{\mathbb{G}^k} \det{}_\alpha \left(K(x_i, x_j)\right)_{1\leq i,j\leq k} \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k) \\
&\leq |\alpha|^k\, k! \int_{\mathbb{G}^k} \prod_{i=1}^k K(x_i, x_i)\, \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k) = k!\left(|\alpha|\,\mathbf{E}[\Phi(\mathbb{G})]\right)^k.
\end{aligned}$$

It follows from the above inequality that, for any $s \in \mathbb{R}_+^*$,

$$\mathbf{E}\left[(1+s)^{\Phi(\mathbb{G})}\right] = \sum_{k\in\mathbb{N}} \frac{1}{k!}\mathbf{E}\left[\Phi(\mathbb{G})^{(k)}\right] s^k \leq \sum_{k\in\mathbb{N}} \left(s|\alpha|\,\mathbf{E}[\Phi(\mathbb{G})]\right)^k,$$

where the first equality is due to Lemma 13.A.15. Therefore, the radius of convergence $R_{\mathcal{G}_{\Phi(\mathbb{G})}}$ of the generating function $\mathcal{G}_{\Phi(\mathbb{G})}$ is larger than or equal to $1 + \frac{1}{|\alpha|\mathbf{E}[\Phi(\mathbb{G})]}$. Proposition 4.3.20(ii) implies that the distribution of Φ is characterized by its factorial moment measures which are uniquely determined by μ and K. □

Corollary 5.3.9. *Let* $\alpha \in \mathbb{R}^*$ *and let* Φ *be an* α*-determinantal point process on a l.c.s.h. space* $\mathbb{G}$ *with background measure* μ *and kernel* K *where* μ *is* *a locally finite measure on $\mathbb{G}$. Assume moreover that, for any $D \in \mathcal{B}_c(\mathbb{G})$, $\int_D K(x,x)\,\mu(\mathrm{d}x) < \infty$ and for any $k \geq 2$, the matrix $(K(x_i,x_j))_{1\leq i,j\leq k}$ is Hermitian nonnegative-definite for μ^k-almost all $(x_1,\ldots,x_k) \in \mathbb{G}^k$. Then, for any $D \in \mathcal{B}_c(\mathbb{G})$, $\mathbf{E}[\Phi(D)] < \infty$ and*

$$\mathbf{E}\left[\Phi(D)^{(k)}\right] \leq k!\,(|\alpha|\,\mathbf{E}[\Phi(D)])^k, \quad \text{for all } k \in \mathbb{N}^*,$$

and

$$\mathbf{E}\left[(1+s)^{\Phi(D)}\right] \leq \frac{1}{1-s\,|\alpha|\,\mathbf{E}[\Phi(D)]}, \quad \text{for all } s \in \left[0, \frac{1}{|\alpha|\,\mathbf{E}[\Phi(D)]}\right).$$

Moreover, the radius of convergence $R_{\mathcal{G}_{\Phi(D)}}$ of the generating function $\mathcal{G}_{\Phi(D)}$ is larger than or equal to $1+\frac{1}{|\alpha|\mathbf{E}[\Phi(D)]}$. Further, the distribution of Φ is uniquely determined by μ and K.

Proof. By lemma 5.3.6, for any $D \in \mathcal{B}_c(\mathbb{G})$, the restriction of Φ to D is an α-determinantal point process which we denote by Φ_D. Applying Proposition 5.3.8 to Φ_D gives the announced inequalities and shows that the distribution of Φ is uniquely determined by μ restricted to D and K restricted to D^2. This being true for any $D \in \mathcal{B}_c(\mathbb{G})$, Corollary 1.3.4 allows one to conclude the proof. □

Corollary 5.3.10. *Let $\alpha \in \mathbb{R}^*$, μ a locally finite measure on a l.c.s.h. space $\mathbb{G}$, and K and $\tilde{K}$ two μ-indistinguishable measurable functions from $\mathbb{G}^2$ to $\mathbb{C}$. Assume moreover that K satisfies the conditions of Corollary 5.3.9. Then two α-determinantal point processes with background measure μ and respective kernels K and $\tilde{K}$ have the same distribution.*

Proof. This follows from Corollary 5.3.9 and Proposition 5.3.7(iii). □

5.3.3 Generating function and Laplace transform

We give now expansions of the generating function and Laplace transform of a finite α-determinantal point process. Recall Definition 4.3.12 of the generating function and Laplace transform of finite point processes.

Proposition 5.3.11. Generating function of finite α-determinantal point process. *Under the conditions of Proposition 5.3.8, the α-determinantal point process Φ is almost surely finite and we have the following expansions of its generating function and Laplace transform. Let $R_{\mathcal{G}_{\Phi(\mathbb{G})}}$ be the radius of convergence of the generating function $\mathcal{G}_{\Phi(\mathbb{G})}$.*

(i) For all bounded measurable functions $v : \mathbb{G} \to \mathbb{C}$ such that $\|1-v\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}} - 1$,

$$\mathcal{G}_\Phi(v) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left[\prod_{i=1}^{k}(1-v(x_i))\right] \det\nolimits_\alpha (K(x_i,x_j))_{1\leq i,j\leq k} \\ \mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k). \tag{5.3.5}$$

(ii) *For any measurable function $f:\mathbb{G}\to\bar{\mathbb{C}}$ such that $\left\|1-e^{-f}\right\|_{\infty}<R_{\mathcal{G}_{\Phi(\mathbb{G})}}-1$,*

$$\mathcal{L}_{\Phi}(f)=1+\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k!}\int_{\mathbb{G}^{k}}\varphi(x_{1})\ldots\varphi(x_{k})\det\nolimits_{\alpha}(K(x_{i},x_{j}))_{1\leq i,j\leq k}\,\mu(\mathrm{d}x_{1})\ldots\mu(\mathrm{d}x_{k}), \tag{5.3.6}$$

where $\varphi(x)=1-e^{-f(x)}$.

(iii) *The void probability of Φ equals*

$$\mathbf{P}(\Phi(D)=0)=1+\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k!}\int_{D^{k}}\det\nolimits_{\alpha}(K(x_{i},x_{j}))_{1\leq i,j\leq k}\,\mu(\mathrm{d}x_{1})\cdots\mu(\mathrm{d}x_{k}), \tag{5.3.7}$$

for any $D\in\mathcal{B}(\mathbb{G})$ such that $R_{\mathcal{G}_{\Phi(D)}}>2$.

Moreover, the above two series are absolutely convergent.

Proof. The fact that Φ is almost surely finite comes from the following assumption in Proposition 5.3.8

$$\mathbf{E}[\Phi(\mathbb{G})]=M_{\Phi}(\mathbb{G})=\int_{\mathbb{G}}K(x,x)\,\mu(\mathrm{d}x)<\infty,$$

where the second equality is due to (5.3.2). (i) Observe that, by Proposition 5.3.8, $R_{\mathcal{G}_{\Phi(\mathbb{G})}}\geq 1+\frac{1}{|\alpha|\mathbf{E}[\Phi(\mathbb{G})]}>1$ (where $R_{\mathcal{G}_{\Phi(\mathbb{G})}}$ is the radius of convergence of the generating function $\mathcal{G}_{\Phi(\mathbb{G})}$). Then Proposition 4.3.15(i) gives the announced expansion. (ii) This follows from Corollary 4.3.16. (iii) This follows from Proposition 4.3.1. $\square$

We deduce now the generating function and Laplace transform of a general α-determinantal point process.

Corollary 5.3.12. Generating function of α-determinantal point processes. *Under the conditions of Corollary 5.3.9, the following results hold true. For any $D\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$, let $R_{\mathcal{G}_{\Phi(D)}}$ be the radius of convergence of the generating function $\mathcal{G}_{\Phi(D)}$.*

(i) *The expansion (5.3.5) of the generation function holds for any measurable function $v:\mathbb{G}\to\mathbb{C}$ such that the support D of $1-v$ is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ and $\|1-v\|_{\infty}<R_{\mathcal{G}_{\Phi(D)}}-1$.*

(ii) *The expansion (5.3.6) of the Laplace transform holds for any measurable function $f:\mathbb{G}\to\bar{\mathbb{C}}$ whose support D is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ and such that $\left\|1-e^{-f}\right\|_{\infty}<R_{\mathcal{G}_{\Phi(D)}}-1$.*

(iii) *The expansion (5.3.7) of the void probability holds for any $D\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ such that $R_{\mathcal{G}_{\Phi(D)}}>2$.*

Moreover, the series in the above two expansions are absolutely convergent.

Proof. (i) Let D be the support of $1-v$. By Lemma 5.3.6(i), the restriction of Φ to D is a determinantal point process which we denote by Φ_D. Applying Proposition 5.3.11(i) to Φ_D gives the announced expansion. (ii) This follows from Proposition 5.3.11(ii) with the same argument as above for D being the support of f. (iii) This is immediate from Proposition 5.3.11(iii). $\square$

5.3.4 Permanental point process as Cox point process

Permanental point processes is a special class of α-determinantal point processes ($\alpha = 1$) which are also Gaussian Cox point processes. For required results concerning the symmetric complex Gaussian random variables; see Section 14.G.

Proposition 5.3.13. *[49, Proposition 4.9.2] Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let $\{Z(x)\}_{x\in\mathbb{G}}$ be a symmetric complex Gaussian stochastic process (cf. Definition 14.G.1(iii)). Let Φ be a Cox point process on $\mathbb{G}$ directed by the measure $\Lambda(\mathrm{d}x) = |Z(x)|^2\mu(\mathrm{d}x)$ (Φ is called* Gaussian Cox point process*) Then Φ is a permanental point process with kernel*

$$K(x,y) = \mathbf{E}\left[Z(x)\,Z(y)^*\right], \quad x,y\in\mathbb{G}.$$

Proof. For any locally finite measure ζ, let Φ_ζ be a Poisson point process of intensity measure ζ. Then by Definition 2.3.1, the Cox point process Φ is the mixture Φ_Λ. Given Λ, it follows from Proposition 2.3.25 that the k-th factorial moment measure of Φ_Λ is given by

$$\begin{aligned}
M_{\Phi_\Lambda^{(k)}}(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k) &= \Lambda(\mathrm{d}x_1)\ldots\Lambda(\mathrm{d}x_k)\\
&= |Z(x_1)|^2\ldots|Z(x_k)|^2\,\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k).
\end{aligned}$$

Thus, for any $B\in\mathcal{B}(\mathbb{G})^{\otimes k}$,

$$\begin{aligned}
M_{\Phi^{(k)}}(B) &= \mathbf{E}\left[M_{\Phi_\Lambda^{(k)}}(B)\right]\\
&= \mathbf{E}\left[\mathbf{E}\left[M_{\Phi_\Lambda^{(k)}}(B)\ \mid \Lambda\right]\right]\\
&= \mathbf{E}\left[\int_B |Z(x_1)|^2\ldots|Z(x_k)|^2\,\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k)\right]\\
&= \int_B \mathbf{E}\left[|Z(x_1)|^2\ldots|Z(x_k)|^2\right]\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k)\\
&= \int_B \mathrm{per}\left(\mathbf{E}\left[Z(x_i)\,Z(x_j)^*\right]\right)_{1\le i,j\le k}\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k),
\end{aligned}$$

where the last equality follows from Wick's formula (14.G.1). $\square$

The converse of the above proposition is also true: a permanental point process with regular kernel can be constructed as some Gaussian Cox process.

Theorem 5.3.14. *[49, Corollary 4.9.3] Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be a regular kernel on $\mathbb{G}$. Then there exists a permanental point process with background measure μ and kernel K.*

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding respective eigenvalues. Since K is regular, it is μ-indistinguishable from its canonical version (5.2.1); that is

$$\tilde{K}(x, y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x)\varphi_n(y)^*, \quad x, y \in \mathbb{G}_1,$$

where $\mathbb{G}_1 = \left\{x \in \mathbb{G} : \sum_{n\in\mathbb{N}^*} \lambda_n |\varphi_n(x)|^2 < \infty\right\}$. By Proposition 5.3.7(iii), it is enough to show that there exists a permanental point process with background measure μ and kernel $\tilde{K}$. Let $\{Y_n\}_{n\in\mathbb{N}^*}$ be independent standard complex Gaussian random variables (cf. Definition 14.G.1(i)) and let $\{Z(x)\}_{x\in\mathbb{G}}$ be defined by

$$Z(x) = \begin{cases} 0, & x \in \mathbb{G}\backslash\mathbb{G}_1, \\ \sum_{n\in\mathbb{N}^*} \sqrt{\lambda_n}\varphi_n(x)Y_n, & x \in \mathbb{G}_1. \end{cases}$$

Observe that, for each $x \in \mathbb{G}_1$, the series in the right-hand side of the above equality converges in $L^2_{\mathbb{C}}(\mathbf{P}, \Omega)$ by [21, Theorem 1.3.15(a) p.70]. Then $Z(x)$ belongs to the Hilbert subspace $\mathcal{H}$ of $L^2_{\mathbb{C}}(\mathbf{P}, \Omega)$ generated by $\{Y_n\}_{n\in\mathbb{N}^*}$; cf. Definition 14.G.3. Moreover, $\mathcal{H}$ is symmetric complex Gaussian by Lemma 14.G.4. Thus the stochastic process $\{Z(x)\}_{x\in\mathbb{G}}$ is symmetric complex Gaussian. Since $\{Y_n\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $\mathcal{H}$, then for any $x, y \in \mathbb{G}_1$,

$$\mathbf{E}\left[Z(x) Z(y)^*\right] = \sum_{n\in\mathbb{N}^*} \sqrt{\lambda_n}\varphi_n(x)\sqrt{\lambda_n}\varphi_n(y)^* = \tilde{K}(x, y),$$

where the first equality is due to [21, Theorem 1.3.15(d) p.71]. Let Φ be a Cox point process on $\mathbb{G}$ directed by the measure $\Lambda(\mathrm{d}x) = |Z(x)|^2 \mu(\mathrm{d}x)$. Then, by Proposition 5.3.13, Φ is a permanental point process with kernel $\tilde{K}(x, y)$ with respect to the background measure μ. □

We now extend the above result to locally regular kernels.

Theorem 5.3.15. Existence of permanental point process with locally regular kernels. *Let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be locally regular on $\mathbb{G}$. Then there exists a unique (in distribution) permanental point process Φ on $\mathbb{G}$ with background measure μ and kernel K.*

Proof of Theorem 5.3.15

We shall proceed as we did for determinantal point processes in Section 5.2.4; we begin by two preliminary results.

Lemma 5.3.16. Kolmogorov consistency conditions for permanental point process. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, $K : \mathbb{G}^2 \to \mathbb{C}$*

regularizable on $\mathbb{G}$, and $\tilde{K}$ a canonical version of K. Let $D \in \mathcal{B}_c(\mathbb{G})$, K_D the restriction of K to $D \times D$, and $\widetilde{K_D}$ a canonical version of K_D. Let Φ be a permanental point process on $\mathbb{G}$ with kernel $\tilde{K}$ and background measure μ and let Φ^D be a permanental point process on D with kernel $\widetilde{K_D}$ and background measure μ restricted to D. Then Φ^D has the same distribution as the restriction of Φ to D.

Proof. The proof follows the same lines as that of Lemma 5.2.19(iii) by invoking Theorem 5.3.14 for the existence of Φ and Φ^D and Corollary 5.3.10 for uniqueness in distribution. □

Lemma 5.3.17. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and $K : \mathbb{G}^2 \to \mathbb{C}$ locally regularizable on $\mathbb{G}$. For any $D \in \mathcal{B}_c(\mathbb{G})$, let K_D be the restriction of K to $D \times D$ and let $\widetilde{K_D}$ be a canonical version of K_D. Then there exists a permanental point process Φ on $\mathbb{G}$ with background measure μ such that for any $D \in \mathcal{B}_c(\mathbb{G})$, the restriction of Φ to D has kernel $\widetilde{K_D}$.*

Proof. The proof follows the same lines as that of Lemma 5.2.20 by invoking Theorem 5.3.14 for the existence of Φ^D and Corollary 5.3.10 for uniqueness in distribution. □

Proof of Theorem 5.3.15. The proof follows the same lines as that of Theorem 5.2.17(i). Since K is locally regular, then for any $D \in \mathcal{B}_c(\mathbb{G})$, K_D is indistinguisable from $\widetilde{K_D}$. Lemma 5.3.17 and Proposition 5.3.7(iii) show that there exists a permanental point process Φ on $\mathbb{G}$ with background measure μ such that for any $D \in \mathcal{B}_c(\mathbb{G})$, the restriction of Φ to D has kernel K_D. Lemma 5.3.6(ii) implies that Φ has kernel K on $\mathbb{G}$. Uniqueness follows again from Corollary 5.3.9. □

5.3.5 Existence of α-determinantal point processes for $\alpha \in \{\pm 1/m : m \in \mathbb{N}^*\}$

We consider now the problem of existence of α-determinantal point processes for values of $\alpha \neq 1$ and -1 (the case $\alpha = 1$ or -1 are already treated in Sections 5.2 and 5.3.4 respectively). Below, an α-determinantal point process is constructed for any $\alpha \in \{-1/m : m \in \mathbb{N}^*\}$ (resp. $\{1/m : m \in \mathbb{N}^*\}$) as a superposition of determinantal (resp. permanental) point processes.

Theorem 5.3.18. Existence of α-determinantal point processes. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$. Let $\alpha \in \mathbb{R}$ and $K : \mathbb{G}^2 \to \mathbb{C}$ be such that:*

(a) either $\alpha \in \{-1/m : m \in \mathbb{N}^*\}$ and K is locally α-regular on $\mathbb{G}$;

(b) or $\alpha \in \{1/m : m \in \mathbb{N}^*\}$ and K is locally regular on $\mathbb{G}$.

Then the following results hold.

(i) There exists an α-determinantal point process Φ_α on $\mathbb{G}$ with background measure μ and kernel K.

(ii) The distribution of Φ_α is uniquely determined by μ and K.

(iii) Φ_α may be constructed as the superposition of $m = 1/|\alpha|$ independent sign (α)*-determinantal point processes with background measure μ and kernel $|\alpha| K$.*

Proof. Observe that:

(a) If $\alpha = -1/m$ ($m \in \mathbb{N}^*$), then $-\alpha K$ is locally (-1)-regular on $\mathbb{G}$, hence, by Theorem 5.2.17(i), there exists a determinantal point process Φ on $\mathbb{G}$ with background measure μ and kernel $-\alpha K$.

(b) If $\alpha = 1/m$ ($m \in \mathbb{N}^*$), then αK is locally regular on $\mathbb{G}$, hence by Theorem 5.3.15, there exists a permanental point process Φ on $\mathbb{G}$ with background measure μ and kernel αK.

Let Φ_α be the superposition of m independent point processes with the same distribution as Φ. By Proposition 5.3.5 Φ_α is α-determinantal with background measure μ and kernel K. Uniqueness follows from Corollary 5.3.9. □

Corollary 5.3.19. *In the conditions of Theorem 5.3.18(a), for any $B \in \mathcal{B}_c(\mathbb{G})$, the radius of convergence $R_{\mathcal{G}_{\Phi_\alpha(D)}}$ of the generating function $\mathcal{G}_{\Phi_\alpha(D)}$ is infinite.*

Proof. Let $\alpha = -1/m$ for some $m \in \mathbb{N}^*$ and let Φ be a determinantal point processes with background measure μ and kernel $|\alpha| K$. By Theorem 5.3.18(iii),

$$\mathcal{G}_{\Phi_\alpha(D)} = \left(\mathcal{G}_{\Phi(D)}\right)^m .$$

Then by [59, Proposition 1.1.4 p.4], $R_{\mathcal{G}_{\Phi_\alpha(D)}} \geq R_{\mathcal{G}_{\Phi(D)}} = \infty$ where the last equality is due to Lemma 5.1.12(ii). □

For more general values of α, the existence of the corresponding α-determinantal point processes is not garanteed for all locally regular kernel K. Indeed, since the factorial moment density should be nonnegative, $\det_\alpha$ in the right-hand side of (5.3.1) should be nonnegative. This holds true for any nonnegative-definite matrix only for the specific values of α considered above.

Remark 5.3.20. *Cf. [10, Theorem 2.3].*

(i) $\det_\alpha(A) \geq 0$ *for any* complex Hermitian *nonnegative-definite matrix A iff*

$$\alpha \in \{\pm 1/m : m \in \mathbb{N}\} \cup \{0\} . \tag{5.3.8}$$

(ii) $\det_\alpha(A) \geq 0$ *for any* real symmetric *nonnegative-definite matrix A iff*

$$\alpha \in \{-1/m : m \in \mathbb{N}\} \cup \{2/m : m \in \mathbb{N}\} \cup \{0\} .$$

Remark 5.3.21. Bibliographic notes. *[68, §3] shows the existence of α-determinantal point processes for any $\alpha \in \mathbb{R}_+^*$ with some specific kernels, besides the Poisson kernel (5.1.3).*

5.4 Laplace transform and Janossy measures revisited

In this section, we shall express the Laplace transform of α-determinantal point processes as a *determinant* of some associated operator (called *Fredholm determinant*; see Definition 16.B.4). We shall also give the Janossy measures of α-determinantal point processes.

5.4.1 Laplace transform as operator determinant

Lemma 5.4.1. Laplace transform as operator determinant. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$. Let $\alpha \in \mathbb{R}$ and $K : \mathbb{G}^2 \to \mathbb{C}$ be such that:*

(a) either $\alpha \in \{-1/m : m \in \mathbb{N}^*\}$ and K is α-regular on $\mathbb{G}$;

(b) or $\alpha \in \{1/m : m \in \mathbb{N}^*\}$ and K is regular on $\mathbb{G}$.

Let Φ_α be an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel K. Then for any measurable function $f : \mathbb{G} \to \bar{\mathbb{R}}_+$,

$$\mathcal{L}_{\Phi_\alpha}(f) = \det(\mathcal{I} + \alpha\mathcal{K}_\varphi)^{-1/\alpha}, \tag{5.4.1}$$

where $\varphi = 1 - e^{-f}$ and $\mathcal{K}_\varphi$ is the integral operator with kernel $K_\varphi(x, y) = \sqrt{\varphi(x)}K(x, y)\sqrt{\varphi(y)}$; provided the function f satisfies in the case (b) that

$$\left\|1 - e^{-f}\right\|_\infty < \min\left(R_{\mathcal{G}_{\Phi_\alpha(\mathbb{G})}} - 1, 1/\left\|\alpha\mathcal{K}_\mathbb{G}\right\|\right). \tag{5.4.2}$$

Proof. The existence of Φ_α is proved in Theorem 5.3.18. Since $|K_\varphi(x, y)| \le |K(x, y)|$, then K_φ is square integrable with respect to μ^2. Thus the associated integral operator $\mathcal{K}_\varphi$ is well defined by Lemma 16.A.9(i). Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_\mathbb{C}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_\mathbb{G}$ and let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding eigenvalues. Then, a canonical version $\tilde{K}$ of K is given by (5.2.1)

$$\tilde{K}(x, y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x)\varphi_n(y)^*, \quad x, y \in \mathbb{G}_1,$$

where $\mathbb{G}_1$ be given by (5.2.2). Then $\mathcal{K}_\varphi$ is the integral operator with kernel K_φ defined by

$$\begin{aligned} K_\varphi(x, y) &= \sqrt{\varphi(x)}\tilde{K}(x, y)\sqrt{\varphi(y)} \\ &= \sum_{n\in\mathbb{N}^*} \lambda_n \sqrt{\varphi(x)}\varphi_n(x)\varphi_n(y)^*\sqrt{\varphi(y)}, \end{aligned}$$

for all $x, y \in \mathbb{G}_1$. Observe that

$$\int_\mathbb{G} |K_\varphi(x, x)|\,\mu(\mathrm{d}x) = \sum_{n\in\mathbb{N}^*} \lambda_n \int_\mathbb{G} \varphi(x)\,|\varphi_n(x)|^2\,\mu(\mathrm{d}x) \le \sum_{n\in\mathbb{N}^*} \lambda_n < \infty.$$

Then, by Proposition 16.A.26(iii), $\mathcal{K}_\varphi$ is trace class and

$$\mathrm{tr}\left(\mathcal{K}_\varphi\right) = \int_{\mathbb{G}} K_\varphi(x,x)\mu\left(\mathrm{d}x\right).$$

(i) *Case* $\alpha = -1$. By Proposition 16.B.10,

$$\begin{aligned}
&\det\left(\mathcal{I} - \mathcal{K}_\varphi\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \det\left(K_\varphi\left(x_i, x_j\right)\right)_{1\le i,j\le k} \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left(1-e^{-f(x_1)}\right)\ldots\left(1-e^{-f(x_k)}\right) \det\left(\tilde{K}\left(x_i, x_j\right)\right)_{1\le i,j\le k} \\
&\qquad \mu\left(\mathrm{d}x_1\right)\cdots\mu\left(\mathrm{d}x_k\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left(1-e^{-f(x_1)}\right)\ldots\left(1-e^{-f(x_k)}\right) \det\left(K\left(x_i, x_j\right)\right)_{1\le i,j\le k} \\
&\qquad \mu\left(\mathrm{d}x_1\right)\cdots\mu\left(\mathrm{d}x_k\right),
\end{aligned}$$

where the last equality is due to Proposition 5.1.11(i). The right-hand side of the above relation equals $\mathcal{L}_{\Phi_\alpha}(f)$ by Proposition 5.1.18(ii). (ii) *Case* $\alpha = -1/m$, for some $m \in \mathbb{N}^*$. Let Φ be a determinantal point processes with background measure μ and kernel $|\alpha| K$. By Theorem 5.3.18(iii) and Proposition 1.3.15

$$\mathcal{L}_{\Phi_\alpha}(f) = \left[\mathcal{L}_\Phi(f)\right]^m = \det\left(\mathcal{I} + \alpha\mathcal{K}_\varphi\right)^{-1/\alpha},$$

where the second equality is due to Item (i). (iii) *Case* $\alpha = 1/m$, for some $m \in \mathbb{N}^*$. Since $\|\mathcal{K}_\varphi\| \le \left\|1 - e^{-f}\right\|_\infty \|\alpha\mathcal{K}_{\mathbb{G}}\| < 1$, where the second inequality is due to (5.4.2), then by Proposition 16.B.13(i),

$$\begin{aligned}
&\det\left(\mathcal{I} + \alpha\mathcal{K}_\varphi\right)^{-1/\alpha} \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \det_\alpha\left(K_\varphi\left(x_i, x_j\right)\right)_{1\le i,j\le k} \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left(1-e^{-f(x_1)}\right)\ldots\left(1-e^{-f(x_k)}\right) \det_\alpha\left(\tilde{K}\left(x_i, x_j\right)\right)_{1\le i,j\le k} \\
&\qquad \mu\left(\mathrm{d}x_1\right)\cdots\mu\left(\mathrm{d}x_k\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{G}^k} \left(1-e^{-f(x_1)}\right)\ldots\left(1-e^{-f(x_k)}\right) \det_\alpha\left(K\left(x_i, x_j\right)\right)_{1\le i,j\le k} \\
&\qquad \mu\left(\mathrm{d}x_1\right)\cdots\mu\left(\mathrm{d}x_k\right),
\end{aligned}$$

where the last equality is due to Proposition 5.3.7(i). The right-hand side of the above relation equals $\mathcal{L}_{\Phi_\alpha}(f)$ by Proposition 5.3.11(ii) and the assumption $\left\|1 - e^{-f}\right\|_\infty < R_{\mathcal{G}_{\Phi(\mathbb{G})}} - 1$ which is due to (5.4.2). □

Note that both side of equality (5.4.1) are well defined for any measurable function $f : \mathbb{G} \to \bar{\mathbb{R}}_+$. In fact, we will show in Proposition 5.4.5 below that condition (5.4.2) is actually not necessary for equality (5.4.1) to hold. For this, we shall relate the determinant to Janossy measures.

5.4.2 Janossy measures of α-determinantal point processes; $\alpha \in \{1/m : m \in \mathbb{N}^*\}$

We now introduce the notion of α-inverse of integral operators for $\alpha \in \mathbb{R}_+$.

Definition 5.4.2. α-Inverse of integral operators. *Let $\mathbb{G}$ be a l.c.s.h. space, $\alpha \in \mathbb{R}_+$, and $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function.*

(i) If K is regular on $\mathbb{G}$, then we define the α-inverse *of the associated integral operator $\mathcal{K}_{\mathbb{G}}$ by*
$$\mathcal{L}_\alpha^{\mathbb{G}} := (\mathcal{I} + \alpha\mathcal{K}_{\mathbb{G}})^{-1}\mathcal{K}_{\mathbb{G}}. \tag{5.4.3}$$

(ii) If K is locally regular on $\mathbb{G}$, then for each $D \in \mathcal{B}_c(\mathbb{G})$ we define the α-inverse *of $\mathcal{K}_D$ by*
$$\mathcal{L}_\alpha^{D} := (\mathcal{I} + \alpha\mathcal{K}_{D})^{-1}\mathcal{K}_{D}. \tag{5.4.4}$$

The following lemma studies the properties of the α-inverse operator (5.4.3). In particular, the first result justifies the naming of the operator $\mathcal{L}_\alpha^{\mathbb{G}}$ as an α-inverse of $\mathcal{K}_{\mathbb{G}}$.

Lemma 5.4.3. α-Inverse operator properties. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, $\alpha \in \mathbb{R}_+$, $K : \mathbb{G}^2 \to \mathbb{C}$ be regular on $\mathbb{G}$, and $\mathcal{L}_\alpha^{\mathbb{G}}$ the α-inverse of the associated integral operator $\mathcal{K}_{\mathbb{G}}$. Then the following results hold true.*

(i) $(\mathcal{I} + \alpha\mathcal{K}_{\mathbb{G}})\left(\mathcal{I} - \alpha\mathcal{L}_\alpha^{\mathbb{G}}\right) = \mathcal{I}$.

(ii) The operator $\mathcal{L}_\alpha^{\mathbb{G}}$ defined by (5.4.3) is trace class and has nonnegative eigenvalues. Moreover,
$$\left\|\mathcal{L}_\alpha^{\mathbb{G}}\right\| = \frac{\|\mathcal{K}_{\mathbb{G}}\|}{1 + \alpha\|\mathcal{K}_{\mathbb{G}}\|}. \tag{5.4.5}$$

(iii) Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding eigenvalues. Then $\mathcal{L}_\alpha^{\mathbb{G}}$ is an integral operator with kernel $\tilde{L}_\alpha^{\mathbb{G}}$ equal to the canonical kernel associated to $\{\varphi_n\}_{n\in\mathbb{N}^*}$ and $\left\{\frac{\lambda_n}{1+\alpha\lambda_n}\right\}_{n\in\mathbb{N}^*}$.

(iv) Let $g : \mathbb{G} \to \mathbb{R}_+$ be a bounded measurable function, then $\mathcal{L}_\alpha^{\mathbb{G}} g$ and $g\mathcal{L}_\alpha^{\mathbb{G}}$ are trace class, where $\mathcal{L}_\alpha^{\mathbb{G}} g$ is the composition of $\mathcal{L}_\alpha^{\mathbb{G}}$ with the operator of multiplication by g and similarly for $g\mathcal{L}_\alpha^{\mathbb{G}}$.

Proof. (i) Using (5.4.3), we get

$$\left(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}}\right)\left(\mathcal{I}-\alpha\mathcal{L}_{\alpha}^{\mathbb{G}}\right)=\left(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}}\right)\left[\mathcal{I}-\alpha(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})^{-1}\mathcal{K}_{\mathbb{G}}\right]=\mathcal{I}.$$

(ii) A straightforward calculation shows that, for any $\lambda\neq-1/\alpha$ and $\varphi\in L_{\mathbb{C}}^{2}(\mu,\mathbb{G})$,

$$\mathcal{K}_{\mathbb{G}}\varphi=\lambda\varphi\Leftrightarrow\mathcal{L}_{\alpha}^{\mathbb{G}}\varphi=\frac{\lambda}{1+\alpha\lambda}\varphi. \tag{5.4.6}$$

By Proposition 16.A.13(i), there exists an orthonormal basis of $L_{\mathbb{C}}^{2}(\mu,\mathbb{G})$ composed of eigenvectors $\{\varphi_n\}_{n\in\mathbb{N}^*}$ of $\mathcal{K}_{\mathbb{G}}$ with respective eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$. The above display shows that $\{\varphi_n\}_{n\in\mathbb{N}^*}$ are also eigenvectors of $\mathcal{L}_{\alpha}^{\mathbb{G}}$ with corresponding eigenvalues $\left\{\frac{\lambda_n}{1+\alpha\lambda_n}\right\}_{n\in\mathbb{N}^*}$ which are nonnegative. Since $\mathcal{K}_{\mathbb{G}}$ is trace class, then $\sum_{n\in\mathbb{N}^*}\lambda_n<\infty$. Clearly $\sum_{n\in\mathbb{N}^*}\frac{\lambda_n}{1+\alpha\lambda_n}<\infty$, and hence $\mathcal{L}_{\alpha}^{\mathbb{G}}$ is trace class. By Proposition 16.A.13(iv)

$$\begin{aligned}\left\|\mathcal{L}_{\alpha}^{\mathbb{G}}\right\|&=\sup_{n\in\mathbb{N}^*}\frac{\lambda_n}{1+\alpha\lambda_n}\\&=\sup_{n\in\mathbb{N}^*}\frac{1}{\alpha}\left(1-\frac{1}{1+\alpha\lambda_n}\right)\\&=\frac{1}{\alpha}\left(1-\frac{1}{1+\alpha\sup_{n\in\mathbb{N}^*}\lambda_n}\right)=\frac{\|\mathcal{K}_{\mathbb{G}}\|}{1+\alpha\|\mathcal{K}_{\mathbb{G}}\|}.\end{aligned}$$

(iii) We have shown in (ii) that $\{\varphi_n\}_{n\in\mathbb{N}^*}$ are eigenvectors of $\mathcal{L}_{\alpha}^{\mathbb{G}}$ with respective eigenvalues $\left\{\frac{\lambda_n}{1+\alpha\lambda_n}\right\}_{n\in\mathbb{N}^*}$. (iv) Recall $\mathcal{L}_{\alpha}^{\mathbb{G}}g$ is bounded (cf. Example 16.B.8), then Lemma 16.A.17 allows one to conclude. □

Remark 5.4.4. *Let K be as in Lemma 5.4.3. In general, the restriction property (16.A.26) doesn't hold for the α-inverse operator $\mathcal{L}_{\alpha}^{D}$ defined by (5.4.4); that is*

$$\mathcal{L}_{\alpha}^{D}\neq P_D\mathcal{L}_{\alpha}^{\mathbb{G}}P_D,$$

where P_D is the projection operator (see (16.A.25)).

Proposition 5.4.5. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, $\alpha\in\mathbb{R}_+^*$, and $K:\mathbb{G}^2\to\mathbb{C}$ be regular on $\mathbb{G}$. Let $f:\mathbb{G}\to\mathbb{R}_+$ be a measurable function, $\varphi=1-e^{-f}$, and $\mathcal{K}_\varphi$ be the integral operator with kernel $K_\varphi(x,y)=\sqrt{\varphi(x)}K(x,y)\sqrt{\varphi(y)}$. Then, the following results hold true.*

(i)

$$\det(\mathcal{I}+\alpha\mathcal{K}_\varphi)=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})\det(\mathcal{I}-\alpha e^{-f/2}\mathcal{L}_{\alpha}^{\mathbb{G}}e^{-f/2}), \tag{5.4.7}$$

where $\mathcal{L}_{\alpha}^{\mathbb{G}}$ is the α-inverse of $\mathcal{K}_{\mathbb{G}}$ given by (5.4.3).

(ii)

$$\begin{aligned}\det\left(\mathcal{I}+\alpha\mathcal{K}_\varphi\right)^{-1/\alpha}&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})^{-1/\alpha}\\&\quad+\sum_{k\in\mathbb{N}^*}\frac{1}{k!}\int_{\mathbb{G}^k}e^{-\sum_{i=1}^{k}f(x_i)}J_{\alpha,k}^{\mathbb{G}}\left(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k\right),\end{aligned}$$

where $J_{\alpha,k}^{\mathbb{G}}$ *is given by*

$$J_{\alpha,k}^{\mathbb{G}}\left(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k\right)= \qquad (5.4.8)$$
$$\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})\det{}_{\alpha}\left(\tilde{L}_{\alpha}^{\mathbb{G}}\left(x_i,x_j\right)\right)_{1\leq i,j\leq k}\ \mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right),$$

with $\tilde{L}_{\alpha}^{\mathbb{G}}$ *being the canonical kernel associated to* $\mathcal{L}_{\alpha}^{\mathbb{G}}$.

(iii) If $\alpha\in\{1/m:m\in\mathbb{N}^*\}$*, then there exists a finite point process* Φ_α *on* $\mathbb{G}$ *with Janossy measures* $\left\{J_{\alpha,k}^{\mathbb{G}}\right\}_{k\in\mathbb{N}^*}$ *given by (5.4.8) and such that*

$$\mathbf{P}(\Phi_\alpha(\mathbb{G})=0)=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})^{-1/\alpha}.$$

Moreover, the Laplace transform of Φ_α *is given by*

$$\mathcal{L}_{\Phi_\alpha}\left(f\right)=\det\left(\mathcal{I}+\alpha\mathcal{K}_{\varphi}\right)^{-1/\alpha},$$

for any measurable function $f:\mathbb{G}\to\mathbb{R}_+$.

(iv) For $\alpha=1$*, the number of atoms* $\Phi_1(\mathbb{G})$ *has the same distribution as* $\sum_{n\in\mathbb{N}^*}Z_n$ *where* $\{Z_n\}_{n\in\mathbb{N}^*}$ *are independent geometric random variables with respective parameters* $\{1/\left(1+\lambda_n\right)\}_{n\in\mathbb{N}^*}$ *(i.e.,* $\mathbf{P}\left(Z_n=k\right)=\frac{1}{1+\lambda_n}\left(\frac{\lambda_n}{1+\lambda_n}\right)^k$ *for any* $k\in\mathbb{N}$*), with* $\{\lambda_n\}_{n\in\mathbb{N}^*}$ *being the eigenvalues of* $\mathcal{K}_{\mathbb{G}}$ *accounting for their multiplicities.*

(v) For any $\alpha\in\{1/m:m\in\mathbb{N}^*\}$*,*

$$R_{\mathcal{G}_{\Phi_\alpha(\mathbb{G})}}=1+\frac{1}{\|\alpha\mathcal{K}_{\mathbb{G}}\|}.$$

(vi) For any $\alpha\in\{1/m:m\in\mathbb{N}^*\}$*, the point process* Φ_α *is* α*-determinantal on* $\mathbb{G}$ *with background measure* μ *and kernel* K.

Proof. (i) Observe that φ is bounded, then

$$\begin{aligned}\det(\mathcal{I}+\alpha\mathcal{K}_{\varphi})&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}}\varphi)\\&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}}-\alpha\mathcal{K}_{\mathbb{G}}e^{-f})\\&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})\det(\mathcal{I}-\alpha(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})^{-1}\mathcal{K}_{\mathbb{G}}e^{-f})\\&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})\det(\mathcal{I}-\alpha\mathcal{L}_{\alpha}^{\mathbb{G}}e^{-f})\\&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})\det(\mathcal{I}-\alpha e^{-f/2}\mathcal{L}_{\alpha}^{\mathbb{G}}e^{-f/2}).\end{aligned}$$

where the first equality is due to (16.B.9), the third equality is due to (16.B.5) and the fact that $\mathcal{L}_{\alpha}^{\mathbb{G}}e^{-f}$ is trace class by Lemma 5.4.3(iv), and the last equality is due to (16.B.4). (ii) By Lemma 5.4.3(iv), the operator $e^{-f/2}\mathcal{L}_{\alpha}^{\mathbb{G}}e^{-f/2}$ is trace class. Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu,\mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding eigenvalues. By Lemma 5.4.3(iii), the operator $e^{-f/2}\mathcal{L}^{\mathbb{G}}_{\alpha}e^{-f/2}$ is an integral operator with kernel

$$L(x,y)=\sum_{n\in\mathbb{N}^*}\frac{\lambda_n}{1+\alpha\lambda_n}e^{-f(x)/2}\varphi_n(x)\,\varphi_n(y)^*\,e^{-f(y)/2}.$$

Then by Proposition 16.A.26(ii), the operator $e^{-f/2}\mathcal{L}^{\mathbb{G}}_{\alpha}e^{-f/2}$ is Hermitian and nonnegative-definite. By Proposition 5.2.9(iv), L is indistinguishable from its canonical version. Since f is nonnegative, then $\left\|e^{-f/2}\right\|_{\infty}\leq 1$. On the other hand by (5.4.5), $\left\|\alpha\mathcal{L}^{\mathbb{G}}_{\alpha}\right\|<1$ and therefore $\left\|\alpha e^{-f/2}\mathcal{L}^{\mathbb{G}}_{\alpha}e^{-f/2}\right\|<1$. Then applying Proposition 16.B.13(i) to the operator $e^{-f/2}\mathcal{L}^{\mathbb{G}}_{\alpha}e^{-f/2}$, we get

$$\begin{aligned}
&\det(\mathcal{I}-\alpha e^{-f/2}\mathcal{L}^{\mathbb{G}}_{\alpha}e^{-f/2})^{-1/\alpha}\\
&=1+\sum_{k\in\mathbb{N}^*}\frac{1}{k!}\int_{\mathbb{G}^k}\det\nolimits_{\alpha}\left(L(x_i,x_j)\right)_{1\leq i,j\leq k}\;\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k) \qquad (5.4.9)\\
&=1+\sum_{k\in\mathbb{N}^*}\frac{1}{k!}\int_{\mathbb{G}^k}\det\nolimits_{\alpha}\left(\tilde{L}^{\mathbb{G}}_{\alpha}(x_i,x_j)\right)_{1\leq i,j\leq k}\;e^{-\sum_{i=1}^k f(x_i)}\;\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k),
\end{aligned}$$

where $\tilde{L}^{\mathbb{G}}_{\alpha}(x,y)=\sum_{n\in\mathbb{N}^*}\frac{\lambda_n}{1+\alpha\lambda_n}\varphi_n(x)\,\varphi_n(y)^*$ is a canonical kernel associated to $\mathcal{L}^{\mathbb{G}}_{\alpha}$ as provided in Lemma 5.4.3(iii). Combining the above equation with (5.4.7) and then using (5.4.8), we get

$$\begin{aligned}
\det(\mathcal{I}+\alpha\mathcal{K}_{\varphi})^{-1/\alpha}&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})^{-1/\alpha}\\
&\quad+\sum_{k\in\mathbb{N}^*}\frac{1}{k!}\int_{\mathbb{G}^k}e^{-\sum_{i=1}^k f(x_i)}\;J^{\mathbb{G}}_{\alpha,k}(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k).
\end{aligned}$$

(iii) **Step 1**. Consider first the case $\alpha=1$. By Lemma 16.A.25(v), for μ^k-almost all $x_1,\ldots,x_k\in\mathbb{G}$, the matrix $\left(\tilde{L}^{\mathbb{G}}_1(x_i,x_j)\right)_{1\leq i,j\leq k}$ is nonnegative-definite. Therefore, by Lemma 15.A.7, $\det_1\left(\tilde{L}^{\mathbb{G}}_1(x_i,x_j)\right)_{1\leq i,j\leq k}\geq 0$ since $\det_1=\mathrm{per}$. Thus $J^{\mathbb{G}}_{1,k}$ defined by (5.4.8) is indeed a measure on $\mathbb{G}^k$. Moreover, applying (ii) with $f=0$, we get

$$\det(\mathcal{I}+\mathcal{K}_{\mathbb{G}})^{-1}+\sum_{k\in\mathbb{N}^*}\frac{1}{k!}J^{\mathbb{G}}_{1,k}\left(\mathbb{G}^k\right)=\det(\mathcal{I})=1.$$

Then, by Corollary 4.3.8, there exists a finite point process Φ on $\mathbb{G}$ with Janossy measures $\left(J^{\mathbb{G}}_{1,k}\right)_{k\in\mathbb{N}^*}$ and such that $\mathbf{P}(\Phi(\mathbb{G})=0)=\det(\mathcal{I}+\mathcal{K}_{\mathbb{G}})^{-1}$. This point process has Laplace transform given by, for any measurable function $f:\mathbb{G}\to\mathbb{R}_+$,

$$\begin{aligned}
\mathcal{L}_{\Phi}(f)&=\det(\mathcal{I}+\mathcal{K}_{\mathbb{G}})^{-1}+\sum_{k\in\mathbb{N}^*}\frac{1}{k!}\int_{\mathbb{G}^k}e^{-\sum_{i=1}^k f(x_i)}J^{\mathbb{G}}_{1,k}(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k)\\
&=\det(\mathcal{I}+\mathcal{K}_{\varphi})^{-1},
\end{aligned}$$

where the first equality follows from Lemma 4.3.14(iii) and the second equality follows from (ii). **Step 2**. Consider now some $\alpha \in \{1/m : m \in \mathbb{N}^*\}$. Observe that αK is locally regular on $\mathbb{G}$. Then by Step 1, there exists a point process Φ such that

$$\mathcal{L}_{\Phi}(f) = \det(\mathcal{I} + \alpha \mathcal{K}_{\varphi})^{-1}.$$

Let Φ_α be the sum of m independent point processes with the same distribution as Φ. By Proposition 1.3.15, its Laplace transform equals

$$\mathcal{L}_{\Phi_\alpha}(f) = [\mathcal{L}_{\Phi}(f)]^m = \det(\mathcal{I} + \alpha \mathcal{K}_{\varphi})^{-1/\alpha}.$$

The sum of m independent copies of the finite random variable $\Phi(\mathbb{G})$ is also finite. Then Φ_α is a finite point process. By Lemma 4.3.14(iii), for any measurable function $f : \mathbb{G} \to \mathbb{R}_+$,

$$\mathcal{L}_{\Phi_\alpha}(f) = \mathbf{P}(\Phi_\alpha(\mathbb{G}) = 0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{G}^k} e^{-\sum_{i=1}^k f(x_i)} J_k(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_k),$$

where J_k is the kth Janossy meassure of Φ_α. On the other hand,

$$\begin{aligned}
\mathcal{L}_{\Phi_\alpha}(f) &= \det(\mathcal{I} + \alpha \mathcal{K}_{\varphi})^{-1/\alpha} \\
&= \det(\mathcal{I} + \alpha \mathcal{K}_{\mathbb{G}})^{-1/\alpha} + \sum_{k \in \mathbb{N}^*} \frac{1}{k!} \int_{\mathbb{G}^k} e^{-\sum_{i=1}^k f(x_i)} J_{\alpha,k}^{\mathbb{G}}(\mathrm{d}x_1 \times \cdots \times \mathrm{d}x_k),
\end{aligned}$$

where the second equality follows from Item (ii). Identifying the terms of the above two expansions concludes the proof. (iv) Let $t \in \mathbb{R}_+$ and $f(x) = t$ for all $x \in \mathbb{G}$, then

$$\begin{aligned}
\mathcal{L}_{\Phi_1(D)}(t) &= \mathcal{L}_{\Phi_1}(f) \\
&= \det\left(\mathcal{I} + \left(1 - e^{-t}\right)\mathcal{K}_D\right)^{-1} \\
&= \prod_{n \in \mathbb{N}^*} \left(1 + \left(1 - e^{-t}\right)\lambda_n\right)^{-1} \\
&= \prod_{n \in \mathbb{N}^*} \mathcal{L}_{Z_n}(t),
\end{aligned} \tag{5.4.10}$$

where the second equality follows from (5.4.1) with $\alpha = 1$, the third equality is due to (16.B.6), and the fourth equality is due to the fact that

$$\begin{aligned}
\mathcal{L}_{Z_n}(t) &= \mathbf{E}\left[e^{-tZ_n}\right] \\
&= \frac{1}{1 + \lambda_n} \sum_{k \in \mathbb{N}} \left(\frac{\lambda_n}{1 + \lambda_n}\right)^k e^{-tk} \\
&= \left(1 + \lambda_n - \lambda_n e^{-t}\right)^{-1}.
\end{aligned}$$

(v) *Case* $\alpha = 1$. Let $\{\lambda_n\}_{n \in \mathbb{N}^*}$ be the eigenvalues of $\mathcal{K}_{\mathbb{G}}$ accounting for their multiplicities. By Item (iii), $\Phi_1(\mathbb{G}) = \sum_{n \in \mathbb{N}^*} Z_n$ where $\{Z_n\}_{n \in \mathbb{N}^*}$ are independent geometric random variables with respective parameters $\{1/(1 + \lambda_n)\}_{n \in \mathbb{N}^*}$.

Observe from (13.A.14) that

$$R_{\mathcal{G}_{Z_n}} = 1 + \frac{1}{\lambda_n}.$$

Moreover, for any $n \in \mathbb{N}^*$ and $x \in \mathbb{R}_+$,

$$\begin{aligned}
\mathcal{G}_{Z_n}(x) &= \mathbf{E}\left[x^{Z_n}\right] \\
&= \frac{1}{1+\lambda_n} \sum_{k \in \mathbb{N}} \left(\frac{\lambda_n}{1+\lambda_n}\right)^k x^k \\
&= \begin{cases} (1+\lambda_n - \lambda_n x)^{-1}, & \text{if } x \in \left[0, 1 + \frac{1}{\lambda_n}\right), \\ \infty, & \text{if } x \geq 1 + \frac{1}{\lambda_n}. \end{cases}
\end{aligned}$$

Since for any $x \in \mathbb{R}_+$,

$$\mathcal{G}_{\Phi_1(\mathbb{G})}(x) = \mathbf{E}\left[x^{\Phi_1(\mathbb{G})}\right] = \prod_{n \in \mathbb{N}^*} \mathcal{G}_{Z_n}(x),$$

then $\mathcal{G}_{\Phi_1(\mathbb{G})}(x) = \infty$ for any $x \geq 1 + \frac{1}{\lambda_n}$ for some $n \in \mathbb{N}^*$. Thus

$$R_{\mathcal{G}_{\Phi_1(\mathbb{G})}} \leq \inf_{n \in \mathbb{N}^*} \left(1 + \frac{1}{\lambda_n}\right) = 1 + \frac{1}{\|\mathcal{K}_{\mathbb{G}}\|},$$

where the last equality is due to Proposition 16.A.13(iv). It remains to prove the converse inequality. Observe from (5.4.10) that for any $t \in \mathbb{R}_+$,

$$\mathcal{L}_{\Phi_1(\mathbb{G})}(t) = A(B(t)),$$

where

$$A(x) = \det(\mathcal{I} + x\mathcal{K}_{\mathbb{G}})^{-1}, \quad B(t) = 1 - e^{-t}.$$

By Proposition 16.B.13(i), $A(x)$ admits a series expansion with radius of convergence $R_A \geq 1/\|\mathcal{K}_{\mathbb{G}}\|$. On the other hand,

$$B(t) = -\sum_{n=1}^{\infty} \frac{(-t)^n}{n!}, \quad t \in \mathbb{R},$$

has a radius of convergence $R_B = \infty$. By Lemma 15.B.1, $A(B(t))$ admits a convergent power series expansion for any $t \in \mathbb{R}$ such that $\sum_{n=0}^{\infty} \frac{|t|^n}{n!} < R_A$. Thus $\mathcal{L}_{\Phi_1(\mathbb{G})}(t) = A(B(t))$ admits a convergent power series expansion for any $t \in \mathbb{R}_+$ such that $e^t - 1 < R_A$. Thus

$$R_{\mathcal{L}_{\Phi_1(\mathbb{G})}} \geq \log(1 + R_A) \geq \log\left(1 + \frac{1}{\|\mathcal{K}_{\mathbb{G}}\|}\right).$$

Proposition 13.B.4(iii), then implies

$$R_{\mathcal{G}_{\Phi_1(\mathbb{G})}} = \exp\left(R_{\mathcal{L}_{\Phi_1(\mathbb{G})}}\right) \geq 1 + \frac{1}{\|\mathcal{K}_{\mathbb{G}}\|}.$$

Case $\alpha \in \{1/m : m \in \mathbb{N}^*\}$. Recall from Theorem 5.3.18(iii) that Φ_α may be constructed as the superposition of m independent permanental point processes with background measure μ and kernel αK. Then

$$\Phi_\alpha(\mathbb{G}) = X_1 + \ldots + X_m,$$

where $X_1, \ldots, X_m$ are m independent random variables such that

$$R_{\mathcal{G}_{X_i}} = 1 + \frac{1}{\|\alpha \mathcal{K}_{\mathbb{G}}\|}.$$

Since for any $x \in \mathbb{R}_+$,

$$\mathcal{G}_{\Phi_\alpha(\mathbb{G})}(x) = \mathbf{E}\left[x^{\Phi_\alpha(\mathbb{G})}\right] = \prod_{i=1}^{m} \mathcal{G}_{X_i}(x),$$

then, by [59, Proposition 1.1.4 p.4],

$$R_{\mathcal{G}_{\Phi_\alpha(\mathbb{G})}} \geq \min_{i \in \{1,\ldots,m\}} \left\{R_{\mathcal{G}_{X_i}}\right\} = 1 + \frac{1}{\|\alpha \mathcal{K}_{\mathbb{G}}\|}.$$

On the other hand, if $x > 1 + 1/\|\alpha \mathcal{K}_{\mathbb{G}}\|$, then $\mathcal{G}_{X_i}(x) = \infty$ which implies $\mathcal{G}_{\Phi_\alpha(\mathbb{G})}(x) = \infty$. Thus

$$R_{\mathcal{G}_{\Phi_\alpha(\mathbb{G})}} \leq 1 + \frac{1}{\|\alpha \mathcal{K}_{\mathbb{G}}\|}.$$

Combining the above two inequalities concludes the proof. (vi) By Theorem 5.3.18, there exists an α-determinantal point process $\tilde{\Phi}_\alpha$ on $\mathbb{G}$ with background measure μ and kernel K. On the other hand, by Item (iii), there exists a point process Φ_α whose Laplace transform is given by $\mathcal{L}_{\Phi_\alpha}(f) = \det(\mathcal{I} + \alpha \mathcal{K}_\varphi)^{-1/\alpha}$, for any measurable function $f : \mathbb{G} \to \mathbb{R}_+$. Lemma 5.4.1(b) shows that the Laplace transform of $\tilde{\Phi}_\alpha$ coincide with that of Φ_α on the set of measurable functions $f : \mathbb{G} \to \bar{\mathbb{R}}_+$ which satisfy condition (5.4.2). Observe from Proposition 5.3.8 that the moment measures of $\tilde{\Phi}_\alpha$ are finite and $R_{\mathcal{G}_{\tilde{\Phi}_\alpha(\mathbb{G})}} \geq 1 + \frac{1}{|\alpha| \mathbf{E}\left[\tilde{\Phi}(\mathbb{G})\right]}$ which implies $R_{\mathcal{L}_{\Phi_\alpha}(\tilde{\Phi})} > 0$ (cf. Definition 13.B.3) by Proposition 13.B.4(iii). Then by Corollary 1.3.12, $\Phi_\alpha \stackrel{\text{dist.}}{=} \tilde{\Phi}_\alpha$. □

Corollary 5.4.6. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$ and $K : \mathbb{G}^2 \to \mathbb{C}$ be locally regular on $\mathbb{G}$. For any $\alpha \in \{1/m : m \in \mathbb{N}^*\}$, let Φ_α be an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel K. Then the following results hold.*

(i) The Laplace transform $\mathcal{L}_{\Phi_\alpha}(f)$ is given by (5.4.1) for any measurable function $f : \mathbb{G} \to \bar{\mathbb{R}}_+$ whose support is in $\mathcal{B}_c(\mathbb{G})$.

(ii) For any $D \in \mathcal{B}_c(\mathbb{G})$, the restriction of Φ_α to D admits the Janossy measures $\left\{J^D_{\alpha,k}\right\}_{k \in \mathbb{N}^*}$ given by (5.4.8) and has void probability $\mathbf{P}(\Phi_\alpha(D) = 0) = \det(\mathcal{I} + \alpha \mathcal{K}_D)^{-1/\alpha}$.

(iii) *For $\alpha = 1$ and any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$, the number of atoms $\Phi_1(D)$ has the same distribution as $\sum_{n\in\mathbb{N}^*} Z_n$ where $\{Z_n\}_{n\in\mathbb{N}^*}$ are independent geometric random variables with respective parameters $\{1/(1+\lambda_n)\}_{n\in\mathbb{N}^*}$ (i.e., $\mathbf{P}(Z_n = k) = \frac{1}{1+\lambda_n}\left(\frac{\lambda_n}{1+\lambda_n}\right)^k$ for any $k \in \mathbb{N}$), with $\{\lambda_n\}_{n\in\mathbb{N}^*}$ being the eigenvalues of $\mathcal{K}_D$ accounting for their multiplicities.*

(iv) *For any $\alpha \in \{1/m : m \in \mathbb{N}^*\}$ and any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$,*

$$R_{\mathcal{G}_{\Phi_\alpha(D)}} = 1 + \frac{1}{\|\alpha\mathcal{K}_D\|}.$$

Proof. The existence of Φ_α follows from Theorem 5.3.18(i). Applying Proposition 5.4.5 to the restriction of Φ_α to any $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$ gives the announced results. □

5.4.3 Janossy measures of α-determinantal point processes; $\alpha \in \{-1/m : m \in \mathbb{N}^*\}$

We will define the α-inverse of integral operators for $\alpha \in \mathbb{R}_-^*$. In this regard we introduce the notion of *strictly* α-regular kernel.

Definition 5.4.7. Strictly α-regular kernel; $\alpha \in \mathbb{R}_-^*$. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function, and $\alpha \in \mathbb{R}_-^*$.*

- *K is called* strictly α-regular *on $\mathbb{G}$ iff K is α-regular and the eigenvalues of $\mathcal{K}_{\mathbb{G}}$ are strictly smaller than $-1/\alpha \in \mathbb{R}_+^*$.*
- *K is called* locally strictly α-regular *on $\mathbb{G}$ iff K is strictly α-regular on all $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$.*

The following definition extends Definition 5.4.2 for negative values of α.

Definition 5.4.8. α-Inverse of integral operators; $\alpha \in \mathbb{R}_-^*$. *Let $\mathbb{G}$ be a l.c.s.h. space, $\alpha \in \mathbb{R}_-^*$, and $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function.*

(i) *If K is strictly α-regular on $\mathbb{G}$, then we define the* α-inverse *of associated integral operator $\mathcal{K}_{\mathbb{G}}$ by*

$$\mathcal{L}_\alpha^{\mathbb{G}} := (\mathcal{I} + \alpha\mathcal{K}_{\mathbb{G}})^{-1}\mathcal{K}_{\mathbb{G}}. \tag{5.4.11}$$

(ii) *If K is locally strictly α-regular on $\mathbb{G}$, then for each $D \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$ we define the* α-inverse *of $\mathcal{K}_D$ by*

$$\mathcal{L}_\alpha^{D} := (\mathcal{I} + \alpha\mathcal{K}_D)^{-1}\mathcal{K}_D. \tag{5.4.12}$$

Similarly to Lemma 5.4.3, we give now the properties of the α-inverse operator for negative values of α.

Lemma 5.4.9. α-Inverse operator properties; $\alpha \in \mathbb{R}_{-}^{*}$. *Let μ be a locally finite measure on a l.c.s.h space $\mathbb{G}$, let $\alpha \in \mathbb{R}_{-}^{*}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be strictly α-regular on $\mathbb{G}$. Then the following results hold true.*

(i) $(\mathcal{I} + \alpha\mathcal{K}_{\mathbb{G}})\left(\mathcal{I} - \alpha\mathcal{L}_{\alpha}^{\mathbb{G}}\right) = \mathcal{I}$.

(ii) The operator $\mathcal{L}_{\alpha}^{\mathbb{G}}$ is trace class and has nonnegative eigenvalues.

(iii) Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding eigenvalues. Then $\mathcal{L}_{\alpha}^{\mathbb{G}}$ is an integral operator with kernel $\tilde{L}_{\alpha}^{\mathbb{G}}$ equal to the canonical kernel associated to $\{\varphi_n\}_{n\in\mathbb{N}^*}$ and $\left\{\frac{\lambda_n}{1+\alpha\lambda_n}\right\}_{n\in\mathbb{N}^*}$.

(iv) Let $g : \mathbb{G} \to \mathbb{R}_+$ be a bounded measurable function, then $\mathcal{L}_{\alpha}^{\mathbb{G}} g$ and $g\mathcal{L}_{\alpha}^{\mathbb{G}}$ are trace class.

Proof. The proof follows the same lines as that of Lemma 5.4.3, except that $\sum_{n\in\mathbb{N}^*} \frac{\lambda_n}{1+\alpha\lambda_n} < \infty$ has to be checked as follows. Since $\sum_{n\in\mathbb{N}^*} \lambda_n < \infty$, then $\lambda_n \to 0$ as $n \to \infty$, thus $\frac{\lambda_n}{1+\alpha\lambda_n} \sim \lambda_n$ from which the announced result follows. □

Proposition 5.4.10. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, let $\alpha \in \mathbb{R}_{-}^{*}$ and let $K : \mathbb{G}^2 \to \mathbb{C}$ be strictly α-regular on $\mathbb{G}$. Let $f : \mathbb{G} \to \mathbb{R}_+$ be a measurable function, $\varphi = 1 - e^{-f}$ and $\mathcal{K}_{\varphi}$ be the integral operator with kernel $K_{\varphi}(x, y) = \sqrt{\varphi(x)} K(x, y) \sqrt{\varphi(y)}$. Then, the following results hold true.*

(i)

$$\det(\mathcal{I} + \alpha\mathcal{K}_{\varphi}) = \det(\mathcal{I} + \alpha\mathcal{K}_{\mathbb{G}}) \det(\mathcal{I} - \alpha e^{-f/2}\mathcal{L}_{\alpha}^{\mathbb{G}} e^{-f/2}), \qquad (5.4.13)$$

where $\mathcal{L}_{\alpha}^{\mathbb{G}}$ is the α-inverse of $\mathcal{K}_{\mathbb{G}}$ given by (5.4.11).

(ii)

$$\begin{aligned}\det(\mathcal{I} + \alpha\mathcal{K}_{\varphi})^{-1/\alpha} &= \det(\mathcal{I} + \alpha\mathcal{K}_{\mathbb{G}})^{-1/\alpha} \\ &\quad + \sum_{k\in\mathbb{N}^*} \frac{1}{k!} \int_{\mathbb{G}^k} e^{-\sum_{i=1}^{k} f(x_i)} J_{\alpha,k}^{\mathbb{G}}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_k),\end{aligned}$$

where $J_{\alpha,k}^{\mathbb{G}}$ is given by

$$\begin{aligned}J_{\alpha,k}^{\mathbb{G}}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_k) = \det(\mathcal{I} + \alpha\mathcal{K}_{\mathbb{G}}) \det{}_{\alpha}\left(\tilde{L}_{\alpha}^{\mathbb{G}}(x_i, x_j)\right)_{1\le i,j\le k} \\ \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_k),\end{aligned} \qquad (5.4.14)$$

with $\tilde{L}_{\alpha}^{\mathbb{G}}$ being the canonical kernel associated to $\mathcal{L}_{\alpha}^{\mathbb{G}}$.

(iii) *Let Φ_α be an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel K. Then, Φ_α has Janossy measures $\left\{J^{\mathbb{G}}_{\alpha,k}\right\}_{k\in\mathbb{N}^*}$ given by (5.4.14); moreover $\mathbf{P}(\Phi_\alpha(\mathbb{G})=0)=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})^{-1/\alpha}$.*

Proof. (i) The proof follows the same lines as that of Lemma 5.4.5(i). (ii) The arguments are the same as those in the proof of Lemma 5.4.5(ii), except that the inequality $\left\|\alpha\mathcal{L}^{\mathbb{G}}_\alpha\right\|<1$ does not hold now. So one should apply Proposition 16.B.13(ii) to prove (5.4.9). (iii) The existence of Φ_α follows from Theorem 5.3.18(i). Let $f:\mathbb{G}\to\bar{\mathbb{R}}_+$ be a measurable function, $\varphi=1-e^{-f}$, and $\mathcal{K}_\varphi$ the integral operator with kernel $\sqrt{\varphi(x)}K(x,y)\sqrt{\varphi(y)}$. Then

$$\begin{aligned}\mathcal{L}_{\Phi_\alpha}(f)&=\det(\mathcal{I}+\alpha\mathcal{K}_\varphi)^{-1/\alpha}\\&=\det(\mathcal{I}+\alpha\mathcal{K}_{\mathbb{G}})^{-1/\alpha}+\sum_{k\in\mathbb{N}^*}\frac{1}{k!}\int_{\mathbb{G}^k}e^{-\sum_{i=1}^k f(x_i)}J^{\mathbb{G}}_{\alpha,k}(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k),\end{aligned}$$

where the first equality follows from Lemma 5.4.1 and the second equality is due to Item (ii). On the other hand, by Lemma 4.3.14(iii),

$$\mathcal{L}_{\Phi_\alpha}(f)=\mathbf{P}(\Phi_\alpha(\mathbb{G})=0)+\sum_{k=1}^{\infty}\frac{1}{k!}\int_{\mathbb{G}^k}e^{-\sum_{i=1}^k f(x_i)}J_k(\mathrm{d}x_1\times\cdots\times\mathrm{d}x_k),$$

where $\{J_k\}_{k\in\mathbb{N}^*}$ are the Janossy measures of Φ_α. Identifying the terms of the above two expansions concludes the proof. □

Corollary 5.4.11. *Let μ be a locally finite measure on a l.c.s.h. space $\mathbb{G}$, let $\alpha\in\{-1/m:m\in\mathbb{N}^*\}$, K be locally strictly α-regular on $\mathbb{G}$, and Φ_α be an α-determinantal point process on $\mathbb{G}$ with background measure μ and kernel K. Then the following results hold.*

(i) *The Laplace transform $\mathcal{L}_{\Phi_\alpha}(f)$ is given by (5.4.1) for any measurable function $f:\mathbb{G}\to\bar{\mathbb{R}}_+$ whose support is in $\mathcal{B}_{\mathrm{c}}(\mathbb{G})$.*

(ii) *For any $D\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$, the restriction of Φ_α to D admits the Janossy measures $\left\{J^{D}_{\alpha,k}\right\}_{k\in\mathbb{N}^*}$ given by (5.4.14) and has void probability $\mathbf{P}(\Phi_\alpha(D)=0)=\det(\mathcal{I}+\alpha\mathcal{K}_D)^{-1/\alpha}$.*

Proof. The existence of Φ_α follows from Theorem 5.3.18(i). Applying Proposition 5.4.10 to the restriction of Φ_α to any $D\in\mathcal{B}_{\mathrm{c}}(\mathbb{G})$ gives the announced results. □

Remark 5.4.12. *The expression of the Janossy measures (5.4.14) in the particular case $\alpha=-1$ (i.e., for a determinantal point process with a locally strictly (-1)-regular kernel) is consistent with (5.2.13). Indeed, let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu,D)$ composed of eigenvectors of $\mathcal{K}_D$ with respective* *eigenvalues* $\{\lambda_n\}_{n\in\mathbb{N}^*}$. *Invoking Lemma 16.A.25(vi), we get*

$$\begin{aligned}\det(\mathcal{I}-\mathcal{K}_D)\det{}_\alpha\left(\tilde{L}_\alpha^D(x_i,x_j)\right)_{1\leq i,j\leq k}\\ = \prod_{n\in\mathbb{N}^*}(1-\lambda_n)\sum_{1\leq n_1<\ldots<n_k}\frac{\lambda_{n_1}}{1-\lambda_{n_1}}\cdots\frac{\lambda_{n_k}}{1-\lambda_{n_k}}\left|\det\left(\varphi_{n_j}(x_i)\right)_{1\leq i,j\leq k}\right|^2,\end{aligned}$$

which is equal to (5.2.13).

5.5 Palm distributions

We will show that the reduced Palm version of a determinantal point process is again a determinantal point process. It will be useful to see the restriction of a kernel to a finite set as a matrix.

Notation 5.5.1. *Let* $\mathbb{G}$ *be a set and* $K:\mathbb{G}^2\to\mathbb{C}$. *For any* $u=(u_1,\ldots,u_n)\in\mathbb{G}^n$ $(n\in\mathbb{N}^*)$, *we consider matrix*

$$K_u=\left(K(u_i,u_j)\right)_{1\leq i,j\leq n}.$$

Similarly, for any $u\in\mathbb{G}^n, v\in\mathbb{G}^m$ $(n,m\in\mathbb{N}^*)$, *let*

$$K_{uv}=\left(K(u_i,v_j)\right)_{1\leq i\leq n,1\leq j\leq m}.$$

Theorem 5.5.2. Palm version of a determinantal point process. *Let* μ *be a locally finite measure on a l.c.s.h. space* $\mathbb{G}$, *let* $K:\mathbb{G}^2\to\mathbb{C}$ *be a measurable function and let* Φ *be a determinantal point process on* $\mathbb{G}$ *with background measure* μ *and kernel* K. *Assume that* Φ *has* σ*-finite moment measures.*

(i) Then for $M_{\Phi^{(n)}}$*-almost all* $u=(u_1,\ldots,u_n)\in\mathbb{G}^n$, *the* n*-th reduced Palm version of* Φ *at* u, $\Phi_u^!$ *is a determinantal point process with background measure* μ *and kernel* K^u *defined on* $\mathbb{G}^2$ *by*

$$K^u(x,y)=K(x,y)-\sum_{i,j=1}^n K(x,u_i)(K_u)^{-1}(u_i,u_j)K(u_j,y),\quad (x,y)\in\mathbb{G}^2. \tag{5.5.1}$$

(ii) For any $v=(v_1,\ldots,v_m)\in\mathbb{G}^m$, *the matrix* $(K^u)_v=(K^u(v_i,v_j))_{1\leq i,j\leq m}$ *is the Schur complement (15.A.6) of* K_u *in* $K_{u\cup v}$; *that is*

$$(K^u)_v=K_v-K_{vu}(K_u)^{-1}K_{uv}. \tag{5.5.2}$$

(iii) In the particular case $n=1$,

$$K^u(x,y)=K(x,y)-\frac{K(x,u)K(u,y)}{K(u,u)},\quad (x,y)\in\mathbb{G}^2. \tag{5.5.3}$$

Proof. Let $n, m \in \mathbb{N}^*$. It follows from Proposition 3.3.9 that for, $A \in \mathcal{B}(\mathbb{G})^{\otimes n}, B \in \mathcal{B}(\mathbb{G})^{\otimes m}$,

$$M_{\Phi^{(n+m)}}(A \times B) = \int_A M_{\Phi_u^{!(m)}}(B) M_{\Phi^{(n)}}(\mathrm{d}u).$$

Then $M_{\Phi^{(n+m)}}$ a mixture of $\left\{M_{\Phi_u^{!(m)}}(\cdot)\right\}_{u \in \mathbb{G}^n}$ with respect to $M_{\Phi^{(n)}}$ (cf. Definition 14.D.2). Since for any $A \in \mathcal{B}(\mathbb{G})^{\otimes n}$,

$$M_{\Phi^{(n)}}(A) = \int_A \det(K_u) \mu^n(\mathrm{d}u),$$

then

$$M_{\Phi^{(n)}}(\{u \in \mathbb{G}^n : \det(K_u) = 0\}) = 0.$$

Then the matrix K_u is invertible for $M_{\Phi^{(n)}}$-almost all $u \in \mathbb{G}^n$. (In particular, the components $u_1, \ldots, u_n$ are pairwise distinct.) On the other hand, for any $A \in \mathcal{B}(\mathbb{G})^{\otimes n}, B \in \mathcal{B}(\mathbb{G})^{\otimes m}$,

$$\begin{aligned}
&M_{\Phi^{(n+m)}}(A \times B) \\
&= \int_{A \times B} \det(K_{u \cup v}) \mu^n(\mathrm{d}u) \mu^m(\mathrm{d}v) \\
&= \int_{A \times B} \det(K_u) \det(K_v^u) \mu^n(\mathrm{d}u) \mu^m(\mathrm{d}v) \\
&= \int_A \left(\int_B \det(K_v^u) \mu^m(\mathrm{d}v)\right) M_{\Phi^{(n)}}(\mathrm{d}u),
\end{aligned}$$

where the second equality follows from the Schur complement formula (15.A.5) with the matrix K_v^u being the Schur complement (15.A.6) of K_u in $K_{u \cup v}$; that is

$$K_v^u = K_v - K_{vu}(K_u)^{-1} K_{uv}.$$

Note that K_v^u may be seen as the restriction of the function K^u defined on $\mathbb{G}^2$ by (5.5.1) to $v \times v$. Then $M_{\Phi^{(n+m)}}$ is also a mixture of $\{\Gamma(u, \cdot)\}_{u \in \mathbb{G}^n}$ with respect to $M_{\Phi^{(n)}}$, where

$$\Gamma(u, B) = \int_B \det(K_v^u) \mu^m(\mathrm{d}v), \quad u \in \mathbb{G}^n, B \in \mathcal{B}(\mathbb{G})^{\otimes m}.$$

Then by the uniqueness of the kernel stated in Theorem 14.D.14(ii), for $M_{\Phi^{(n)}}$-almost all $u \in \mathbb{G}^n$,

$$M_{\Phi_u^{!(m)}}(B) = \int_B \det(K_v^u) \mu^m(\mathrm{d}v), \quad B \in \mathcal{B}(\mathbb{G})^{\otimes m}.$$

By the very Definition 5.1.1 of determinantal point processes, the above equality implies that $\Phi_u^!$ is a determinantal point process with background measure μ and kernel K^u. $\square$

Remark 5.5.3. Bibliographic notes. *Theorem 5.5.2 is an extension of [88, Theorem 6.5] where only the case $n = 1$ is considered.*

5.6 Stationary determinantal point processes on $\mathbb{R}^d$

The following proposition gives a necessary and sufficient condition for a determinantal point process (whose kernel satisfies conditions of Corollary 5.1.14) to be stationary.

Proposition 5.6.1. Stationary determinantal point process. *Let Φ be a determinantal point process on $\mathbb{R}^d$ with background measure ℓ^d (Lebesgue measure on $\mathbb{R}^d$) and kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ which satisfies the conditions of Corollary 5.1.14. Then Φ is stationary if and only if for any $k \in \mathbb{N}^*$ and any $t \in \mathbb{R}^d$,*

$$\det\left(K\left(x_i + t, x_j + t\right)\right)_{1 \leq i,j \leq k} = \det\left(K\left(x_i, x_j\right)\right)_{1 \leq i,j \leq k}, \tag{5.6.1}$$

for ℓ^{dk}-almost all $(x_1, \ldots, x_k) \in \left(\mathbb{R}^d\right)^k$.

Proof. By Definition 2.3.4, Φ is stationary if and only if $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ and $S_t\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k - t}$ have the same distribution for any $t \in \mathbb{R}^d$. Fix some $t \in \mathbb{R}^d$ and observe that

$$S_t\Phi\left(B\right) = \Phi\left(B + t\right), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right)$$

and more generally, for any $k \in \mathbb{N}^*$,

$$\left(S_t\Phi\right)^{(k)}\left(B\right) = \Phi^{(k)}\left(B + t\right), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right)^{\otimes k}.$$

Then for any $k \in \mathbb{N}^*$ and any $B \in \mathcal{B}\left(\mathbb{R}^d\right)^{\otimes k}$,

$$\begin{aligned}
\mathbf{E}\left[\left(S_t\Phi\right)^{(k)}\left(B\right)\right] &= \mathbf{E}\left[\Phi^{(k)}\left(B + t\right)\right] \\
&= \int_{B+t} \det\left(K\left(x_i, x_j\right)\right)_{1\leq i,j\leq k} \, \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= \int_B \det\left(K\left(y_i + t, y_j + t\right)\right)_{1\leq i,j\leq k} \, \mathrm{d}y_1 \ldots \mathrm{d}y_k,
\end{aligned}$$

where the third equality is due to the change of variable $x \to y = x - t$. (i) *Sufficiency.* If (5.6.1) holds true, then the above display shows that $S_t\Phi$ is a determinantal point process on $\mathbb{R}^d$ with background measure ℓ^d and kernel K. It follows from Corollary 5.1.14 that Φ and $S_t\Phi$ have the same distribution. (ii) *Necessity.* If Φ is stationary, then for any $k \in \mathbb{N}^*$, $t \in \mathbb{R}^d$, and $B \in \mathcal{B}\left(\mathbb{R}^d\right)^{\otimes k}$, $\mathbf{E}\left[\left(S_t\Phi\right)^{(k)}\left(B\right)\right] = \mathbf{E}\left[\Phi^{(k)}\left(B\right)\right]$, which combined with the above display shows that

$$\begin{aligned}
&\int_B \det\left(K\left(x_i + t, x_j + t\right)\right)_{1\leq i,j\leq k} \, \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&\quad = \int_B \det\left(K\left(x_i, x_j\right)\right)_{1\leq i,j\leq k} \, \mathrm{d}x_1 \ldots \mathrm{d}x_k.
\end{aligned}$$

This being true for any $B \in \mathcal{B}\left(\mathbb{R}^d\right)^{\otimes k}$, then equality (5.6.1) holds for ℓ^{dk}-almost all $(x_1, \ldots, x_k) \in \left(\mathbb{R}^d\right)^k$. $\square$

5.6.1 Ginibre determinantal point process

Definition 5.6.2. Ginibre kernel. *The* Ginibre kernel *is the function K defined on $\mathbb{C}\times\mathbb{C}$ by*

$$K(z_1, z_2) = \frac{1}{\pi} e^{z_1 z_2^*} e^{-\frac{1}{2}\left(|z_1|^2+|z_2|^2\right)}, \quad z_1, z_2 \in \mathbb{C}, \tag{5.6.2}$$

where z^* is the complex-conjugate of $z \in \mathbb{C}$.

Let ℓ^2 be the Lebesgue measure on $\mathbb{C}$ (identified with $\mathbb{R}^2$); i.e., $\ell^2(\mathrm{d}z) = \mathrm{d}x\mathrm{d}y$ when $z = x + iy$. Obviously, the Ginibre kernel is continuous on $\mathbb{C}\times\mathbb{C}$ and locally square integrable with respect to $\ell^2 \otimes \ell^2$. Using the expansion $e^{z_1 z_2^*} = \sum_{n\in\mathbb{N}} \frac{(z_1 z_2^*)^n}{n!}$, we deduce the following expansion for Ginibre kernel

$$K(z_1, z_2) = \sum_{n\in\mathbb{N}} \phi_n(z_1)\,\phi_n(z_2)^*, \quad z_1, z_2 \in \mathbb{C}, \tag{5.6.3}$$

where the functions ϕ_n are defined on $\mathbb{C}$ by

$$\phi_n(z) = \frac{z^n}{\sqrt{\pi n!}} e^{-\frac{1}{2}|z|^2}, \quad n \in \mathbb{N}, z \in \mathbb{C}. \tag{5.6.4}$$

For any $D \in \mathcal{B}(\mathbb{C})$, recall the notation $L^2_{\mathbb{C}}\left(\ell^2, D\right)$ for the set of measurable functions $f : D \to \mathbb{C}$ which are square-integrable with respect to ℓ^2.

Lemma 5.6.3. *The family $\{\phi_n\}_{n\in\mathbb{N}}$ defined by (5.6.4) is an orthonormal family of $L^2_{\mathbb{C}}\left(\ell^2, \mathbb{C}\right)$.*

Proof. Observe that for any $n, m \in \mathbb{N}$,

$$\begin{aligned}
\int_{\mathbb{C}} \phi_n(z)\,\phi_m(z)^*\,\ell^2(\mathrm{d}z) &= \frac{1}{\pi n!}\int_{\mathbb{C}} z^n (z^*)^m e^{-|z|^2}\ell^2(\mathrm{d}z) \\
&= \frac{1}{\pi\sqrt{n!}\sqrt{m!}}\int_{\mathbb{R}_+\times[0,2\pi[} r^{n+m+1} e^{-r^2} e^{-(n-m)\theta}\mathrm{d}r\mathrm{d}\theta \\
&= \frac{1}{\pi\sqrt{n!}\sqrt{m!}}\left(\int_0^\infty r^{n+m+1}e^{-r^2}\mathrm{d}r\right)\left(\int_0^{2\pi} e^{-(n-m)\theta}\mathrm{d}\theta\right) \\
&= \mathbf{1}_{\{n=m\}},
\end{aligned}$$

where for the second equality we make the change of variable $z \to (r = |z|, \theta = \arg z)$, and the third equality is due to the fact that $\int_0^\infty r^{2n+1}e^{-r^2}\mathrm{d}r = \frac{n!}{2}$, which in turn follows from [42, §3.326.2 p.337]. ☐

Lemma 5.6.4. Restriction of Ginibre kernel to an annulus. *Let $0 \le a < b$, $D = \{z \in \mathbb{C} : a \le |z| \le b\}$, and*

$$\phi_n^D(z) = \frac{\phi_n(z)}{\sqrt{\lambda_n^D}}\mathbf{1}_D(z), \quad z \in \mathbb{C}, n \in \mathbb{N}, \tag{5.6.5}$$

where ϕ_n is defined by (5.6.4) and

$$\lambda_n^D = \frac{\gamma\left(n+1, b^2\right) - \gamma\left(n+1, a^2\right)}{n!}, \quad n \in \mathbb{N}, \tag{5.6.6}$$

with $\gamma(\alpha, x) = \int_0^x e^{-v} v^{\alpha-1} \mathrm{d}v, \alpha \in \mathbb{R}_+^*, x \in \mathbb{R}_+$ being the lower incomplete gamma function. Then the following results hold.

(i) $\left\{\phi_n^D\right\}_{n\in\mathbb{N}}$ is an orthonormal family of $L^2_{\mathbb{C}}\left(\ell^2, D\right)$. Moreover,

$$\sum_{n\in\mathbb{N}} \lambda_n^D = b^2 - a^2 < \infty.$$

(ii) Let K be the Ginibre Kernel defined by (5.6.2). The restriction K_D of K to $D \times D$ is a canonical kernel associated to $\left\{\phi_n^D\right\}_{n\in\mathbb{N}^*}$ and $\left\{\lambda_n^D\right\}_{n\in\mathbb{N}^*}$.

Proof. (i) Observe first that for any $n, m \in \mathbb{N}$,

$$\begin{aligned}\int_D \phi_n(z)\, \phi_m(z)^* \,\ell^2(\mathrm{d}z) &= \frac{1}{\pi n!} \int_D z^n \left(z^*\right)^m e^{-|z|^2} \ell^2(\mathrm{d}z) \\ &= \frac{1}{\pi\sqrt{n!}\sqrt{m!}} \left(\int_a^b r^{n+m+1} e^{-r^2} \mathrm{d}r\right) \left(\int_0^{2\pi} e^{-(n-m)\theta} \mathrm{d}\theta\right),\end{aligned}$$

where for the second equality we make the change of variable $z \to (r = |z|, \theta = \arg z)$. Making the change of variable $r \to t = r^2$, we get

$$\int_a^b r^{2n+1} e^{-r^2} \mathrm{d}r = \frac{1}{2} \int_{a^2}^{b^2} t^n e^{-t} \mathrm{d}t = \frac{1}{2}\left[\gamma\left(n+1, b^2\right) - \gamma\left(n+1, a^2\right)\right]$$

where γ is the lower incomplete gamma function. Combining the above two equalities, we get

$$\int_D \phi_n(z)\, \phi_m(z)^* \,\ell^2(\mathrm{d}z) = \mathbf{1}_{\{n=m\}} \frac{\gamma\left(n+1, b^2\right) - \gamma\left(n+1, a^2\right)}{n!} = \mathbf{1}_{\{n=m\}} \lambda_n^D,$$

from which we deduce that $\left\{\phi_n^D\right\}_{n\in\mathbb{N}}$ is an orthonormal family. Observe that

$$\begin{aligned}\sum_{n\in\mathbb{N}} \lambda_n^D &= \sum_{n\in\mathbb{N}} \frac{\gamma\left(n+1, b^2\right) - \gamma\left(n+1, a^2\right)}{n!} \\ &= \sum_{n\in\mathbb{N}} \frac{1}{n!} \int_{a^2}^{b^2} t^n e^{-t} \mathrm{d}t \\ &= \int_{a^2}^{b^2} \left(\sum_{n\in\mathbb{N}} \frac{1}{n!} t^n\right) e^{-t} \mathrm{d}t = b^2 - a^2 < \infty.\end{aligned}$$

(ii) It follows from (5.6.3) that

$$K_D(z_1, z_2) = \sum_{n\in\mathbb{N}} \lambda_n^D \phi_n^D(z_1)\, \phi_n^D(z_2)^*, \quad z_1, z_2 \in D,$$

where ϕ_n^D are given by (5.6.5) and λ_n^D are given by (5.6.6). Thus K_D is a canonical kernel associated to $\{\phi_n^D\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n^D\}_{n\in\mathbb{N}^*}$; cf. Definition 5.2.1. □

Corollary 5.6.5. *The Ginibre Kernel K defined by (5.6.2) is locally (-1)-regular on $\mathbb{C}$.*

Proof. Let $0 \le a < b$, $D = \{z \in \mathbb{C} : a \le |z| \le b\}$, K_D be the restriction of K to $D \times D$, and $\mathcal{K}_D$ be the integral operator (16.A.24). By Lemma 5.6.4(ii), K_D is a canonical kernel associated to $\{\phi_n^D\}_{n\in\mathbb{N}^*}$ given by (5.6.5) and $\{\lambda_n^D\}_{n\in\mathbb{N}^*}$ given by (5.6.6). Then by Proposition 16.A.26, $\mathcal{K}_D$ is Hermitian, nonnegative-definite and trace class. Moreover $\{\lambda_n^D\}_{n\in\mathbb{N}^*}$ are the non-null eigenvalues of the integral operator $\mathcal{K}_D$ with respective eigenvectors $\{\varphi_n^D\}_{n\in\mathbb{N}^*}$. By (5.6.6), $0 < \lambda_n^D \le 1$ for any $n \in \mathbb{N}$. Then K is (-1)-regularizable on D. Since K is moreover continuous, then by Corollary 5.2.13(ii), K is (-1)-regular on D. By Lemma 5.2.19(ii), K is (-1)-regular on any locally compact subset of D. Observing that any locally compact subset of $\mathbb{C}$ is bounded and therefore included in some annulus D allows one to conclude. □

By Theorem 5.2.17, there exists a determinantal point process Φ on $\mathbb{C}$ with background measure ℓ^2 and kernel K given defined by (5.6.2) which we call the *Ginibre point process*.

Proposition 5.6.6. *The Ginibre point process is stationary.*

Proof. Observe that for any $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} K(z_1 + t, z_2 + t) &= \frac{1}{\pi} e^{(z_1+t)(z_2+t)^*} e^{-\frac{1}{2}\left(|z_1+t|^2 + |z_2+t|^2\right)} \\ &= e^{\frac{1}{2}(t^* z_1 - t z_1^*)} K(z_1, z_2)\, e^{\frac{1}{2}(t z_2^* - t^* z_2)}, \end{aligned}$$

Then for any $k \in \mathbb{N}$, any $t \in \mathbb{C}$, and any $(z_1, \ldots, z_k) \in \mathbb{C}^k$,

$$\begin{aligned} &\det\left(K(z_i + t, z_j + t)\right)_{1\le i,j\le k} \\ &\quad = \left(\prod_{i=1}^{k} e^{\frac{1}{2}(t^* z_i - t z_i^*)}\right) \det\left(K(z_i, z_j)\right)_{1\le i,j\le k} \left(\prod_{j=1}^{k} e^{\frac{1}{2}(t z_j^* - t^* z_j)}\right) \\ &\quad = \det\left(K(z_i, z_j)\right)_{1\le i,j\le k}. \end{aligned}$$

Invoking Proposition 5.6.1 allows one to conclude. □

5.6.2 Shift-invariant kernel

A function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfying

$$K(x,y) = K(x+t, y+t), \quad x,y,t \in \mathbb{R}^d,$$

will be called *shift-invariant*. For such function, letting

$$C(x) = K(x,0), \quad x \in \mathbb{R}^d, \tag{5.6.7}$$

we get

$$K(x,y) = C(x-y), \quad x,y \in \mathbb{R}^d. \tag{5.6.8}$$

Obviously Condition (5.6.1) holds true when the kernel K is shift-invariant. Nevertheless, the Ginibre point process shows that shift-invariance of the kernel is not necessary for stationarity to hold.

We will show that the condition of existence of a determinantal point process in Theorem 5.2.17 becomes simpler in the particular case of shift-invariant kernels.

Definition 5.6.7. *Let $C : \mathbb{R}^d \to \mathbb{C}$ and let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be defined by (5.6.8).*

1. *C is called* Hermitian *iff K is so; cf. Definition 16.A.30(i). Note that C is* Hermitian *iff $C(-x) = C(x)^*$ for any $x \in \mathbb{R}^d$.*

2. *C is called* nonnegative-definite *iff K is so; cf. Definition 16.A.30(ii).*

Recall the notation $L^2_{\mathbb{C}}(\ell^d, \mathbb{R}^d)$ for the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ which are square-integrable with respect to ℓ^d. Recall that any function $f \in L^2_{\mathbb{C}}(\ell^d, \mathbb{R}^d)$ admits a well defined *Fourier transform* (see [21, § 1.4.1 p.73] which extends to $d \geq 2$), which we shall denote by $\hat{f}$ or $\mathfrak{F}f$.

Theorem 5.6.8. *Let $C : \mathbb{R}^d \to \mathbb{C}$ be a square integrable (with respect to Lebesgue measure), Hermitian, nonnegative-definite and continuous function, $\hat{C}$ be its Fourier transform, and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be defined by (5.6.8). Then the following results hold.*

(i) K is locally regular with respect to the Lebesgue measure.

(ii) If there exists some $\alpha \in \mathbb{R}^*_-$ such that $0 \leq \hat{C}(\xi) \leq -1/\alpha$ for any $\xi \in \mathbb{R}^d$, then K is locally α-regular with respect to the Lebesgue measure.

Proof. Let $D \in \mathcal{B}_c(\mathbb{R}^d)$ and let $L^2_{\mathbb{C}}(\ell^d, D)$ be the set of square integrable functions from D to $\mathbb{C}$. Observe first that K is square integrable on $D \times D$, since

$$\begin{aligned}
\int_{D\times D} |K(x,y)|^2 \,\mathrm{d}x\mathrm{d}y &= \int_{D\times D} |C(x-y)|^2 \,\mathrm{d}x\mathrm{d}y \\
&= \int_D \left(\int_{D-y} |C(u)|^2 \,\mathrm{d}u \right) \mathrm{d}y \\
&\leq \int_D \left(\int_{\mathbb{R}^d} |C(u)|^2 \,\mathrm{d}u \right) \mathrm{d}y = \left(\int_{\mathbb{R}^d} |C(u)|^2 \,\mathrm{d}u \right) \ell(D),
\end{aligned}$$

where the second equality is due to the change of variable $x \to u = x - y$ Let $\mathcal{K}_D$ be the integral operator associated to the restriction of K to $D \times D$ (with respect to Lebesgue measure). (i) Since K is Hermitian, then by Lemma 16.A.9(vi), $\mathcal{K}_D$ is Hermitian. Since K is moreover continuous and nonnegative-definite, then by Theorem 16.A.32, $\mathcal{K}_D$ is nonnegative-definite. It remains to show that $\mathcal{K}_D$ is trace class. Since K is continuous and $\mathcal{K}_D$ is Hermitian and nonnegative-definite, then by Theorem 16.A.28,

$$K(x,y) = \sum_{n \in \mathbb{N}^*} \lambda_n \varphi_n(x) \varphi_n(y)^*, \quad x, y \in D,$$

where $\{\varphi_n\}_{n \in \mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}\left(\ell^d, D\right)$ composed of eigenvectors of $\mathcal{K}_D$ and $\{\lambda_n\}_{n \in \mathbb{N}^*}$ are the corresponding eigenvalues. By Lemma 16.A.25(ii),

$$\sum_{n \in \mathbb{N}^*} \lambda_n = \int_D K(x,x) \mathrm{d}x = C(0) \, \ell(D) < \infty.$$

Then Proposition 16.A.26(iii) shows that $\mathcal{K}_D$ is trace class. (ii) We have to show that the eigenvalues of $\mathcal{K}_D$ are not larger than $-1/\alpha$. Observe from (16.A.24) that for any $f \in L^2_{\mathbb{C}}\left(\ell^d, D\right)$,

$$\mathcal{K}_D f(x) = \mathbf{1}_{\{x \in D\}} \int_D K(x,y) f(y) \mathrm{d}y = \mathbf{1}_{\{x \in D\}} \int_D C(x-y) f(y) \mathrm{d}y, \quad x \in \mathbb{R}^d.$$

Thus

$$\mathcal{K}_D f = \mathbf{1}_D \times (C \star f).$$

Since $D \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$, then by the Cauchy-Schwarz inequality $f \in L^1_{\mathbb{C}}\left(\ell^d, D\right)$ and therefore the convolution $C \star f$ is almost-everywhere well-defined and $C \star f \in L^2_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right)$; cf. [21, Theorem 1.4.4 p.75] (which extends to $d \geq 2$). Moreover, its Fourier transform equals

$$\mathfrak{F}(C \star f) = \hat{C} \times \hat{f}. \tag{5.6.9}$$

Since $\hat{C}$ is bounded, the muliplication operator $\mathcal{L}$, defined for any $g \in L^2_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right)$ by

$$\mathcal{L} g = \hat{C} \times g,$$

is well defined; i.e., $\mathcal{L} g \in L^2_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right)$ as shown in Example 16.B.8. For any $f \in L^2_{\mathbb{C}}\left(\ell^d, D\right)$, we deduce from (5.6.9) that

$$\mathcal{L} \hat{f} = \hat{C} \times \hat{f} = \mathfrak{F}(C \star f).$$

Thus

$$C \star f = \mathfrak{F}^{-1} \mathcal{L} \hat{f} = \mathfrak{F}^{-1} \mathcal{L} \mathfrak{F} f,$$

and therefore,

$$\mathcal{K}_D f = \mathbf{1}_D \times \left(\mathfrak{F}^{-1} \mathcal{L} \mathfrak{F} f\right), \quad f \in L^2_{\mathbb{C}}\left(\ell^d, D\right).$$

Since C is Hermitian, then its Fourier transform $\hat{C}$ is real. By [58, Lemma 4.55 p.166], the spectrum of $\mathcal{L}$ equals the essential image of $\hat{C}$ defined as

$$B = \left\{y \in \mathbb{R} : \ell^d \left(\hat{C}^{-1}\left(]y-\varepsilon, y+\varepsilon[\right)\right) > 0 \text{ for every } \varepsilon > 0\right\}.$$

Since this latter is a subset of the closure of $\hat{C}\left(\mathbb{R}^d\right)$ (indeed, let $y \in B$ then for any $n \in \mathbb{N}^*$ there exist $x_n \in \hat{C}^{-1}\left(]y-1/n, y+1/n[\right)$; obviously $\hat{C}\left(x_n\right) \to y$, then y belongs to the closure of $\hat{C}\left(\mathbb{R}^d\right)$), it follows that the spectrum of $\mathcal{L}$ is included in $[0, -1/\alpha]$. Since $\mathfrak{F}^{-1}\mathfrak{F}$ is the identity on $L^2_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right)$, then the spectrum of $\mathfrak{F}^{-1}\mathcal{L}\mathfrak{F}$ equals that of $\mathcal{L}$. Since $L^2_{\mathbb{C}}\left(\ell^d, D\right)$ is a Hilbert space and $\mathcal{K}_D$ is Hermitian, then the spectral radius of $\mathcal{K}_D$ equals its norm; cf. [82, Theorem VI.6 p.192]. Observing that

$$\|\mathcal{K}_D\| \leq \|\mathbf{1}_D\| \times \left\|\mathfrak{F}^{-1}\mathcal{L}\mathfrak{F}\right\| \leq -1/\alpha,$$

allows one to conclude. □

Remark 5.6.9. *The arguments of Point (ii) in the proof of Theorem 5.6.8 are inspired from [63, Appendix H]. Related statements are given in [88, Lemma 5.1 p.440] and [91, §3 p.958] without proof.*

Definition 5.6.10. *If Φ is a determinantal point process on $\mathbb{R}^d$ with shift-invariant kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, we will also refer to C defined by (5.6.7) as* kernel *of the determinantal point process.*

Example 5.6.11. Gaussian determinantal point process *[63, p.16]. Let $C : \mathbb{R}^d \to \mathbb{C}$ be defined by*

$$C(x) = \lambda \exp(-\|x\|^2/\zeta^2), \qquad x \in \mathbb{R}^d, \tag{5.6.10}$$

where λ and ζ are two given positive real numbers. Obviously C is square integrable, continuous and Hermitian. Its Fourier transform $\hat{C}$ is given by, for any $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\hat{C}(\xi) &= \int_{\mathbb{R}^d} C(x) e^{-2i\pi\langle x, \xi\rangle} \mathrm{d}x \\
&= \lambda \prod_{k=1}^d \int_{\mathbb{R}} e^{-x_k^2/\zeta^2} e^{-2i\pi x_k \xi_k} \mathrm{d}x_k \\
&= \lambda \prod_{k=1}^d \zeta\sqrt{\pi} \int_{\mathbb{R}} e^{-\pi y_k^2} e^{-2i\pi y_k\left(\zeta\sqrt{\pi}\xi_k\right)} \mathrm{d}y_k \\
&= \lambda \prod_{k=1}^d \zeta\sqrt{\pi} e^{-\pi\left(\zeta\sqrt{\pi}\xi_k\right)^2} = \lambda(\sqrt{\pi}\zeta)^d \exp(-\|\pi\zeta\xi\|^2),
\end{aligned}$$

where for the third equality we make the change of variable $x_k \to y_k = \frac{x_k}{\zeta\sqrt{\pi}}$ and the fourth equality is due to [21, Example 1.1.5 p.7]. Observe that $\hat{C}$ is nonnegative, integrable and bounded. Then by Proposition 5.6.13, C is nonnegative-definite.

If

$$\lambda \leq \lambda_{\max} := (\sqrt{\pi}\zeta)^{-d},$$

then by Theorems 5.6.8 and 5.2.17, there exists a stationary determinantal point process Φ *with kernel (5.6.10) called a* Gaussian determinantal point process*.*

Example 5.6.12. Cauchy determinantal point process *[63, p.17]. Let* $C : \mathbb{R}^d \to \mathbb{C}$ *be defined by*

$$C(x) = \frac{\lambda}{(1 + \|x/\zeta\|^2)^{\eta + d/2}}, \qquad x \in \mathbb{R}^d,$$

where λ*,* ζ *and* η *are given positive real numbers. Obviously,* C *is continuous, square integrable and Hermitian. Moreover, if* $\eta < d/2$*, the Fourier transform* $\hat{C}$ *is given by*

$$\hat{C}(\xi) = \lambda \frac{2^{1-\eta}(\sqrt{\pi}\zeta)^d}{\Gamma(\eta + d/2)} \|2\pi\zeta\xi\|^{\eta} K_{\eta}(\|2\pi\zeta\xi\|), \quad \xi \in \mathbb{R}^d,$$

where K_η *is the* modified Bessel function of the second kind*; see [42, Equation (8.407.1) p.901].*

Assume that $\eta \in [1/2, d/2)$*. Then, by [63, Equation (K.3) p.51]*

$$\hat{C}(\xi) \leq \lambda \frac{\Gamma(\eta)(\sqrt{\pi}\zeta)^d}{\Gamma(\eta + d/2)}.$$

If

$$\lambda \leq \lambda_{\max} := \frac{\Gamma(\eta + d/2)}{\Gamma(\eta)(\sqrt{\pi}\zeta)^d},$$

then by Theorems 5.6.8 and 5.2.17, there exists a stationary determinantal point process Φ *with kernel (5.6.10) called a* Cauchy determinantal point process*.*

We will show that starting from any bounded probability density function φ on $\mathbb{R}^d$, there exist a determinantal point process with kernel equal to the inverse Fourier transform of φ (up to a multiplicative factor).

Proposition 5.6.13. *Let* $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ *be integrable and bounded. Then the inverse Fourier transform of* φ*, that is* $C = \mathfrak{F}^{-1}(\varphi)$*, is square integrable, Hermitian, nonnegative-definite and continuous.*

Proof. Letting $M = \sup\{\varphi(\xi) : \xi \in \mathbb{R}^d\}$, we get

$$\int_{\mathbb{R}^d} |\varphi(\xi)|^2 \,\mathrm{d}\xi \leq M \int_{\mathbb{R}^d} |\varphi(\xi)| \,\mathrm{d}\xi < \infty,$$

i.e., φ is square integrable. Then $C = \mathfrak{F}^{-1}(\varphi)$ is square integrable. Since φ is integrable, then C is continuous; cf. [21, Theorem 2.1.1 p.92]. Let $a =$

$\int_{\mathbb{R}^d} |\varphi(\xi)| \mathrm{d}\xi$ and let X be an $\mathbb{R}^d$-valued random variable with probability density function φ/a. Since

$$C(x) = \int_{\mathbb{R}^d} \varphi(\xi) e^{2i\pi\langle x,\xi\rangle} \mathrm{d}\xi,$$

then C/a is the characteristic function of X. If follows that C is Hermitian and nonnegative-definite. □

It follows from the above proposition and Theorems 5.6.8 and 5.2.17 that any bounded probability density functions on $\mathbb{R}^d$ allows one to define a family of determinantal point processes. Here is an illustration.

Corollary 5.6.14. Isotropic probability density functions *[63, p.20]. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable bounded function such that $\int_0^\infty r^{d-1} f(r) \mathrm{d}r < \infty$, for some $d \in \mathbb{N}^*$. Let*

$$\varphi_\lambda(x) = \lambda \frac{f(\|x\|)}{c}, \tag{5.6.11}$$

where

$$c = \int_{\mathbb{R}^d} f(\|x\|) \mathrm{d}x = \frac{d\pi^{d//2}}{\Gamma(d/2+1)} \int_0^\infty r^{d-1} f(r) \mathrm{d}r, \tag{5.6.12}$$

and

$$0 \le \lambda \le \lambda_{\max} := \frac{c}{\|f\|_\infty}.$$

Then there exists a stationary determinantal point process Φ_λ with kernel $C_\lambda = \mathfrak{F}^{-1}(\varphi_\lambda)$ (where $\mathfrak{F}^{-1}$ is the inverse Fourier transform).

Proof. Making the change of coordinates from Cartesian to spherical (2.7.1), we get

$$c = \int_{\mathbb{R}^d} f(\|x\|) \mathrm{d}x = \sigma_d \int_0^\infty r^{d-1} f(r) \mathrm{d}r,$$

where $\sigma_d = \frac{d\pi^{d//2}}{\Gamma(d/2+1)}$ is the surface area of the unit ball in $\mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} \varphi_\lambda(x) \mathrm{d}x = \lambda.$$

Moreover, $\lambda \le \lambda_{\max} = \frac{c}{\|f\|_\infty}$ implies

$$\|\varphi_\lambda\|_\infty \le 1.$$

Applying Proposition 5.6.13 and then Theorems 5.6.8 and 5.2.17 allows one to conclude. □

Here is an example illustrating Corollary 5.6.14.

Example 5.6.15. Generalized gamma determinantal point process *[63, p.20]*. *Let X be a random variable on $\mathbb{R}_+$ with* Gamma distribution *of parameters $\gamma, \beta \in \mathbb{R}_+^*$; i.e., with probability density function*

$$h(x) = \frac{1}{\Gamma(\gamma)\beta^\gamma} x^{\gamma-1} e^{-x/\beta}, \quad x \in \mathbb{R}_+^*, \tag{5.6.13}$$

cf. [89, p.156] (γ is called the shape parameter *and β the* scale parameter*). Let f be the probability density function of $Y = X^{1/\mu}$ where $\mu \in \mathbb{R}_+^*$; that is*

$$f(y) = \mu y^{\mu-1} h(y^\mu) = \frac{\mu}{\Gamma(\gamma)\beta^\gamma} y^{\gamma\mu-1} e^{-y^\mu/\beta}, \quad y \in \mathbb{R}_+^*.$$

By (5.6.12),

$$\begin{aligned} c &= \frac{d\pi^{d//2}}{\Gamma(d/2+1)} \frac{\mu}{\Gamma(\gamma)\beta^\gamma} \int_0^\infty r^{\mu\gamma+d-2} e^{-r^\mu/\beta} \mathrm{d}r \\ &= \frac{d\pi^{d//2}}{\Gamma(d/2+1)} \frac{\mu}{\Gamma(\gamma)\beta^\gamma} \frac{\beta^{\gamma+\frac{d-1}{\mu}}}{\mu} \int_0^\infty u^{\gamma+\frac{d-1}{\mu}-1} e^{-u} \mathrm{d}u \\ &= \frac{d\pi^{d//2}\Gamma\left(\gamma+\frac{d-1}{\mu}\right)}{\Gamma(d/2+1)\Gamma(\gamma)} \beta^{\frac{d-1}{\mu}}, \end{aligned}$$

where the second equality is due to the change of variable $r \to u = r^\mu/\beta$.

By (5.6.11),

$$\begin{aligned} \varphi_\lambda(x) &= \lambda \frac{f(\|x\|)}{c} \\ &= \lambda \frac{\Gamma(d/2+1)\mu}{d\pi^{d//2}\Gamma\left(\gamma+\frac{d-1}{\mu}\right)} \frac{1}{\beta^{\gamma+\frac{d-1}{\mu}}} \|x\|^{\gamma\mu-1} e^{-\|x\|^\mu/\beta} \\ &= \lambda \frac{\Gamma(d/2+1)\mu\alpha^d}{d\pi^{d//2}\Gamma\left(\gamma+\frac{d-1}{\mu}\right)} \|\alpha x\|^{\gamma\mu-1} e^{-\|\alpha x\|^\mu/\beta}, \end{aligned}$$

where $\alpha = \beta^{-1/\mu}$. By Corollary 5.6.14, there exists a stationary determinantal point process Φ_λ with kernel $C_\lambda = \mathfrak{F}^{-1}(\varphi_\lambda)$ called a generalized gamma determinantal point process *(where $\mathfrak{F}^{-1}$ is the inverse Fourier transform).*

5.7 Discrete determinantal point processes

We assume in the present section that the space $\mathbb{G}$ is discrete (i.e., finite or countable) and that the associated Borel σ-algebra is the family of all subsets of $\mathbb{G}$. All determinatal point processes considered in the present section will have as background measure the *counting measure* $\nu_{\mathbb{G}}$ on $\mathbb{G}$; that is $\nu_{\mathbb{G}} = \sum_{x\in\mathbb{G}} \delta_x$.

5.7.1 Characterization

The following lemma characterizes simple point process on discrete set in terms of its factorial moment measures.

Lemma 5.7.1. Simple point process on discrete set. *Let Φ be a point process on a discrete set $\mathbb{G}$. Then the following results hold.*

(i) Φ is simple iff

$$\mathbf{P}\left(\Phi\left(\{x\}\right) \geq 2\right) = 0, \quad \forall x \in \mathbb{G}, \tag{5.7.1}$$

or, equivalently,

$$M_\Phi\left(\{x\}\right) = \mathbf{P}(x \in \Phi), \quad \forall x \in \mathbb{G}. \tag{5.7.2}$$

(ii) Φ is simple iff

$$M_{\Phi^{(k)}}\left(\{(x_1, \ldots, x_k)\}\right) = \mathbf{P}((x_1, \ldots, x_k) \in \Phi^{(k)}), \tag{5.7.3}$$

for all $k \in \mathbb{N}^*$ and all $x_1, \ldots, x_k \in \mathbb{G}$, or, equivalently,

$$M_{\Phi^{(k)}}\left(\{(x_1, \ldots, x_k)\}\right) = \mathbf{P}(\{x_1, \ldots, x_k\} \subset \Phi \textit{ and } (x_1, \ldots, x_k) \in \mathbb{G}^{(k)}), \tag{5.7.4}$$

for all $k \in \mathbb{N}^*$ and all $x_1, \ldots, x_k \in \mathbb{G}$, where

$$\mathbb{G}^{(k)} = \left\{(x_1, \ldots, x_k) \in \mathbb{G}^n : x_i \neq x_j, \textit{ for any } i \neq j\right\}.$$

Proof. (i) *Condition (5.7.1).* The condition is obviously necessary. For sufficiency, observe that

$$\begin{aligned}
\mathbf{P}\left(\exists x \in \mathbb{G}, \Phi(\{x\}) \geq 2\right) &= \mathbf{E}\left[\mathbf{1}\left\{\exists x \in \mathbb{G}, \Phi(\{x\}) \geq 2\right\}\right] \\
&\leq \mathbf{E}\left[\sum_{x \in \mathbb{G}} \mathbf{1}\left\{\Phi(\{x\}) \geq 2\right\}\right] \\
&\leq \sum_{x \in \mathbb{G}} \mathbf{P}\left(\Phi\left(\{x\}\right) \geq 2\right).
\end{aligned}$$

Condition (5.7.2). If Φ is simple, then for all $x \in \mathbb{G}$, $\Phi\left(\{x\}\right) = \mathbf{1}\left\{x \in \Phi\right\}$, which implies (5.7.2) by taking expectation. Conversely, if Φ is not simple, then by (5.7.1), there exists some $x \in \mathbb{G}$ such that $\mathbf{P}\left(\Phi\left(\{x\}\right) \geq 2\right) \neq 0$, thus $M_\Phi\left(\{x\}\right) = \mathbf{E}\left[\Phi\left(\{x\}\right)\right] > \mathbf{P}\left(\Phi\left(\{x\}\right) \geq 1\right) = \mathbf{P}(x \in \Phi)$. (ii) *Necessity.* Assume that Φ is simple. Then Equation (14.E.1) implies

$$\Phi^{(k)} = \sum_{\substack{x_1, \ldots, x_k \in \Phi: \\ (x_1, \ldots, x_k) \in \mathbb{G}^{(k)}}} \delta_{(x_1, \ldots, x_k)},$$

which shows in particular that $\Phi^{(k)}$ is simple. Applying (5.7.2) to $\Phi^{(k)}$ gives (5.7.3). Moreover, for any $(x_1, \ldots, x_k) \in \mathbb{G}^k$,

$$\mathbf{1}\left\{(x_1, \ldots, x_k) \in \Phi^{(k)}\right\} = \mathbf{1}\left\{\{x_1, \ldots, x_k\} \subset \Phi \text{ and } (x_1, \ldots, x_k) \in \mathbb{G}^{(k)}\right\}.$$

Taking expectation, we get (5.7.4). *Sufficiency.* Either (5.7.3) or (5.7.4) with $k = 1$ implies (5.7.2) which shows that Φ is simple by Item (i). □

The following proposition characterizes determinantal point process on discrete set.

Proposition 5.7.2. *Let $\mathbb{G}$ be a discrete set, μ be the counting measure on $\mathbb{G}$, $K : \mathbb{G}^2 \to \mathbb{C}$, and Φ be a point process on $\mathbb{G}$. Then Φ is a determinantal point process with kernel K and background measure $\nu_{\mathbb{G}}$ iff*

$$\mathbf{P}((x_1, \ldots, x_k) \in \Phi^{(k)}) = \det \left(K\left(x_i, x_j\right)\right)_{1 \le i,j \le k}, \tag{5.7.5}$$

for any $k \in \mathbb{N}^*$ and any $x_1, \ldots, x_k \in \mathbb{G}$, or, equivalently, Φ is simple and

$$\mathbf{P}(u \subset \Phi) = \det\left(K_u\right), \tag{5.7.6}$$

for any $u \subset \mathbb{G}$, where K_u is the matrix obtained by restricting K to $u \times u$.

Proof. Since μ is the counting measure on $\mathbb{G}$, then by Definition 5.1.1, Φ is a determinantal point process with kernel K and background measure $\nu_{\mathbb{G}}$ iff

$$M_{\Phi^{(k)}}\left(\{(x_1, \ldots, x_k)\}\right) = \det \left(K\left(x_i, x_j\right)\right)_{1 \le i,j \le k}, \tag{5.7.7}$$

for all $k \in \mathbb{N}^*$ and all $x_1, \ldots, x_k \in \mathbb{G}$. (i) *Necessity.* Assume that Φ is a determinantal point process with kernel K and background measure $\nu_{\mathbb{G}}$. Then Φ is simple by Lemma 5.1.4. Combining the above equality with Lemma 5.7.1(ii) allows one to conclude. (ii) *Sufficiency.*

- Assume that (5.7.5) holds. Then for any $x \in \mathbb{G}$,

$$\begin{aligned}\mathbf{P}\left(\Phi\left(\{x\}\right) \ge 2\right) &= \mathbf{P}((x, x) \in \Phi^{(2)}) \\ &= \det \begin{pmatrix} K(x,x) & K(x,x) \\ K(x,x) & K(x,x) \end{pmatrix} = 0.\end{aligned}$$

  It follows from Lemma 5.7.1(i) that Φ is simple. Then combining Equations (5.7.3) and (5.7.5) implies (5.7.7).

- If Φ is simple and (5.7.6) holds true, then Equation (5.7.4) implies (5.7.7).

□

5.7.2 Regularity

Since $\mathbb{G}$ is discrete, any function $K : \mathbb{G}^2 \to \mathbb{C}$ is measurable and may be considered as a matrix indexed by $\mathbb{G}$. Let $\ell^2_{\mathbb{C}}\left(\mathbb{G}^2\right)$ be the set of such matrices $K = \left(K_{ij}\right)_{i,j \in \mathbb{G}}$ such that

$$\sum_{i,j \in \mathbb{G}} \left|K_{ij}\right|^2 < \infty.$$

For any $K \in \ell^2_{\mathbb{C}}\left(\mathbb{G}^2\right)$, the integral operator $\mathcal{K}_{\mathbb{G}}$ defined by (16.A.1) is indeed the linear mapping on $\ell^2_{\mathbb{C}}(\mathbb{G}) = \left\{u \in \mathbb{C}^{\mathbb{G}} : \sum_{i\in\mathbb{G}} |u_i|^2 < \infty\right\}$ associated to the matrix K; i.e.

$$\mathcal{K}_{\mathbb{G}}(u) = K \times u, \quad u \in \ell^2_{\mathbb{C}}(\mathbb{G}),$$

where $\times$ is the matrix multiplication.

For any finite subset D of $\mathbb{G}$, the restriction K_D of K to $D \times D$ is the submatrix of K with indexes in D. The integral operator $\mathcal{K}_D$ defined by (16.A.24) is indeed the linear mapping on $\mathbb{C}^D$ associated to the matrix K_D; i.e.

$$\mathcal{K}_D(u) = K_D \times u, \quad u \in \mathbb{C}^D.$$

In particular, the eigenvalues (and eigenvectors) of $\mathcal{K}_D$ are the same as those of the matrix K_D.

The conditions in Definition 5.2.11 for regularity simplify in the discrete case as follows.

Proposition 5.7.3. *Let $\mathbb{G}$ be a discrete set and let $K : \mathbb{G}^2 \to \mathbb{C}$.*

(i) K is regular iff $K \in \ell^2_{\mathbb{C}}\left(\mathbb{G}^2\right)$ is Hermitian, nonnegative-definite, and with finite trace.

(ii) K is locally regular iff it is Hermitian and nonnegative-definite.

Proof. (i) Necessity is obvious. It remains to show sufficiency. Let $K \in \ell^2_{\mathbb{C}}\left(\mathbb{G}^2\right)$ be Hermitian, nonnegative-definite, and with finite trace. Since, K is Hermitian, then K is diagonalizable by Proposition 16.A.13(i). Since $\mathbb{G}$ is discrete and μ is the counting measure on $\mathbb{G}$, Equality (16.A.14) holds for all $(x, y) \in \mathbb{G}^2$. Then K equals its canonical version. Since K is nonnegative-definite, its eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$ are nonnegative. Since K has finite trace, $\sum_{n\in\mathbb{N}^*} \lambda_n < \infty$. Then Corollary 16.A.18(i) shows that $\mathcal{K}_{\mathbb{G}}$ is trace class. (ii) By Item (i), K is locally regular iff for any finite subset D of $\mathbb{G}$, the restriction K_D of K to $D \times D$ is Hermitian and nonnegative-definite. This is the case iff K is Hermitian and nonnegative-definite by Lemma 5.2.12(i)-(ii). □

Corollary 5.7.4. *Let $\mathbb{G}$ be a discrete set, $K : \mathbb{G}^2 \to \mathbb{C}$ and $\alpha \in \mathbb{R}^*_-$.*

(i) K is α-regular (resp. strictly α-regular) iff $K \in \ell^2_{\mathbb{C}}\left(\mathbb{G}^2\right)$ is Hermitian, nonnegative-definite, with finite trace, and eigenvalues smaller or equal than $-1/\alpha$ (resp. strictly smaller than $-1/\alpha$).

(ii) K is locally α-regular (resp. locally strictly α-regular) iff it is Hermitian, nonnegative-definite, and the eigenvalues of each finite submatrix of K are smaller or equal than $-1/\alpha$ (resp. strictly smaller than $-1/\alpha$).

Proof. This follows from Proposition 5.7.3 and the fact that the eigenvalues of $\mathcal{K}_{\mathbb{G}}$ are the same as those of the matrix K. □

5.7.3 Janossy measures

Proposition 5.7.5. *Let $\mathbb{G}$ be a finite set.*

(i) Let $K : \mathbb{G}^2 \to \mathbb{C}$ be Hermitian, nonnegative-definite, and with eigenvalues strictly smaller than 1, and Φ be a determinantal point process on $\mathbb{G}$ with kernel K and background measure $\nu_{\mathbb{G}}$. Then the Janossy measures of Φ are given by

$$J_k(x_1, \ldots, x_k) = \det(I + L)^{-1} \det(L(x_i, x_j))_{1 \leq i,j \leq k}, \tag{5.7.8}$$

for any $k \in \mathbb{N}^*$ and any $x_1, \ldots, x_k \in \mathbb{G}$, where

$$L := (I - K)^{-1} K. \tag{5.7.9}$$

Moreover,

$$\mathbf{P}(\Phi = u) = \det(I + L)^{-1} \det(L_u), \tag{5.7.10}$$

for any $u \subset \mathbb{G}$, where L_u is the submatrix of L indexed by $u \times u$.

(ii) Let $L : \mathbb{G}^2 \to \mathbb{C}$ be Hermitian and nonnegative-definite. Then there exists a simple point process Φ on $\mathbb{G}$ with distribution (5.7.10). Moreover, Φ is a determinantal point process on $\mathbb{G}$ with kernel

$$K := I - (I + L)^{-1} \tag{5.7.11}$$

and background measure $\nu_{\mathbb{G}}$.

Proof. (i) Since K is strictly (-1)-regular, Theorem 5.2.17 shows that there exists a determinantal point process Φ on $\mathbb{G}$ with kernel K and background measure the counting measure on $\mathbb{G}$. Note that $L = (I - K)^{-1} K$ is the (-1)-inverse of K; see Definition 5.4.2. By Lemma 5.4.9,

$$\det(I - K) = \det(I + L)^{-1}.$$

The Janossy measures of Φ are given by (5.4.14) which combined with the above display proves (5.7.8). On the other hand, by (4.3.6), for any $x = (x_1, \ldots, x_k) \in \mathbb{G}^k$,

$$J_k(x) = k! p_k \Pi_k(x),$$

where p_k and Π_k are defined respectively by (4.3.4) and (4.3.2). In particular, if $x_1, \ldots, x_k$ are pairwise distinct, then

$$\begin{aligned}
\Pi_k(x) &= \frac{1}{k!} \sum_{\sigma} \mathbf{P}\left(\left(X_{\sigma(1)}(\Phi), \ldots, X_{\sigma(k)}(\Phi)\right) = x \mid \Phi(\mathbb{G}) = k\right) \\
&= \frac{1}{k!} \mathbf{P}\left(\exists \sigma : \left(X_{\sigma(1)}(\Phi), \ldots, X_{\sigma(k)}(\Phi)\right) = x \mid \Phi(\mathbb{G}) = k\right) \\
&= \frac{1}{k!} \mathbf{P}\left(\Phi = \sum_{i=1}^{k} \delta_{x_i} \mid \Phi(\mathbb{G}) = k\right) \\
&= \frac{1}{k!} \mathbf{P}\left(\Phi = \{x_1, \ldots, x_k\} \mid \Phi(\mathbb{G}) = k\right),
\end{aligned}$$

where the second equality is due to the fact that Φ is simple by Lemma 5.1.4. Combining the above two equalities with (5.7.8) gives (5.7.10). (ii) The matrix K defined by (5.7.11) is strictly (-1)-regular by Corollary 5.7.4(i). Applying Item (i) to this matrix K and observing that $(I-K)^{-1}K = L$ gives the announced results. □

Note the difference between Equation (5.7.10) and Equation (5.7.6) which gives the probability that $x_1, \ldots, x_k$ are atoms of Φ (for some pairwise distinct $x_1, \ldots, x_k \in \mathbb{G}$).

Remark 5.7.6. Bibliographic notes. *Proposition 5.7.5(ii) is due to [17, Proposition 1.2] (see also [61, Theorem 2.2]). In these references, a point process Φ satisfying (5.7.10) is called* L-ensemble.

Observe from (5.7.10) that $\sum_{u\subset\mathbb{G}} \det(L_u) = \det(I+L)$. Indeed this holds for arbitrary matrices (not necessarily Hermitian nonnegative-definite); see Equation (15.A.11).

5.7.4 Palm version

Proposition 5.7.7. Conditioning versus Palm. *Let $\mathbb{G}$ be a finite set, $K : \mathbb{G}^2 \to \mathbb{C}$ be Hermitian, nonnegative-definite, with eigenvalues strictly smaller than 1, Φ be a determinantal point process on $\mathbb{G}$ with kernel K and background measure $\nu_{\mathbb{G}}$, $u \subset \mathbb{G}$, and K_u be the submatrix of K indexed by $u \times u$. Assume that K_u is invertible and let Φ^u be the point process composed of the atoms of Φ falling in u^{c} given that $u \subset \Phi$. Then the following results hold.*

(i) $\Phi^u \overset{\text{dist.}}{=} \Phi^!_u$ *the reduced Palm version of Φ at u. In particular, Φ^u is a determinantal point process on u^{c} with background measure $\nu_{u^{\mathrm{c}}} = \sum_{x\in u^{\mathrm{c}}} \delta_x$ and kernel K^u the Schur complement (15.A.6) of K_u in K; that is*

$$K^u = K_{u^{\mathrm{c}}} - K_{u^{\mathrm{c}}u} (K_u)^{-1} K_{uu^{\mathrm{c}}}, \tag{5.7.12}$$

where $K_{u^{\mathrm{c}}u}$ and $K_{uu^{\mathrm{c}}}$ are the submatrices of K indexed by $u^{\mathrm{c}} \times u$ and $u \times u^{\mathrm{c}}$ respectively.

(ii) The determinantal point process Φ^u admits also the following kernel (with background measure $\nu_{u^{\mathrm{c}}}$)

$$\tilde{K}^u = I - \left((I^u + L)^{-1}\right)_{u^{\mathrm{c}}}, \tag{5.7.13}$$

where $L = (I-K)^{-1}K$ and $I^u : \mathbb{G}^2 \to \mathbb{C}$ is defined by $I^u(x,y) = \mathbf{1}\{x = y \in u^{\mathrm{c}}\}$.

(iii) For any $v \subset u^{\mathrm{c}}$,

$$\mathbf{P}(\Phi^u = v) = \frac{\det(L_{u\cup v})}{\det(I^u + L)}. \tag{5.7.14}$$

Proof. (i) For any $v \subset u^{\mathrm{c}}$,

$$\begin{aligned}\mathbf{P}(v \subset \Phi^u) &= \mathbf{P}(u \cup v \subset \Phi \mid u \subset \Phi) \\ &= \frac{\mathbf{P}(u \cup v \subset \Phi)}{\mathbf{P}(u \subset \Phi)} \\ &= \frac{\det(K_{u \cup v})}{\det(K_u)} \\ &= \det(K_v^u),\end{aligned}$$

where the third equality is due to (5.7.6) and the fourth equality follows from the Schur complement formula (15.A.5) with the matrix K_v^u being the Schur complement (15.A.6) of K_u in $K_{u \cup v}$; that is

$$K_v^u = K_v - K_{vu}(K_u)^{-1} K_{uv}.$$

Theorem 5.5.2 permits to conclude. (ii) Let $L := (I - K)^{-1} K$. For any $v \subset u^{\mathrm{c}}$,

$$\begin{aligned}\mathbf{P}(\Phi^u = v) &= \mathbf{P}(\Phi = u \cup v \mid u \subset \Phi) \\ &= \frac{\mathbf{P}(\Phi = u \cup v)}{\mathbf{P}(u \subset \Phi)} \\ &= \frac{\det(L_{u \cup v})}{\det(I + L)\det(K_u)},\end{aligned} \tag{5.7.15}$$

where the third equality is due to (5.7.10) and (5.7.6). By the Schur complement formula (15.A.5) for $\det(L_{u \cup v})$, we get

$$\mathbf{P}(\Phi^u = v) = \frac{\det(L_u)}{\det(I + L)\det(K_u)} \det(L_v^u),$$

where L_v^u is the Schur complement (15.A.6) of L_u in $L_{u \cup v}$; that is

$$L_v^u = L_v - L_{vu}(L_u)^{-1} L_{uv}.$$

Note that $L_v^u = \left(\tilde{L}^u\right)_v$ where

$$\tilde{L}^u := L_{u^{\mathrm{c}}} - L_{u^{\mathrm{c}}u}(L_u)^{-1} L_{uu^{\mathrm{c}}}. \tag{5.7.16}$$

Then by Proposition 5.7.5(ii), Φ^u has kernel

$$\tilde{K}^u := I - \left(I + \tilde{L}^u\right)^{-1}.$$

Observe that

$$\begin{aligned}I + \tilde{K}^u &= \left(I + \tilde{L}^u\right)^{-1} \\ &= \left(I + L_{u^{\mathrm{c}}} - L_{u^{\mathrm{c}}u}(L_u)^{-1} L_{uu^{\mathrm{c}}}\right)^{-1} \\ &= \left((I^u + L)^{-1}\right)_{u^{\mathrm{c}}},\end{aligned}$$

where the second equality is due to (5.7.16) and the third equality follows from the formula of the inverse of a partitioned matrix [48, §0.7.3 p.18] applied to

$$I^u + L = \begin{pmatrix} L_u & L_{uu^c} \\ L_{u^c u} & I + L_{u^c} \end{pmatrix}.$$

(iii) Invoking (15.A.10), we get

$$\sum_{v \subset u^c} \det(L_{u \cup v}) = \det(I^u + L).$$

Then adding (5.7.15) over all $v \subset u^c$, we get

$$\det(I + L)\det(K_u) = \det(I^u + L).$$

Combining the above display and (5.7.15) implies the announced result. □

Observe that the kernels K^u and $\tilde{K}^u$ given respectively by (5.7.12) and (5.7.13) are not necessarily equal; see Remark 5.1.2.

Remark 5.7.8. Bibliographic notes. *The statement in Proposition 5.7.7 that Φ^u is a determinantal point process with kernel given by (5.7.13) is due to Borodin and Rains [17, Proposition 1.2]; but they use different arguments.*

Here is their proof of Proposition 5.7.7(ii). For any $v \subset u^c$,

$$\begin{aligned} \mathbf{P}(\Phi^u \subset v) &= \sum_{\beta \subset v} \mathbf{P}(\Phi^u = \beta) \\ &= \det(I^u + L)^{-1} \sum_{\beta \subset v} \det(L_{u \cup \beta}) \\ &= \frac{\det(I^u + L)_{u \cup v}}{\det(I^u + L)}, \end{aligned} \tag{5.7.17}$$

where the second equality is due to (5.7.14) and the third equality is due to (15.A.10) applied to the matrix $A = L_{u \cup v}$ which gives

$$\sum_{\beta \subset v} \det(L_{u \cup \beta}) = \det\left(L_{u \cup v} + (I^u)_{u \cup v}\right) = \det(I^u + L)_{u \cup v}.$$

On the other hand, for any $v \subset u^{\mathrm{c}}$,

$$\begin{aligned}
\det\left(\tilde{K}^{u}\right)_{v} &= \det\left(I - \left((I^{u}+L)^{-1}\right)_{u^{\mathrm{c}}}\right)_{v} \\
&= \sum_{\gamma\subset v}(-1)^{|\gamma|}\det\left((I^{u}+L)^{-1}\right)_{\gamma} \\
&= \sum_{\gamma\subset v}(-1)^{|\gamma|}\frac{\det(I^{u}+L)_{\gamma^{\mathrm{c}}}}{\det(I^{u}+L)} \\
&= \sum_{\gamma\subset v}(-1)^{|\gamma|}\frac{\det(I^{u}+L)_{u\cup(u^{\mathrm{c}}\cap\gamma^{\mathrm{c}})}}{\det(I^{u}+L)} \\
&= \sum_{\gamma\subset v}(-1)^{|\gamma|}\mathbf{P}\left(\Phi^{u}\subset u^{\mathrm{c}}\cap\gamma^{\mathrm{c}}\right) \\
&= \sum_{\gamma\subset v}(-1)^{|\gamma|}\mathbf{P}\left(\Phi^{u}(\gamma)=0\right) \\
&= 1-\mathbf{P}\left(\bigcup_{x\in v}\{\Phi^{u}(x)=0\}\right) = \mathbf{P}\left(v\subset\Phi^{u}\right),
\end{aligned}$$

where the second equality is due to (15.A.11) applies to the matrix $A = -\left((I^{u}+L)^{-1}\right)_{u^{\mathrm{c}}}$, *the third equality follows from (15.A.8), the fifth equality is due to (5.7.17), and the last but one equality is due to the inclusion-exclusion formula for the probability of the union of events. Invoking Proposition 5.7.2 (and in particular formula (5.7.6)) concludes the proof.*

5.8 Exercises

5.8.1 For Section 5.1

Exercise 5.8.1. *Let K be as in Definition 5.2.1. Show that the matrix* $(K(x_i,x_j))_{1\le i,j\le k}$ *is Hermitian nonnegative-definite for all* $x_1,\ldots,x_k\in\mathbb{G}$.

Solution 5.8.1. *We use the same notation as in Definition 5.2.1. Letting*

$$\tilde{\varphi}_n(x) = \varphi_n(x)\mathbf{1}\{x\in\mathbb{G}_1\}, \quad x\in\mathbb{G},$$

we get

$$K(x,y) = \sum_{n\in\mathbb{N}^*}\lambda_n\tilde{\varphi}_n(x)\tilde{\varphi}_n(y)^*, \quad x,y\in\mathbb{G}.$$

Then for any $x_1,\ldots,x_k\in\mathbb{G}$, *the matrix* $(K(x_i,x_j))_{1\le i,j\le k}$ *is Hermitian nonnegative-definite; as a sum of Hermitian nonnegative-definite matrices.*

5.8.2 For Section 5.2

Exercise 5.8.2. *Let Φ be a determinantal point process. Show that* $\mathbf{P}(\Phi(D)\ge 2)\le \mathbf{E}[\Phi(D)]^2$ *for all* $D\in\mathcal{B}(\mathbb{G})$.

Solution 5.8.2. *Since Φ is a determinantal point process, then by Lemma 5.1.4, Φ is simple. Then for any $D \in \mathcal{B}(\mathbb{G})$,*

$$\mathbf{P}(\Phi(D) \geq 2) \leq \mathbf{E}\left[\Phi^{(2)}\left(D^2\right)\right] \leq \mathbf{E}[\Phi(D)]^2,$$

where the first inequality is due to Exercise 2.7.13 and the last inequality is due to (5.1.12).

5.8.3 For Section 5.3

Exercise 5.8.3. *Let $\alpha \in \mathbb{R}$ and $K : \mathbb{G}^2 \to \mathbb{C}$. Show that the function $(x_1, \ldots, x_k) \mapsto \det_\alpha (K(x_i, x_j))_{1 \leq i,j \leq k}$ defined on $\mathbb{G}^k$ is symmetric; i.e., invariant with respect to any permutation of the components. (This verification is motivated by the context of Definition 5.3.1 since the k-th factorial moment measure of any point process is a symmetric measure.)*

Solution 5.8.3. *For any permutation σ of $\{1, \ldots, k\}$,*

$$\begin{aligned}
\det_\alpha \left(K\left(x_{\sigma(i)}, x_{\sigma(j)}\right)\right)_{1 \leq i,j \leq k} &= \sum_{\pi \in S_k} \alpha^{k-\mathrm{cyc}(\pi)} \prod_{i=1}^{k} K\left(x_{\sigma(i)}, x_{\pi(\sigma(i))}\right) \\
&= \sum_{\pi \in S_k} \alpha^{k-\mathrm{cyc}(\pi)} \prod_{l=1}^{k} K\left(x_l, x_{\pi(l)}\right) \\
&= \det_\alpha (K(x_i, x_j))_{1 \leq i,j \leq k},
\end{aligned}$$

where the second equality follows from the change of variable $l = \sigma(i)$.

5.8.4 For Section 5.6

Exercise 5.8.4. *Let $C : \mathbb{R}^d \to \mathbb{C}$ be defined by*

$$C(x) = \frac{\lambda}{(1 + \|x/\zeta\|^2)^{\eta + d/2}}, \qquad x \in \mathbb{R}^d,$$

where λ, ζ and η are given positive real numbers. Prove that $C \in L^1_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right) \cap L^2_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right)$ is square integrable. If $\eta < d/2$, show that the Fourier transform $\hat{C}$ is given by, for any $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\hat{C}(\xi) &= \int_{\mathbb{R}^d} C(x) e^{-2i\pi\langle x, \xi\rangle} \mathrm{d}x \\
&= \lambda \frac{2^{1-\eta}(\sqrt{\pi}\zeta)^d}{\Gamma(\eta + d/2)} \|2\pi\zeta\xi\|^\eta K_\eta(\|2\pi\zeta\xi\|),
\end{aligned} \tag{5.8.18}$$

where K_η is the modified Bessel function of the second kind*; see Example 5.6.12.*

Solution 5.8.4. *We may assume without loss of generality that* $\lambda = 1$*. Making the change of coordinates from Cartesian to spherical (2.7.1) it is easy to see that* $C \in L^1_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right) \cap L^2_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right)$*. Recall that for any* $f \in L^1_{\mathbb{C}}\left(\ell^d, \mathbb{R}^d\right)$*, the Fourier transform of* $x \mapsto f\left(x/\zeta\right)$ *is* $\xi \mapsto |\zeta|^d \hat{f}\left(\zeta\xi\right)$*. Then we may restrict ourselves to the case* $\zeta = 1$*. Observe that since* C *is even, then* $\hat{C}$ *is real, thus*

$$\hat{C}(\xi) = \int_{\mathbb{R}^d} C(x) \cos\left(2\pi \langle x, \xi\rangle\right) \mathrm{d}x$$

Consider first the case $d = 1$*. Then, for any* $\xi \in \mathbb{R}$*,*

$$\begin{aligned}\hat{C}(\xi) &= \int_{\mathbb{R}} \frac{\cos\left(2\pi x\xi\right)}{\left(1 + |x|^2\right)^{\eta+1/2}} \mathrm{d}x \\ &= 2\int_0^\infty \frac{\cos\left(2\pi x\,|\xi|\right)}{\left(1 + |x|^2\right)^{\eta+1/2}} \mathrm{d}x = \frac{2^{1-\eta}\sqrt{\pi}}{\Gamma\left(\eta + 1/2\right)} \left|2\pi\xi\right|^\eta K_\eta(|2\pi\xi|),\end{aligned}$$

where the last equality is due to [42, Equation (8.432.5) p.907]. This proves (5.8.18) for $d = 1$*.*

Consider now $d \geq 2$*. We follow the arguments in [98, p.363]. Let*

$$\varphi\left(\xi\right) = \|2\pi\xi\|^\eta K_\eta(\|2\pi\xi\|), \quad \xi \in \mathbb{R}^d.$$

Its inverse Fourier transform $f = \mathfrak{F}^{-1}\varphi$ *equals*

$$f\left(x\right) = \int_{\mathbb{R}^d} \varphi\left(\xi\right) e^{2i\pi\langle x,\xi\rangle} \mathrm{d}\xi, \quad x \in \mathbb{R}^d.$$

We make the change of coordinates from Cartesian to spherical (2.7.1) where $r \in \mathbb{R}_+, \phi_1, \ldots, \phi_{d-2} \in [0, \pi]\,, \phi_{d-1} \in [0, 2\pi)$ *whose Jacobian is given by (2.7.2). We may choose the first angular coordinate* ϕ_1 *to be the angle between* x *and* ξ*. Then*

$$\begin{aligned}f\left(x\right) = \int_{\mathbb{R}_+\times[0,\pi]^{d-2}\times[0,2\pi)} &\left(2\pi r\right)^\eta K_\eta(2\pi r) \cos\left(2\pi r\|x\| \cos\left(\phi_1\right)\right) r^{d-1} \\ &\times \sin^{d-2}\phi_1 \sin^{d-3}\phi_2 \ldots \sin\phi_{d-2} \mathrm{d}r\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_{d-1}.\end{aligned}$$

The integral with respect to ϕ_1 *may be solved using [42, Equation (8.411.5) p.902]:*

$$\begin{aligned}&\int_0^\pi \cos\left(2\pi r\|x\| \cos\left(\phi_1\right)\right) \sin^{d-2}\phi_1 \mathrm{d}\phi_1 \\ &\quad = \sqrt{\pi}\Gamma\left(\left(d-1\right)/2\right) \left(\pi r\|x\|\right)^{-(d-2)/2} J_{(d-2)/2}\left(2\pi r\|x\|\right),\end{aligned}$$

where J. is the Bessel function of the first kind; see [42, Equation (8.401) p.900].

Let $\sigma_d = \frac{2\pi^{d//2}}{\Gamma(d/2)}$ *be the surface area of the unit ball in* $\mathbb{R}^d$*, then*

$$\int_{[0,\pi]^{d-3}\times[0,2\pi)} \sin^{d-3}\phi_2 \ldots \sin\phi_{d-2} \mathrm{d}\phi_2 \ldots \mathrm{d}\phi_{d-1} = \sigma_{d-1}.$$

Therefore,

$$\begin{aligned}
f(x) &= 2\pi^{d//2}\int_0^\infty (2\pi r)^\eta K_\eta(2\pi r)\,(\pi r\|x\|)^{-(d-2)/2}\,J_{(d-2)/2}\,(2\pi r\|x\|)\,r^{d-1}\mathrm{d}r \\
&= \left(2\pi^{d//2}\,(2\pi)^{-d}\,(\|x\|/2)^{-(d-2)/2}\right)\times\int_0^\infty t^{\eta+d/2}J_{(d-2)/2}\,(t\|x\|)\,K_\eta(t)\mathrm{d}t \\
&= \left(2\pi^{d//2}\,(2\pi)^{-d}\,(\|x\|/2)^{-(d-2)/2}\right)\times 2^{\eta+(d-2)/2}\|x\|^{(d-2)/2}\frac{\Gamma(\eta+d/2)}{(\|x\|^2+1)^{\eta+d/2}} \\
&= \pi^{-d//2}2^{\eta-1}\frac{\Gamma(\eta+d/2)}{(\|x\|^2+1)^{\eta+d/2}},
\end{aligned}$$

where for the second equality we make the change of variable $r \to t = 2\pi r$ and the third equality is due to [42, Equation (6.576.7) p.676] when $d/2 > \eta$. Observing that

$$f(x) = \pi^{-d//2}2^{\eta-1}\Gamma(\eta+d/2)\,C(x),$$

concludes the proof.

Part II

Stationary random measures and point processes

Chapter 6

Palm theory in the stationary framework

In this part, we consider random measures on the d-dimensional Euclidean space $\mathbb{G} = \mathbb{R}^d$, with the associated Borel σ-algebra $\mathcal{B}\left(\mathbb{R}^d\right)$ for some given $d \in \mathbb{N}^*$. Recall that a stochastic process $\{Z(x)\}_{x\in\mathbb{R}^d}$ with values in some measurable space $(\mathbb{K}, \mathcal{K})$ is said to be *stationary* if

$$\{Z(x+t)\}_{x\in\mathbb{R}^d} \overset{\text{dist.}}{=} \{Z(x)\}_{x\in\mathbb{R}^d}, \quad t \in \mathbb{R}^d, \tag{6.0.1}$$

where $\overset{\text{dist.}}{=}$ means 'has the same probability distribution as'.

We will define the stationarity of random measures and introduce some *stationary framework* on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, which facilitates the analysis of jointly stationary random measures and fields. In particular, it allows one to define *Palm probabilities* of different random measures directly on $(\Omega, \mathcal{A})$, to study the relations between these probabilities, and also to consider the distributions of all random object defined on $(\Omega, \mathcal{A})$ under these Palm probabilities. As we shall see, this gives rise to several useful conservation laws including the *mass transport formula*.

6.1 Palm probabilities in the stationary framework

6.1.1 Stationary framework

Shift operator and stationarity

For all $t \in \mathbb{R}^d$, we introduce a *shift operator* S_t on the set of measures μ on $(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right))$ by

$$S_t\mu(B) := \mu(B+t), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right),$$

where $B+t=\{x+t\in\mathbb{R}^d : x\in B\}$. The shift S_t acts also on functions f defined on $\mathbb{R}^d$ as follows

$$S_t f(x) := f(x+t), \quad x\in\mathbb{R}^d.$$

Remark 6.1.1. *For any $t\in\mathbb{R}^d$,*

$$\begin{aligned}\int f(x)\, S_t\mu(\mathrm{d}x) &= \int f(x)\,\mu(\mathrm{d}x+t)\\ &= \int f(y-t)\mu(\mathrm{d}y) = \int f(x-t)\mu(\mathrm{d}x),\end{aligned}$$

where for the second equality we make the change of variable $y=x+t$. Equivalently,

$$\int f\ S_t(\mathrm{d}\mu) = \int S_{-t}f\ \mathrm{d}\mu. \tag{6.1.1}$$

Also

$$\int S_t f\ S_t(\mathrm{d}\mu) = \int f\ \mathrm{d}\mu. \tag{6.1.2}$$

For a counting measure $\mu=\sum_{k\in\mathbb{Z}}\delta_{x_k}$,

$$S_t\mu = \sum_k \delta_{x_k-t}.$$

Indeed, for all $B\in\mathcal{B}(\mathbb{R}^d)$, $\mu(B)=\sum_{k\in\mathbb{Z}} 1\{x_k\in B\}$ then

$$\begin{aligned}S_t\mu(B) &= \mu(B+t)\\ &= \sum_{k\in\mathbb{Z}} \mathbf{1}\{x_k\in B+t\}\\ &= \sum_{k\in\mathbb{Z}} \mathbf{1}\{x_k-t\in B\} = \sum_{k\in\mathbb{Z}} \delta_{x_k-t}(B).\end{aligned}$$

So the shift S_t acts on a counting measure by translating its points by $-t$.

Observe that the definition of stationarity of a stochastic process $Z=\{Z(x)\}_{x\in\mathbb{R}^d}$ in (6.0.1) may be written as follows

$$\mathbf{P}_{S_tZ} = \mathbf{P}_Z, \quad t\in\mathbb{R}^d,$$

where $S_tZ=\{S_tZ(x)\}_{x\in\mathbb{R}^d}=\{Z(x+t)\}_{x\in\mathbb{R}^d}$.

Definition 6.1.2. Stationary random measure. *A random measure Φ is said* stationary *if*

$$\mathbf{P}_{S_t\Phi} = \mathbf{P}_\Phi, \quad t\in\mathbb{R}^d. \tag{6.1.3}$$

A family of random measures Φ_i and stochastic processes Z_i, $i=1, 2, \ldots$, all defined on the same probability space is called jointly stationary *if the joint distribution of all these random objects is invariant with respect to their shift by any vector $t\in\mathbb{R}^d$; that is*

$$\mathbf{P}_{(S_t\Phi_1, S_t\Phi_2,\ldots,S_tZ_1,S_tZ_2,\ldots)} = \mathbf{P}_{(\Phi_1,\Phi_2,\ldots,Z_1,Z_2,\ldots)}, \quad t\in\mathbb{R}^d.$$

By Corollary 1.3.4, a random measure Φ is stationary if and only if, for all $t \in \mathbb{R}^d, B_1, \ldots, B_n \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\left(\Phi\left(B_1+t\right), \ldots, \Phi\left(B_n+t\right)\right) \stackrel{\text{dist.}}{=} \left(\Phi\left(B_1\right), \ldots, \Phi\left(B_n\right)\right).$$

Example 6.1.3. Shifted comb. *Let U be a random variable uniformly distributed in $[0,1)$. Then the point process on $\mathbb{R}$ defined by*

$$\Phi = \sum_{k \in \mathbb{Z}} \delta_{k+U}$$

is stationary. It is called the shifted comb. *Indeed, for all $t \in \mathbb{R}$, letting $a = \lceil t \rceil - t \in [0,1]$, we get*

$$\begin{aligned} S_t \Phi &= \sum_{k \in \mathbb{Z}} \delta_{k+U-t} \\ &= \sum_{k \in \mathbb{Z}} \delta_{k+U+\lceil t \rceil - t} \\ &= \sum_{k \in \mathbb{Z}} \delta_{k+U+a} \\ &= \sum_{k \in \mathbb{Z}} \delta_{k+U+a-\mathbf{1}\{U+a>1\}} = \sum_{k \in \mathbb{Z}} \delta_{k+\tilde{U}}, \end{aligned}$$

where $\tilde{U} = U + a - \mathbf{1}\left\{U+a>1\right\}$ is uniformly distributed in $[0,1]$.

Example 6.1.4. Shifted grid. *Let (U,V) be independent random variables uniformly distributed in $[0,1)$. Then the point process on $\mathbb{R}^2$ defined by*

$$\Phi = \sum_{(k,n) \in \mathbb{Z}^2} \delta_{(k+U, n+V)}$$

is stationary (the arguments follow the same lines as in Example 6.1.3). It is called the shifted grid. *There is no difficulty extending this to $\mathbb{R}^d$ for all $d \geq 1$.*

Flow and compatibility

Definition 6.1.5. *A family $\left\{\theta_t\right\}_{t \in \mathbb{R}^d}$ is called a* flow *on the measurable space $(\Omega, \mathcal{A})$ if*

(i) $\forall t \in \mathbb{R}^d$, $\theta_t : \Omega \to \Omega$ is bijective;

(ii) $\forall t, u \in \mathbb{R}^d$, $\theta_{t+u} = \theta_t \circ \theta_u$ (therefore $\theta_t^{-1} = \theta_{-t}$ and θ_0 is the identity).

If moreover, $(t, \omega) \mapsto \theta_t(\omega)$ is measurable, then $\{\theta_t\}$ is called a measurable flow*.*

Remark 6.1.6. Random flow composition rule. *Let $\{\theta_t\}_{t\in\mathbb{R}^d}$ be a flow on the measurable space $(\Omega,\mathcal{A})$ and let U and V be two $\mathbb{R}^d$-valued random variables. Consider the mapping $\theta_U : \Omega \to \Omega, \omega \mapsto \theta_{U(\omega)}\omega$ and similarly for θ_V. One should be careful when considering $\theta_U \circ \theta_V$ with the composition rule in Definition 6.1.5(ii); indeed*

$$\begin{aligned}\theta_U \circ \theta_V(\omega) &= \theta_U\left[\theta_{V(\omega)}\omega\right] \\ &= \theta_{U\left[\theta_{V(\omega)}\omega\right]}\theta_{V(\omega)}\omega \\ &= \theta_{U\left[\theta_{V(\omega)}\omega\right]+V(\omega)}\omega \\ &= \theta_{U\circ\theta_V+V}(\omega).\end{aligned} \tag{6.1.4}$$

Lemma 6.1.7. *The family of shifts $\{S_t\}_{t\in\mathbb{R}^d}$ is a measurable flow on the space of locally finite measures $(\bar{\mathbb{M}}(\mathbb{R}^d), \bar{\mathcal{M}}(\mathbb{R}^d))$.*

Proof. (Cf. [79, §II.1]) Note that for all $t,u \in \mathbb{R}^d, \mu \in \bar{\mathbb{M}}(\mathbb{R}^d), B \in \mathcal{B}(\mathbb{R}^d)$, $S_tS_u\mu(B) = S_u\mu(B+t) = \mu(B+t+u) = S_{t+u}\mu(B)$ and that S_0 is identity. Then $\{S_t\}_{t\in\mathbb{R}^d}$ is a flow on $(\bar{\mathbb{M}}(\mathbb{R}^d), \bar{\mathcal{M}}(\mathbb{R}^d))$. It remains to show that the mapping $\mathbb{R}^d \times \bar{\mathbb{M}}(\mathbb{R}^d) \to \bar{\mathbb{M}}(\mathbb{R}^d), (t,\mu) \mapsto S_t\mu$ is measurable. By [52], $\bar{\mathcal{M}}(\mathbb{R}^d)$ is generated by the mappings $\mu \mapsto \mu(f), f \in \mathfrak{F}_c(\mathbb{R}^d)$. Then, it is enough to show that, for all $f \in \mathfrak{F}_c(\mathbb{R}^d)$, the mapping $g : \mathbb{R}^d \times \bar{\mathbb{M}}(\mathbb{R}^d) \to \bar{\mathbb{R}}_+, (t,\mu) \mapsto (S_t\mu)(f)$ is measurable. This mapping f is Carathéodory; i.e., continuous in its first argument (indeed, let $t_n \to t$, then $(S_{t_n}\mu)(f) \to (S_t\mu)(f)$ by dominated convergence theorem) and measurable in its second argument (indeed, observe that $(S_t\mu)(f) = \mu(S_{-t}f)$ and invoke Lemma 1.1.5). Then f is measurable by [2, Lemma 4.51]. □

Definition 6.1.8. *A random measure Φ is* compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ *if*

$$\Phi \circ \theta_t = S_t\Phi, \quad t \in \mathbb{R}^d. \tag{6.1.5}$$

A stochastic process $\{Z(t)\}_{t\in\mathbb{R}^d}$ is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ *if*

$$Z(x) \circ \theta_t = S_tZ(x), \quad x,t \in \mathbb{R}^d,$$

that is

$$Z(x,\theta_t\omega) = Z(x+t,\omega), \quad x,t \in \mathbb{R}^d, \omega \in \Omega. \tag{6.1.6}$$

Lemma 6.1.9. *A stochastic process $\{Z(t)\}_{t\in\mathbb{R}^d}$ is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ iff*

$$Z(t) = Z(0) \circ \theta_t, \quad t \in \mathbb{R}^d,$$

that is,

$$Z(t,\omega) = Z(0,\theta_t\omega), \quad t \in \mathbb{R}^d, \omega \in \Omega. \tag{6.1.7}$$

In particular, $\{Z(t)\}_{t\in\mathbb{R}^d}$ is entirely characterized by the random variable $Z(0)$.

Proof. *Necessity.* It is enough to apply (6.1.6) with $x = 0$. *Sufficiency.* Note that

$$\begin{aligned} Z(x) \circ \theta_t &= Z(0) \circ \theta_x \circ \theta_t \\ &= Z(0) \circ \theta_{x+t} \\ &= Z(x+t) = S_t Z(x) . \end{aligned}$$

□

Example 6.1.10. *Let Φ be a point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. The stochastic process*

$$Z(x) := \inf\{|x - X| : X \in \Phi\}$$

is entirely characterized by the random variable $f = \inf\{|Y| : Y \in \Phi\}$, since

$$\begin{aligned} f(\theta_x \omega) &= \inf\{|Y| : Y \in \Phi \circ \theta_x\} \\ &= \inf\{|Y| : Y \in S_x \Phi\} \\ &= \inf\{|Y - x| : Y \in \Phi\} = Z(x) . \end{aligned}$$

Definition 6.1.11. *Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space with a measurable flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. The probability $\mathbf{P}$ is* invariant *with respect to $\{\theta_t\}_{t\in\mathbb{R}^d}$ if*

$$\mathbf{P} \circ \theta_t^{-1} = \mathbf{P}, \quad t \in \mathbb{R}^d,$$

i.e.,

$$\mathbf{P}(\{\omega \in \Omega : \theta_t \omega \in A\}) = \mathbf{P}(\{\omega \in \Omega : \omega \in A\}), \quad A \in \mathcal{A},$$

(since $\theta_t^{-1}(A) = \{\omega \in \Omega : \omega \in \theta_t^{-1} A\} = \{\omega \in \Omega : \theta_t \omega \in A\}$).

In this case,

- *we say that $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ is a* stationary framework*;*
- *we call $\mathbf{P}$ the* stationary probability *on $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d})$ (to distinguish it from* Palm probabilities *to be defined later on the same space).*

Lemma 6.1.12. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework.*

(i) Let Φ be a random measure defined on $(\Omega, \mathcal{A}, \mathbf{P})$ and compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Then Φ is stationary.

(ii) Let Φ_i and Z_i, $i = 1, 2, \ldots$ be a family of random measures and stochastic processes, respectively, defined on $(\Omega, \mathcal{A}, \mathbf{P})$ compatible with the flow. Then Φ_i and Z_i are jointly stationary.

Proof. (i) Recall that Φ is stationary if and only if $\mathbf{P}_{S_t\Phi} = \mathbf{P}_\Phi$ for all $t \in \mathbb{R}^d$. Now, if Φ is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$, then

$$\mathbf{P}_{S_t\Phi} = \mathbf{P}_{\Phi\circ\theta_t} = \mathbf{P} \circ (\Phi \circ \theta_t)^{-1} = \mathbf{P} \circ \theta_t^{-1} \circ \Phi^{-1} = \mathbf{P} \circ \Phi^{-1} = \mathbf{P}_\Phi .$$

(ii) The proof follows in the same lines as (i). □

Definition 6.1.13. *Let Φ be a stationary random measure. Then the stationary framework $\left(\bar{\mathbb{M}}\left(\mathbb{R}^d\right), \bar{\mathcal{M}}\left(\mathbb{R}^d\right), \{S_t\}_{t\in\mathbb{R}^d}, \mathbf{P}_\Phi\right)$ is called a* canonical stationary framework *for Φ. Indeed, $\{S_t\}_{t\in\mathbb{R}^d}$ is a measurable flow on $(\bar{\mathbb{M}}\left(\mathbb{R}^d\right), \bar{\mathcal{M}}\left(\mathbb{R}^d\right))$ by Lemma 6.1.7 and $\mathbf{P}_\Phi$ is invariant with respect to it by (6.1.3).*

Remark 6.1.14. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Then*

$$S_t\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k - t} = \Phi\circ\theta_t = \sum_{l\in\mathbb{Z}} \delta_{X_l\circ\theta_t}.$$

Then for each $k \in \mathbb{Z}$, there exists some $l \in \mathbb{Z}$ such that $X_l \circ \theta_t = X_k - t$. Nevertheless, in general $X_k \circ \theta_t \neq X_k - t$.

For example, let $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a point process on $\mathbb{R}$ with the usual enumeration convention (1.6.8). The point

$$X_0 \circ \theta_t = \max_{n\in\mathbb{Z}} \{X_n - t : X_n \leq t, \} = \max_{n\in\mathbb{Z}} \{X_n \in (-\infty, t]\} - t$$

is in general different from $X_0 - t$.

Example 6.1.15. *The distribution of a homogeneous Poisson process on $\mathbb{R}^d$ (cf. Definition 2.1.2) is invariant with respect to any shift S_t, $t \in \mathbb{R}^d$. The canonical probability space can serve as a stationary framework for it.*

6.1.2 Palm probability of a random measure

Definition 6.1.16. *The* intensity *of a stationary random measure Φ on $\mathbb{R}^d$ is defined by*

$$\lambda := M_\Phi\left([0,1)^d\right),$$

where M_Φ is the mean measure of Φ.

Lemma 6.1.17. *Let Φ be a stationary random measure on $\mathbb{R}^d$ with intensity λ. Then the following results hold true.*

(i) The mean measure M_Φ is invariant under translations; that is

$$M_\Phi(B+t) = M_\Phi(B), \quad B \in \mathcal{B}\left(\mathbb{R}^d\right), t \in \mathbb{R}^d.$$

(ii) Φ is almost surely the null measure iff $\lambda = 0$.

(iii) If $\lambda < \infty$, then M_Φ is proportional to the Lebesgue measure; more precisely,

$$M_\Phi(B) = \lambda|B|, \quad B \in \mathcal{B}\left(\mathbb{R}^d\right). \tag{6.1.8}$$

(iv) M_Φ is locally finite iff $\lambda < \infty$.

Proof. (i) For any $B \in \mathcal{B}(\mathbb{R}^d)$, $t \in \mathbb{R}^d$,

$$\begin{aligned} M_\Phi(B+t) &= \mathbf{E}[\Phi(B+t)] \\ &= \mathbf{E}[S_t\Phi(B)] \\ &= \mathbf{E}[\Phi(B)] = M_\Phi(B), \end{aligned}$$

where the third equality is due to (6.1.3). (ii) Necessity is obvious. Sufficiency follows from (i) and the fact that every relatively compact in a metric space may be covered by a finite number of balls of radius 1. (iii) If $\lambda = 0$ then the result is obvious by (ii). Assume now that $0 < \lambda < \infty$. Then, it follows from (i) and the fact that every relatively compact in a metric space may be covered by a finite number of balls of radius 1, that M_Φ is locally finite. Thus, by the Haar theorem [80, Theorem 2], M_Φ is proportional to the Lebesgue measure. (iv) Necessity is obvious and sufficiency follows from (iii). □

Proposition 6.1.18. *A Poisson point process on $\mathbb{R}^d$ is stationary if and only if its intensity measure is proportional to the Lebesgue measure.*

Proof. The direct part follows from Lemma 6.1.17(iii) and the fact that the intensity measure of a Poisson point process is locally finite and therefore the intensity λ is necessarily finite. For the converse, observe that $S_t\Phi$ is a Poisson point process with intensity measure S_tM_Φ for all $t \in \mathbb{R}^d$. Since M_Φ is proportional to the Lebesgue measure, then $S_tM_\Phi = M_\Phi$. Since the probability distribution of a Poisson point process is characterized by its intensity measure, it follows that $\mathbf{P}_{S_t\Phi} = \mathbf{P}_\Phi$ for all $t \in \mathbb{R}^d$. □

Lemma 6.1.19. *A stationary point process Φ on $\mathbb{R}^d$ with finite intensity does not have fixed atoms; that is for any $x \in \mathbb{R}^d$, $\mathbf{P}(\Phi(\{x\}) > 0) = 0$.*

Proof. Since $\mathbf{1}\{\Phi(\{x\}) > 0\} \leq \Phi(\{x\})$, then

$$\begin{aligned} \mathbf{P}(\Phi(\{x\}) > 0) &= \mathbf{E}[\mathbf{1}\{\Phi(\{x\}) > 0\}] \\ &\leq \mathbf{E}[\Phi(\{x\})] \\ &= M_\Phi(\{x\}) \\ &= \lambda|\{x\}| = 0, \end{aligned}$$

where the last but one equality is due to (6.1.8). □

Proposition 6.1.20. Existence of the Campbell-Matthes measure and Palm probability. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and Φ be a random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and having intensity $\lambda \in \mathbb{R}_+^*$. Then the following results hold true.*

(i) There exists a unique σ-finite measure $\mathcal{C}$ on $\mathbb{R}^d \times \Omega$ characterized by

$$\mathcal{C}(B \times A) := \mathbf{E}\left[\int_B \mathbf{1}\{\theta_x(\omega) \in A\}\Phi(\mathrm{d}x)\right], \quad B \in \mathcal{B}(\mathbb{R}^d), A \in \mathcal{A}. \quad (6.1.9)$$

(ii) The measure $\mathcal{C}$ defined by (6.1.9) satisfies

$$\mathcal{C}((B+t)\times A)=\mathcal{C}(B\times A)=C(A)|B|,\quad t\in\mathbb{R}^d, B\in\mathcal{B}\left(\mathbb{R}^d\right), A\in\mathcal{A},$$

for some set function C defined on $\mathcal{A}$.

(iii) Let $B\in\mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$ with $|B|>0$. The set function $\mathbf{P}^0$ defined on $\mathcal{A}$ by

$$\mathbf{P}^0(A):=\frac{\mathcal{C}(B\times A)}{\lambda|B|}=\frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}_A\circ\theta_x\Phi\left(\mathrm{d}x\right)\right],\quad A\in\mathcal{A},\qquad(6.1.10)$$

is a probability on $(\Omega,\mathcal{A})$ which does not depend on the particular choice of $B\in\mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$, provided $|B|>0$.

Proof. (i) The proof follows the same lines as that of Lemma 3.1.1. Indeed, let $\mathcal{R}$ be the class of all finite disjoint unions of sets of the form $B\times A$ where $B\in\mathcal{B}\left(\mathbb{R}^d\right), A\in\mathcal{A}$. Note first that $\mathcal{R}$ is an algebra of sets. Moreover $\mathcal{C}_\Phi$ is σ-finite on $\mathcal{R}$. (It is enough to consider a sequence $\{B_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$ covering $\mathbb{R}^d$, and note that $\mathcal{C}(B_n\times\Omega)=M_\Phi(B)<\infty$.) Then it follows from the Carathéodory's extension theorem [44, §13 Theorem A] that $\mathcal{C}$ admits a unique extension on $\sigma\left(\mathcal{R}\right)$ which is precisely the product σ-algebra on $\mathbb{R}^d\times\Omega$. (ii)

$$\begin{aligned}
\mathcal{C}((B+t)\times A)&=\mathbf{E}\left[\int_{\mathbb{R}^d}\mathbf{1}\{x\in B+t,\theta_x(\omega)\in A\}\Phi\left(\mathrm{d}x\right)\right]\\
&=\mathbf{E}\left[\int_{\mathbb{R}^d}\mathbf{1}\{x-t\in B,\theta_x(\omega)\in A\}\Phi\left(\mathrm{d}x\right)\right]\\
&=\mathbf{E}\left[\int_{\mathbb{R}^d}\mathbf{1}\{y\in B,\theta_{y+t}(\omega)\in A\}\Phi(t+\mathrm{d}y)\right]\quad(y:=x-t)\\
&=\mathbf{E}\left[\int_{\mathbb{R}^d}\mathbf{1}\{y\in B,\theta_{y+t}(\omega)\in A\}S_t\Phi\left(\mathrm{d}y\right)\right]\\
&=\mathbf{E}\left[\int_{\mathbb{R}^d}\mathbf{1}\{y\in B,\theta_y\circ\theta_t(\omega)\in A\}\Phi\circ\theta_t(\omega)\left(\mathrm{d}y\right)\right]\\
&=\mathcal{C}(B\times A).
\end{aligned}$$

Thus, for fixed A, the measure $\mathcal{C}(\cdot\times A)$ is invariant under left translation. Moreover, for all $B\in\mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$,

$$\mathcal{C}(B\times A)\leq M_\Phi(B)<\infty.$$

Then, by the Haar theorem [80, Theorem 2], $\mathcal{C}(\cdot\times A)$ is equal to the Lebesgue measure up to a multiplicative constant; i.e.,

$$\mathcal{C}(B\times A)=C(A)|B|,\quad B\in\mathcal{B}\left(\mathbb{R}^d\right), A\in\mathcal{A}.$$

(iii) Since $\mathcal{C}(B\times\emptyset)=0$, then $\mathbf{P}^0(\emptyset)=0$. Moreover, since $\mathcal{C}(B\times\Omega)=M_\Phi(B)=\lambda|B|$, then $\mathbf{P}^0(\Omega)=1$. Finally, σ-additivity of $\mathbf{P}^0$ follows from that of $\mathcal{C}(B\times\cdot)$. □

Definition 6.1.21. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and Φ be a random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and having intensity $\lambda \in \mathbb{R}_+^*$. The measure $\mathcal{C}$ on $\mathbb{R}^d \times \Omega$ characterized by (6.1.9) is the* Campbell-Matthes measure *associated to Φ. The probability $\mathbf{P}^0$ defined by (6.1.9) is the* Palm probability *associated to Φ.*

Remark 6.1.22. Palm terminology and notation. *Observe that there is no risk of confusion with our previous terminology in the general (non-stationary) case. Indeed, we introduced Palm distributions $\mathbf{P}_\Phi^x$ in Section 3.1 and* local *Palm probabilities $\mathbf{P}^x$ in Section 3.1.3. Further, we will see in Theorem 6.1.31(ii) below that the notation in the general and stationary case are consistent.*

Remark 6.1.23. *Note that Equation (6.1.10) may be written as*

$$\mathbf{P}^0(A) = \frac{1}{\lambda|B|}\mathbf{P}\left(\Phi \neq 0\right)\mathbf{E}\left[\int_B \mathbf{1}_A \circ \theta_x \Phi\left(\mathrm{d}x\right)\middle| \Phi \neq 0\right], \quad A \in \mathcal{A}.$$

It follows that the restriction of the probability $\mathbf{P}$ on the event $\{\Phi = 0\}$ does not impact the Palm probability.

Example 6.1.24. Deterministic measure. *Let $\Phi \equiv \ell^d$ the Lebesgue measure. Then, its intensity is $\lambda = 1$ and its Palm probability is*

$$\mathbf{P}^0 = \mathbf{P}.$$

Indeed,

$$\lambda := \ell^d\left([0,1)^d\right) = 1.$$

Moreover, for all $A \in \mathcal{A}$,

$$\begin{aligned}
\mathbf{P}^0(A) &= \mathbf{E}\left[\int_{(0,1]^d} \mathbf{1}_A \circ \theta_x \mathrm{d}x\right] \\
&= \int_{(0,1]^d} \mathbf{E}\left[\mathbf{1}_A \circ \theta_x\right] \mathrm{d}x \\
&= \int_{(0,1]^d} \mathbf{P}\left(A\right) \mathrm{d}x = \mathbf{P}\left(A\right).
\end{aligned}$$

The following propositions give some basic properties of the Palm probability of a stationary random measure.

Proposition 6.1.25. *Let Φ be a stationary random measure on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. The event*

$$A = \{\Phi = 0\} \cup \left\{\Phi\left(\mathbb{R}^d\right) = \infty\right\}$$

is such that $\mathbf{P}\left(A\right) = 1$.

Proof. Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be the underlying stationary framework Let

$$\Omega' = \left\{\omega \in \Omega : \Phi\left(\omega, \mathbb{R}^d\right) = \infty\right\}.$$

Note that this event is θ_t-invariant, for all $t \in \mathbb{R}^d$. Moreover, the random measure Φ' defined by

$$\Phi'(\omega, B) := \frac{\Phi(\omega, B)}{\Phi(\omega, \mathbb{R}^d)} \mathbf{1}\{\omega \in \Omega'\}, \quad B \in \mathcal{B}\left(\mathbb{R}^d\right)$$

is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. The mean measure $M_{\Phi'}$ of Φ' is finite since

$$M_{\Phi'}\left(\mathbb{R}^d\right) = \mathbf{P}\left(\Omega'\right).$$

In particular, the intensity λ' of Φ' is finite. Then, by Lemma 6.1.17(iii), $M_{\Phi'}(B) = \lambda'|B|$, for all $B \in \mathcal{B}\left(\mathbb{R}^d\right)$ which together with the above display shows that $\lambda' = 0$ and therefore $\mathbf{P}\left(\Omega'\right) = 0$. This proof is due to J. Murphy [76]. See also [31, Proposition 12.1.VI] for another proof. □

Proposition 6.1.26. *Let Φ be a stationary random measure on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. Then the following results hold true:*

(i)

$$\mathbf{P}^0(\Phi = 0) = 0.$$

(ii) Let $A \in \mathcal{A}$ be invariant with respect to the flow (i.e., $\theta_t A = A$ for all $t \in \mathbb{R}^d$) such that $\mathbf{P}(A) = 1$. Then $\mathbf{P}^0(A) = 1$.

(iii)

$$\mathbf{P}\left(\{\Phi = 0\} \cup \left\{\Phi\left(\mathbb{R}^d\right) = \infty\right\}\right) = \mathbf{P}^0\left(\Phi\left(\mathbb{R}^d\right) = \infty\right) = 1.$$

(iv) In the particular case $d = 1$,

$$\mathbf{P}\left(\{\Phi = 0\} \cup \left\{\Phi(\mathbb{R}_+^*) = \Phi(\mathbb{R}_-^*) = \infty\right\}\right) = \mathbf{P}^0\left(\Phi(\mathbb{R}_+^*) = \Phi(\mathbb{R}_-^*) = \infty\right) = 1.$$

Proof. (i) Let $B \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ with $|B| > 0$. By (6.1.10),

$$\begin{aligned}\mathbf{P}^0(\Phi = 0) &= \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}_{\{\Phi=0\}} \circ \theta_x \Phi(\mathrm{d}x)\right] \\ &= \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}_{\{\Phi=0\}} \Phi(\mathrm{d}x)\right] = 0,\end{aligned}$$

where the second equality is due to the fact that $\Phi \circ \theta_x$ is the null measure iff Φ is so. (ii) Let $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$ with $|B| > 0$. By (6.1.10),

$$\begin{aligned}
\mathbf{P}^0(A) &= \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}_A \circ \theta_x \Phi\left(\mathrm{d}x\right)\right] \\
&= \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}_A \Phi\left(\mathrm{d}x\right)\right] \\
&= \frac{1}{\lambda|B|}\mathbf{E}\left[\mathbf{1}_A \Phi\left(B\right)\right] \\
&= \frac{1}{\lambda|B|}\mathbf{E}\left[\Phi\left(B\right)\right] = 1,
\end{aligned}$$

where the second equality is due to the invariance of A with respect to the flow and the fourth equality follows from $\mathbf{P}\left(A\right) = 1$. (iii) Let $A = \{\Phi = 0\} \cup \left\{\Phi\left(\mathbb{R}^d\right) = \infty\right\}$. This event is invariant with respect to the flow. Moreover, from Proposition 6.1.25, $\mathbf{P}\left(A\right) = 1$. Then by (ii), $\mathbf{P}^0\left(A\right) = 1$. Invoking (i) proves that $\mathbf{P}^0\left(\Phi\left(\mathbb{R}^d\right) = \infty\right) = 1$. (iv) Let $A = \{\Phi = 0\} \cup \left\{\Phi(\mathbb{R}_+^*) = \Phi(\mathbb{R}_-^*) = \infty\right\}$. This event is invariant with respect to the flow. Moreover, $\mathbf{P}\left(A\right) = 1$; which may be proved in the same way as [8, Property 1.1.2] for point processes. Then by (ii), $\mathbf{P}^0\left(A\right) = 1$. Invoking (i) proves that $\mathbf{P}^0\left(\Phi(\mathbb{R}_+^*) = \Phi(\mathbb{R}_-^*) = \infty\right) = 1$. □

Remark 6.1.27. *Let Φ be a stationary random measure on $\mathbb{R}^d$ such that $\mathbf{P}\left(\Phi \neq 0\right) = 1$. Consider the event $A = \left\{\omega \in \Omega : \Phi(\omega, \mathbb{R}^d) = \infty\right\}$. By Proposition 6.1.26(iii), $\mathbf{P}\left(A\right) = 1$. Observe moreover that A is invariant with respect to the flow. Then, with a possible restriction of Ω to A, one may assume that*

$$\Phi(\omega, \mathbb{R}^d) = \infty, \quad \forall \omega \in \Omega.$$

In the particular case $d = 1$, with a possible restriction of Ω to

$$B = \left\{\omega \in \Omega : \Phi(\omega, \mathbb{R}_+^*) = \Phi(\omega, \mathbb{R}_-^*) = \infty\right\},$$

one may assume that

$$\Phi(\omega, \mathbb{R}_+^*) = \Phi(\omega, \mathbb{R}_-^*) = \infty, \quad \forall \omega \in \Omega.$$

6.1.3 Campbell-Little-Mecke-Matthes theorem

Theorem 6.1.28. Campbell-Little-Mecke-Matthes (C-L-M-M). *Let Φ be a random measure which is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ of a stationary framework $\left(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P}\right)$. If the intensity λ of Φ is positive, then:*

(i) For all measurable functions $f : \mathbb{R}^d \times \Omega \to \bar{\mathbb{R}}_+$,

$$\mathbf{E}\left[\int_{\mathbb{R}^d} f(x, \theta_x\omega)\Phi\left(\mathrm{d}x\right)\right] = \lambda\int_{\mathbb{R}^d} \mathbf{E}^0[f(x,\omega)]\mathrm{d}x. \tag{6.1.11}$$

(ii) *For all measurable functions* $f : \mathbb{R}^d \times \Omega \to \bar{\mathbb{C}}$, *if either of the following conditions*

$$\mathbf{E}\left[\int_{\mathbb{R}^d} |f(x, \theta_x\omega)| \, \Phi(\mathrm{d}x)\right] < \infty, \quad \text{or} \int_{\mathbb{R}^d} \mathbf{E}^0[|f(x, \omega)|]\mathrm{d}x < \infty$$

holds, then the other one holds, and Equality (6.1.11) holds true.

Proof. (i) We first show the announced equality for $f(x, \omega) = \mathbf{1}\{x \in B, \omega \in A\}$. To do so, note that

$$\begin{aligned}\mathbf{E}\left[\int_{\mathbb{R}^d} f(x, \theta_x\omega)\Phi(\mathrm{d}x)\right] &= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\{x \in B, \theta_x\omega \in A\}\Phi(\mathrm{d}x)\right] \\ &= \mathcal{C}(B \times A) = \lambda|B|\mathbf{P}^0(A).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbf{E}^0[f(x, \omega)]\mathrm{d}x &= \int_{\mathbb{R}^d} \mathbf{E}^0[\mathbf{1}\{x \in B, \omega \in A\}]\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \mathbf{1}\{x \in B\}\mathbf{E}^0[\mathbf{1}\{\omega \in A\}]\mathrm{d}x = |B|\mathbf{P}^0(A).\end{aligned}$$

For measurable functions $f : \mathbb{R}^d \times \Omega \to \bar{\mathbb{R}}_+$, the proof follows the same lines as that of Theorem 3.1.9. (ii) Similar arguments to Corollary 3.1.10. $\square$

Corollary 6.1.29. *Let* Φ *be a stationary point process on* $\mathbb{R}^d$ *with intensity* $\lambda \in \mathbb{R}_+^*$. *Then* Φ *is simple under* $\mathbf{P}$ *iff it is simple under* $\mathbf{P}^0$.

Proof. Necessity follows from Proposition 6.1.26(ii). It remains to prove sufficiency. Observe that, for any $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$,

$$\begin{aligned}\mathbf{P}(\exists x \in B : \Phi(\{x\}) \geq 2) &= \mathbf{E}[\mathbf{1}\{\exists x \in B : \Phi(\{x\}) \geq 2\}] \\ &\leq \mathbf{E}\left[\int_B \mathbf{1}\{\Phi(\{x\}) \geq 2\}\Phi(\mathrm{d}x)\right] \\ &= \lambda \int_B \mathbf{E}^0[\mathbf{1}\{\Phi(\{0\}) \geq 2\}]\,\mathrm{d}x,\end{aligned}$$

where the third line is due to (6.1.11). If Φ is simple under $\mathbf{P}^0$, then $\mathbf{P}^0(\Phi(\{0\}) \geq 2) = 0$ and the above integral vanishes which concludes the proof. $\square$

The following corollary shows that, in the particular case of point processes, $\mathbf{P}^0$ may be interpreted as the probability distribution seen from a *typical* point (randomly chosen).

Corollary 6.1.30. *Let* Φ *be a stationary point process on* $\mathbb{R}^d$ *having intensity* $\lambda \in \mathbb{R}_+^*$ *and let* g *be a nonnegative random variable. Then for any* $B \in \mathcal{B}(\mathbb{R}^d)$

such that $0 < |B| < \infty$,

$$\mathbf{E}^0[g] = \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B g(\theta_x\omega)\,\Phi(\mathrm{d}x)\right] = \frac{1}{\lambda|B|}\mathbf{E}\left[\sum_{X\in\Phi\cap B} g(\theta_X\omega)\right]. \tag{6.1.12}$$

Proof. It is enough to apply the Campbell-Little-Mecke-Matthes theorem 6.1.28 (C-L-M-M) to $f(x,\omega) = \mathbf{1}\{x\in B\}g(\omega)$. Indeed,

$$\mathbf{E}\left[\int_{\mathbb{R}^d} f(x,\theta_x\omega)\Phi(\mathrm{d}x)\right] = \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\{x\in B\}g(\theta_x\omega)\,\Phi(\mathrm{d}x)\right] = \mathbf{E}\left[\int_B g(\theta_x\omega)\,\Phi(\mathrm{d}x)\right] = \mathbf{E}\left[\sum_{X\in\Phi\cap B} g(\theta_X\omega)\right]$$

and

$$\int_{\mathbb{R}^d} \mathbf{E}^0[f(x,\omega)]\mathrm{d}x = \int_{\mathbb{R}^d} \mathbf{E}^0[\mathbf{1}\{x\in B\}g(\omega)]\mathrm{d}x = \int_{\mathbb{R}^d} \mathbf{1}\{x\in B\}\mathbf{E}^0[g(\omega)]\mathrm{d}x = |B|\mathbf{E}^0[g].$$

□

The following theorem shows other properties of Palm probability of stationary point processes.

Theorem 6.1.31. *Let Φ be a stationary point process on $\mathbb{R}^d$ having intensity $\lambda\in\mathbb{R}_+^*$.*

(i) Let

$$\Omega_0 := \{\omega\in\Omega : 0\in\Phi(\omega)\}. \tag{6.1.13}$$

Then

$$\mathbf{P}^0(\Omega_0) = 1.$$

That is, under the Palm probability, the point process has almost surely an atom at 0*; this atom is called the* typical point *of the process.*

(ii) Let $\mathbf{P}_\Phi^0$ be the probability distribution of Φ under the Palm probability; that is $\mathbf{P}_\Phi^0 := \mathbf{P}^0\circ\Phi^{-1}$. For any family $\{\mathbf{P}_\Phi^x\}_{x\in\mathbb{R}^d}$ of Palm distributions of Φ (cf. Definition 3.1.2), we have

$$\mathbf{P}_\Phi^0 = \mathbf{P}_\Phi^x\circ S_x^{-1}, \quad \text{for Lebesgue-almost all } x\in\mathbb{R}^d.$$

In particular, the family defined by

$$\mathbf{P}_\Phi^x := \mathbf{P}_\Phi^0\circ S_x, \quad \text{for all } x\in\mathbb{R}^d \tag{6.1.14}$$

is a family of Palm distributions of Φ. Consequently, since S_0 is the identity, the notation is consistent for $x=0$.

(iii) Slivnyak's theorem*: If* Φ *is a homogeneous Poisson point process on* $\mathbb{R}^d$*, then*

$$\mathbf{P}_\Phi^0 = \mathbf{P}_{\Phi+\delta_0}.$$

Proof. (i) Using Corollary 6.1.30 we get

$$\begin{aligned}
\mathbf{P}^0\left(\Phi(\{0\}) \geq 1\right) &= \mathbf{E}^0\left[\mathbf{1}\left\{\Phi(\{0\}) \geq 1\right\}\right] \\
&= \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}\left\{\Phi\left(\theta_x\omega\right)(\{0\}) \geq 1\right\}\Phi\left(\mathrm{d}x\right)\right] \\
&= \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}\left\{S_x\Phi(\{0\}) \geq 1\right\}\Phi\left(\mathrm{d}x\right)\right] \\
&= \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B \mathbf{1}\left\{\Phi(\{x\}) \geq 1\right\}\Phi\left(\mathrm{d}x\right)\right] \\
&= \frac{1}{\lambda|B|}\mathbf{E}\left[\Phi(B)\right] = 1.
\end{aligned}$$

(ii) Applying C-L-M-M theorem 6.1.28 to

$$f(x,\omega) = h\left(x\right)g\left(\Phi\left(\omega\right)\right),$$

we get

$$\begin{aligned}
\lambda\int_{\mathbb{R}^d} h\left(x\right)\mathbf{E}^0[g\left(\Phi\right)]\mathrm{d}x &= \mathbf{E}\left[\int_{\mathbb{R}^d} h\left(x\right)g\left(\Phi\left(\theta_x\omega\right)\right)\Phi\left(\mathrm{d}x\right)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} h\left(x\right)g\left(S_x\Phi\right)\Phi\left(\mathrm{d}x\right)\right] \\
&= \lambda\int_{\mathbb{R}^d\times\mathbb{M}(\mathbb{R}^d)} h\left(x\right)g\left(S_x\mu\right)\mathbf{P}_\Phi^x\left(\mathrm{d}\mu\right)\mathrm{d}x,
\end{aligned}$$

where the last equality is due to the Campbell-Little-Mecke theorem. We deduce that for Lebesgue-almost all $x \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbf{E}^0[g\left(\Phi\right)] &= \int_{\mathbb{M}(\mathbb{R}^d)} g\left(S_x\mu\right)\mathbf{P}_\Phi^x\left(\mathrm{d}\mu\right) \\
&= \int_{\mathbb{M}(\mathbb{R}^d)} g\left(\mu\right)\mathbf{P}_\Phi^x \circ S_x^{-1}\left(\mathrm{d}\mu\right),
\end{aligned}$$

where we make the change of variable $\mu := S_x\mu$. Noting that the left-hand side of the above equality equals

$$\mathbf{E}^0[g\left(\Phi\right)] = \int_{\mathbb{M}(\mathbb{R}^d)} g\left(\mu\right)\mathbf{P}_\Phi^0\left(\mathrm{d}\mu\right),$$

shows that $\mathbf{P}_\Phi^0 = \mathbf{P}_\Phi^x \circ S_x^{-1}$ for Lebesgue-almost all $x \in \mathbb{R}^d$. (iii) If Φ is a homogeneous Poisson point process, then by Slivnyak-Mecke theorem 3.2.4, its

Palm distributions equal $\mathbf{P}_\Phi^x = \mathbf{P}_{\Phi+\delta_x}$ for Lebesgue almost all x. For such x,

$$\begin{aligned}
\mathbf{P}_\Phi^0 &= \mathbf{P}_\Phi^x \circ S_x^{-1} \\
&= \mathbf{P}_{\Phi+\delta_x} \circ S_x^{-1} \\
&= \mathbf{P} \circ (\Phi + \delta_x)^{-1} \circ S_x^{-1} \\
&= \mathbf{P} \circ [S_x \circ (\Phi + \delta_x)]^{-1} \\
&= \mathbf{P} \circ (S_x\Phi + S_x \circ \delta_x)^{-1} \\
&= \mathbf{P} \circ (S_x\Phi + \delta_0)^{-1} = \mathbf{P}_{S_x\Phi+\delta_0}.
\end{aligned}$$

By stationarity of Φ, we have $S_x\Phi \overset{\text{dist.}}{=} \Phi$ and thus $S_x\Phi + \delta_0 \overset{\text{dist.}}{=} \Phi + \delta_0$ which combined with the above equality completes the proof. □

Example 6.1.32. Palm for shifted comb. *Let U be a random variable uniformly distributed in $[0,1)$. We have shown in Example 6.1.3 that the shifted comb $\Phi = \sum_{k\in\mathbb{Z}} \delta_{k+U}$ is stationary. We have already given the Palm distributions $\mathbf{P}_\Phi^x$ in Example 3.2.3. The intensity of Φ equals*

$$\begin{aligned}
\lambda &= \mathbf{E}\left[\Phi\left([0,1)\right)\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{Z}} \mathbf{1}\left\{k + U \in [0,1)\right\}\right] \\
&= \mathbf{E}\left[\mathbf{1}\left\{U \in [0,1)\right\}\right] = 1.
\end{aligned}$$

Then the mean measure of a shifted comb process is the Lebesgue measure. We aim now to compute its Palm probability $\mathbf{P}^0$. By (6.1.12), for any nonnegative random variable g,

$$\mathbf{E}^0\left[g\right] = \mathbf{E}\left[\sum_{X\in\Phi\cap[0,1)} g\left(\theta_X\omega\right)\right] = \mathbf{E}\left[g\left(\theta_U\omega\right)\right].$$

In particular, for any measurable function $f : \mathbb{M}(\mathbb{R}) \to \mathbb{R}_+$,

$$\mathbf{E}^0\left[f\left(\Phi\right)\right] = \mathbf{E}\left[f\left(\theta_U\Phi\right)\right] = f\left(\sum_{k\in\mathbb{Z}} \delta_k\right).$$

We deduce that, under $\mathbf{P}^0$, Φ is deterministic and is equal to $\sum_{k\in\mathbb{Z}} \delta_k$.

6.1.4 Mass transport formula

As we have said in the introduction of the present chapter, the true benefit of the stationary framework is the possibility to study the relations between Palm probabilities corresponding to different random measures or point processes living on the same probability space and being jointly stationarity.

The *mass transport formula* stated in Theorem 6.1.34 below is one such result. Its proof relies on the following preliminary observation.

Lemma 6.1.33. Integral of random product measures. *Consider a stationary framework* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$. *Let* Φ, Φ' *be two random measures on* $\mathbb{R}^d$ *compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^d}$ *and having respective intensities* $\lambda, \lambda' \in \mathbb{R}_+^*$. *Then for all measurable functions* $f : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \bar{\mathbb{R}}_+$,

$$\mathbf{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y,\theta_y\omega)\Phi(\mathrm{d}x)\,\Phi'(\mathrm{d}y)\right] = \lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x+y,y,\omega)\Phi(\mathrm{d}x)\,\mathrm{d}y\right], \tag{6.1.15}$$

where $\mathbf{E}^{0\prime}$ *denotes the expectation with respect to the Palm probability of* Φ'.

Proof. Let $g(y,\omega) = \int_{\mathbb{R}^d} f(x+y,y,\omega)\Phi(\mathrm{d}x)$. Then by Fubini's theorem, the right-hand side of (6.1.15) takes the form

$$\begin{aligned}
\lambda'\mathbf{E}^{0\prime}&\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y,y,\omega)\Phi(\mathrm{d}x)\,\mathrm{d}y\right]\\
&= \lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d} g(y,\omega)\mathrm{d}y\right]\\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} g\left(y,\theta_y\omega\right)\Phi'(\mathrm{d}y)\right]\\
&= \mathbf{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y,y,\theta_y\omega)\Phi\circ\theta_y(\mathrm{d}x)\,\Phi'(\mathrm{d}y)\right]\\
&= \mathbf{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y,\theta_y\omega)\Phi(\mathrm{d}x)\,\Phi'(\mathrm{d}y)\right],
\end{aligned}$$

where the second equality follows from the C-L-M-M theorem 6.1.28 for the random measure Φ' and the fourth equality from the shifting formula (6.1.1). □

Theorem 6.1.34. Mass transport theorem. *Let* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ *be a stationary framework and* Φ, Φ' *be random measures on* $\mathbb{R}^d$ *compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^d}$ *and having respective intensities* $\lambda, \lambda' \in \mathbb{R}_+^*$. *Then, for all measurable functions* $g : \mathbb{R}^d \times \Omega \to \bar{\mathbb{R}}_+$,

$$\lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d} g(y,\omega)\Phi'(\mathrm{d}y)\right] = \lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d} g(-x,\theta_x\omega)\Phi(\mathrm{d}x)\right], \tag{6.1.16}$$

where $\mathbf{E}^0$ *and* $\mathbf{E}^{0\prime}$ *are the expectations with respect to the Palm probabilities of* Φ *and* Φ', *respectively. The above formula is called the* mass transport formula.

Proof. Let

$$f(z,y,\omega) = g(y-z,\theta_{z-y}\omega)\mathbf{1}\left\{y\in(0,1]^d\right\}.$$

Then

$$f(x+y,y,\omega) = g(-x,\theta_x\omega)\mathbf{1}\left\{y\in(0,1]^d\right\}.$$

Thus

$$\begin{aligned}
\lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d} g(-x,\theta_x\omega)\Phi(\mathrm{d}x)\right] &= \lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x+y,y,\omega)\Phi(\mathrm{d}x)\,\mathrm{d}y\right]\\
&= \mathbf{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y,\theta_y\omega)\Phi(\mathrm{d}x)\,\Phi'(\mathrm{d}y)\right],
\end{aligned}$$

where the second equality follows from (6.1.15). Let $h(y,x,\omega) = f(x,y,\theta_{y-x}\omega)$, then we may continue the above equality as follows

$$\begin{aligned}
\lambda'\mathbf{E}^{0\prime}&\left[\int_{\mathbb{R}^d} g(-x,\theta_x\omega)\Phi\,(\mathrm{d}x)\right]\\
&= \mathbf{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} h(y,x,\theta_x\omega)\Phi'\,(\mathrm{d}y)\,\Phi\,(\mathrm{d}x)\right]\\
&= \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} h(y+x,x,\omega)\Phi'\,(\mathrm{d}y)\,\mathrm{d}x\right]\\
&= \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,x+y,\theta_y\omega)\Phi'\,(\mathrm{d}y)\,\mathrm{d}x\right]\\
&= \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} g(y,\theta_{-y}\theta_y\omega)\mathbf{1}\left\{x+y\in(0,1]^d\right\}\Phi'\,(\mathrm{d}y)\,\mathrm{d}x\right]\\
&= \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d} g(y,\omega)\Phi'\,(\mathrm{d}y)\right],
\end{aligned}$$

where the second equality follows from (6.1.15). □

Corollary 6.1.35. Unimodularity. *Let $(\Omega,\mathcal{A},\{\theta_t\}_{t\in\mathbb{R}^d},\mathbf{P})$ be a stationary framework and Φ be a random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and having intensity $\lambda\in\mathbb{R}_+^*$. Then, for all measurable functions $g:\mathbb{R}^d\times\Omega\to\bar{\mathbb{R}}_+$,*

$$\mathbf{E}^0\left[\int_{\mathbb{R}^d} g(y,\omega)\Phi\,(\mathrm{d}y)\right] = \mathbf{E}^0\left[\int_{\mathbb{R}^d} g(-x,\theta_x\omega)\Phi\,(\mathrm{d}x)\right] \tag{6.1.17}$$

where $\mathbf{E}^0$ is the expectation with respect to the Palm probability $\mathbf{P}^0$ of Φ. We say then that $\mathbf{P}^0$ is unimodular *with respect to Φ.*

Proof. This follows from Theorem 6.1.34 with $\Phi=\Phi'$. □

Here is an example which explains why Equation (6.1.16) is called the *mass transport formula*.

Remark 6.1.36. Equivalent form of the mass transport formula. *Let $m(x,y,\omega)$ be a measurable function on $\mathbb{R}^d\times\mathbb{R}^d\times\Omega$ interpreted as the amount of mass sent from x to y on the event ω. We assume that m is compatible with the flow in the following sense*

$$m(x,y,\omega) = m(x-t,y-t,\theta_t),\quad x,y,t\in\mathbb{R}^d.$$

Define $g(y,\omega) := m(0,y,\omega)$ as the amount of mass sent form the origin 0 to y on the event ω. Then by the compatibility of m we have $g(-x,\theta_x\omega) = m(0,-x,\theta_x\omega) = m(x,0,\omega)$ and (6.1.16) is equivalent to

$$\lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d} m(0,y,\omega)\Phi'(\mathrm{d}y)\right] = \lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d} m(x,0,\omega)\Phi(\mathrm{d}x)\right], \tag{6.1.18}$$

which can be interpreted by saying that the proportion between the expected total mass sent from the typical point of Φ (located at the origin under $\mathbf{E}^0$) to all points of Φ' and the expected total mass received by the typical point of Φ' (located at the origin under $\mathbf{E}^{0\prime}$) from all points of Φ is equal to the proportion of the point processes intensities λ' to λ. In particular, if $\lambda = \lambda'$, on average, total mass sent out of the typical point of Φ is equal to the total mass received at the typical point of Φ'.

Example 6.1.37. Bipartite compatible graph. *Consider the context of Theorem 6.1.34 where Φ and Φ' are point processes. This example constructs a* bipartite directed graph *from the points/nodes of Φ to those of Φ' which is* compatible *with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$; i.e., there is an edge from $X \in \Phi(\omega)$ to $Y \in \Phi(\omega)$ iff there is an edge from $X - t \in \Phi(\theta_t\omega)$ to $Y - t \in \Phi'(\theta_t\omega)$.*

Let $\Omega_0 = \{\omega \in \Omega : 0 \in \Phi(\omega)\}$. Recall from Theorem 6.1.31(i) that $\mathbf{P}^0(\Omega_0) = 1$. For all $\omega \in \Omega_0$, we assume that there are directed edges from $0 \in \Phi(\omega)$ to selected nodes Y of $\Phi'(\omega)$. This is equivalent to considering the function g defined on $\mathbb{R}^d \times \Omega$ by

$$g(y,\omega) = \mathbf{1}\{\omega \in \Omega_0\}\,\mathbf{1}\{y \in \Phi'(\omega)\}\,\mathbf{1}\{\exists \text{ a directed edge from } 0 \text{ to } y\}.$$

One defines the out-neighbors *of 0 as the set*

$$h^+(\omega) = \{y \in \Phi'(\omega) : g(y,\omega) = 1\}, \quad \omega \in \Omega_0.$$

By leveraging compatibility, this extends to the following definition of the out-neighbors of any $X \in \Phi$:

$$H^+(X) = X + h^+(\theta_X), \quad \omega \in \Omega, X \in \Phi(\omega).$$

Note that $H^+(X) \subset \Phi'$, $\mathbf{P}$-a.s. If the cardinality of $h^+(\omega)$ is $\mathbf{P}^0$-a.s. finite, this defines a bipartite directed graph *from the points/nodes of Φ to those of Φ'. This graph is compatible with the flow by construction. One can also define the* in-neighbors *of 0 as the set*

$$h^-(\omega) = \{x \in \Phi(\omega) : g(-x, \theta_x\omega) = 1\}, \quad \omega \in \Omega_0.$$

It follows from compatibility and the definition of g that the last set can be rewritten as

$$h^-(\omega) = \begin{cases} \{x \in \Phi(\omega) : \exists \text{ directed edge from } x \text{ to } 0\}, & \text{if } \omega \in \Omega_0', \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\Omega_0' = \{\omega \in \Omega : 0 \in \Phi'(\omega)\}$. The in-neighbors of any $Y \in \Phi'$ can also be defined by

$$H^-(Y) = Y + h^-(\theta_Y), \quad \omega \in \Omega, Y \in \Phi'(\omega).$$

The mass transport formula (6.1.16) then reads

$$\lambda\mathbf{E}^0[\mathrm{card}(h^+(\omega))] = \lambda'\mathbf{E}^{0\prime}[\mathrm{card}(h^-(\omega))],$$

or, equivalently,

$$\lambda\mathbf{E}^0[\mathrm{card}(H^+(0))] = \lambda'\mathbf{E}^{0\prime}[\mathrm{card}(H^-(0))]. \tag{6.1.19}$$

Example 6.1.38. Compatible graph. *Consider the particular case of Example 6.1.37 where $\Phi = \Phi'$. We then get a directed graph on the support of Φ, which is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. For this graph, the mass transport formula gives*

$$\mathbf{E}^0[\mathrm{card}(H^+(0))] = \mathbf{E}^0[\mathrm{card}(H^-(0))]. \tag{6.1.20}$$

Example 6.1.39. Point map. *Consider the particular case of Example 6.1.38 where, in addition to $\Phi = \Phi'$, we have that $\mathbf{P}^0$-a.s., $\mathrm{card}(h^+(\omega)) = 1$. Then $h^+(\omega) = \{h(\omega)\}$ for some point $h(\omega) \in \Phi$. The map $\omega \mapsto h(\omega)$ is called a* point map. *It follows from (6.1.20) that $\mathbf{E}^0[\mathrm{card}(H^-(0))] = 1$.*

6.1.5 Mecke's invariance theorem

Recall that in the stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$, the stationary probability $\mathbf{P}$ is invariant with respect to the flows θ_t, ; that is $\mathbf{P}\theta_t^{-1} = \mathbf{P}$ for any $t \in \mathbb{R}^d$. We address now the question whether Palm probability of a point process has a similar property at least with respect to some particular flow; i.e., whether

$$\mathbf{P}^0\theta_h^{-1} = \mathbf{P}^0$$

holds true for some $h \in \mathbb{R}^d$.

In the present section we consider a point process Φ on $\mathbb{R}^d$ compatible with the flow, having intensity $\lambda \in \mathbb{R}_+^*$ and Palm probability $\mathbf{P}^0$. Let Ω_0 be defined by (6.1.13). We define the notion of point map which was already introduced in Example 6.1.39.

Definition 6.1.40. *A* point map *related to Φ is a measurable mapping $h : \Omega_0 \to \mathbb{R}^d$ such that $h(\omega) \in \Phi(\omega)$ for all $\omega \in \Omega_0$. The associated* point shift *is the mapping H from the support of Φ to itself defined by*

$$H(X, \omega) = X + h(\theta_X\omega), \quad \omega \in \Omega, X \in \Phi(\omega). \tag{6.1.21}$$

The point map h is said to be bijective *if for $\mathbf{P}$-almost all ω, the map $H(\cdot, \omega)$ is a bijection from the support of $\Phi(\omega)$ to itself (which implies in particular that Φ is simple).*

Observe that the point shift is *compatibale* with the flow in the sense that

$$H(X - t, \theta_t\omega) = X - t + h(\theta_{X-t}\theta_t\omega) = H(X, \omega) - t. \tag{6.1.22}$$

Lemma 6.1.41. *Let h be a bijective point map and and let H be the associated point shift (6.1.21). For each $\omega \in \Omega$, let $H^{-1}(\cdot, \omega)$ be the inverse function of $H(\cdot, \omega)$. Then*

$$H^{-1}(X, \omega) = X + H^{-1}(0, \theta_X\omega), \quad \omega \in \Omega, X \in \Phi(\omega). \tag{6.1.23}$$

Moreover,

$$\mathrm{card}(H^{-1}(0, \omega)) = 1, \quad \mathbf{P}^0\text{-a.s.}$$

Proof. Observe from (6.1.22) that

$$H(H^{-1}(X,\omega)-X,\theta_X\omega)=H(H^{-1}(X,\omega),\omega)-X=0.$$

Then

$$H^{-1}(X,\omega)-X=H^{-1}(0,\theta_X\omega),$$

from which (6.1.23) follows. From the very definition of the Palm probability (6.1.10)

$$\begin{aligned}\mathbf{P}^0\left[\operatorname{card}(H^{-1}(0,\omega))=1\right]&=\frac{1}{\lambda}\mathbf{E}\left[\sum_{X\in\Phi\cap(0,1]^d}\mathbf{1}\left\{\operatorname{card}\left(H^{-1}(0,\theta_X\omega)\right)=1\right\}\right]\\&=\frac{1}{\lambda}\mathbf{E}\left[\sum_{X\in\Phi\cap(0,1]^d}\mathbf{1}\left\{\operatorname{card}\left(H^{-1}(X,\omega)-X\right)=1\right\}\right]\\&=\frac{1}{\lambda}\mathbf{E}\left[\sum_{X\in\Phi\cap(0,1]^d}\mathbf{1}\left\{\operatorname{card}\left(H^{-1}(X,\omega)\right)=1\right\}\right]=1,\end{aligned}$$

where the last relation comes from the bijection assumption which implies that $\mathbf{P}$-a.s., for all $X\in\Phi$, $\operatorname{card}\left(H^{-1}(X,\omega)\right)=1$. □

Theorem 6.1.42. Mecke's invariance theorem. *Let $(\Omega,\mathcal{A},\{\theta_t\}_{t\in\mathbb{R}^d},\mathbf{P})$ be a stationary framework, Φ a simple point process compatible with the flow, having intensity $\lambda\in\mathbb{R}_+^*$, and $\mathbf{P}^0$ its Palm probability. Then for any bijective point map $h:\Omega_0\to\mathbb{R}^d$,*

$$\mathbf{P}^0\theta_h^{-1}=\mathbf{P}^0.$$

Proof. The result follows from the mass transport formula (6.1.16), when taking $\Phi=\Phi'$ and picking g of the form

$$g(y,\omega)=f(\theta_y\omega)\mathbf{1}\left\{y=h(\omega)\right\},$$

where f is any measurable function from Ω to $\mathbb{R}_+$. We have,

$$\mathbf{E}^0\left[\int_{\mathbb{R}^d}g(y,\omega)\Phi(\mathrm{d}y)\right]=\mathbf{E}^0\left[f(\theta_h(\omega))\right].$$

In addition,

$$\begin{aligned}\mathbf{E}^0\left[\int_{\mathbb{R}^d}g(-x,\theta_x\omega)\Phi(\mathrm{d}x)\right]&=\mathbf{E}^0\left[\int_{\mathbb{R}^d}f(\omega)\mathbf{1}\left\{-x=h(\theta_x\omega)\right\}\Phi(\mathrm{d}x)\right]\\&=\mathbf{E}^0\left[f(\omega)\Phi(H^{-1}(0,\omega))\right]=\mathbf{E}^0\left[f(\omega)\right],\end{aligned}$$

where the last relation follows from Lemma 6.1.41. □

Here are a few basic examples of bijective point shifts.

Example 6.1.43. Natural point shift on the line. *Let Φ be a stationary simple point process on $\mathbb{R}$ ($d = 1$). Let $A = \left\{\omega \in \Omega : \Phi(\omega, \mathbb{R}_+^*) = \Phi(\omega, \mathbb{R}_-^*) = \infty\right\}$. Observe that for any $\omega \in A$, all the points $T_n(\omega)$ ($n \in \mathbb{Z}$) of Φ with the usual enumeration convention (1.6.8), are finite. Let*

$$h(\omega) = \begin{cases} T_1(\omega), & \omega \in \Omega_0 \cap A, \\ 0, & \omega \in \Omega_0 \cap A^{\mathrm{c}}. \end{cases}$$

Then, the associated point shift is, for all $n \in \mathbb{Z}$,

$$H(T_n, \omega) = \begin{cases} T_{n+1}(\omega), & \omega \in A, \\ T_n(\omega), & \omega \in A^{\mathrm{c}}. \end{cases}$$

Indeed, if $\omega \in A$,

$$\begin{aligned} H(T_n, \omega) &= T_n + h(\theta_{T_n}) \\ &= T_n + (T_{n+1} - T_n) = T_{n+1}, \end{aligned}$$

and if $\omega \in A^{\mathrm{c}}$,

$$H(T_n, \omega) = T_n + h(\theta_{T_n}) = T_n.$$

This point shift is bijective.

Example 6.1.44. MCN point shift. *Let μ be a counting measure on $\mathbb{R}^d$. We say that $x, y \in \mathbb{R}^d$ are* mutual closest neighbors *(MCN) for μ if $x, y \in \operatorname{supp}(\mu)$ and*

$$\begin{cases} \mu\left(B(x, \|x - y\|)\right) = 1 \\ \mu\left(\bar{B}(x, \|x - y\|)\right) = 2 \\ \mu\left(B(y, \|x - y\|)\right) = 1 \\ \mu\left(\bar{B}(y, \|x - y\|)\right) = 2, \end{cases}$$

where $B(x, r)$ denotes the open ball of center x and radius r and $\bar{B}(x, r)$ its closure. For all $x \in \operatorname{supp}(\mu)$ there exists at most one *$y \in \operatorname{supp}(\mu)$ such that (x, y) are mutual closest neighbors. Let Φ be a stationary simple point process on $\mathbb{R}^d$. The* MCN point map *is*

$$h(\omega) = \begin{cases} X, & \text{if } (X, 0) \text{ are MCN in } \Phi(\omega), \\ 0, & \text{if there is not such } X. \end{cases}$$

The associated point shift H, is called the MCN point shift*. For all $X \in$* $\operatorname{supp}(\Phi)$,

$$H(X, \omega) = \begin{cases} Y, & \text{if } (X, Y) \text{ are MCN,} \\ X, & \text{if there is no such } Y. \end{cases}$$

The MCN point shift kernel is involutive *(i.e., $H \circ H$ equals identity) and hence bijective.*

Here is an example of point shift which is not bijective.

Example 6.1.45. Closest neighbor point shift. *Let Φ be a stationary simple point process on $\mathbb{R}^d$ such that $\mathbf{P}(\Phi \neq 0) = 1$. Consider the event $A = \{\omega \in \Omega : \Phi(\omega, \mathbb{R}^d) = \infty\}$. Let $h(\omega)$ be the closest* neighbor *of* 0 *in Φ; that is*

$$h(\omega) = \begin{cases} \arg\min_{X_n \in \Phi \setminus \{0\}} |X_n|, & \omega \in \Omega_0 \cap A, \\ 0, & \omega \in \Omega_0 \cap A^c. \end{cases}$$

The associated point shift is

$$H(X, \omega) = \begin{cases} \arg\min_{X_n \in \Phi \setminus \{X\}} |X_n - X|, & \omega \in A, X \in \Phi(\omega), \\ X, & \omega \in A^c, X \in \Phi(\omega). \end{cases}$$

that is $H(X, \omega)$ is the closest neighbor *of X in Φ. This point shift is not bijective in general since two distinct points of Φ may have the same closest neighbor.*

6.2 Palm inversion formula

We now give a formula allowing one to retrieve the original probability measure $\mathbf{P}$ from the Palm probability $\mathbf{P}^0$. This is called the *inversion formula.* For this, we introduce the notion of Voronoi tessellation.

6.2.1 Voronoi tessellation

Definition 6.2.1. Voronoi tessellation. *Let Φ be a stationary point process on $\mathbb{R}^d$. The set*

$$V(\omega) = \{y \in \mathbb{R}^d : |y| \leq \inf_{X \in \Phi(\omega)} |y - X|\} \tag{6.2.1}$$

is called the virtual cell *of Φ. If for some $\omega \in \Omega$, $\Phi(\omega)$ is the null measure, then $V(\omega) = \mathbb{R}^d$. Let*

$$\tilde{V}(x) := \{y \in \mathbb{R}^d : |y - x| \leq \inf_{X \in \Phi} |y - X|\}. \tag{6.2.2}$$

For each $X \in \Phi$, $\tilde{V}(X)$ is called the Voronoi cell *of X with respect to Φ.*

The Voronoi cells are illustrated in Figure 6.1.

Lemma 6.2.2. *For all $x \in \mathbb{R}^d$,*

$$V(x) := V \circ \theta_x = \tilde{V}(x) - x. \tag{6.2.3}$$

Proof.

$$\begin{aligned} V(x) &= \{y \in \mathbb{R}^d : |y| \leq \inf_{Y \in \Phi \circ \theta_x} |y - Y|\} \\ &= \{y \in \mathbb{R}^d : |y| \leq \inf_{X \in \Phi} |x + y - X|\} \\ &= \{\overline{y} - x \in \mathbb{R}^d : |\overline{y} - x| \leq \inf_{X \in \Phi} |\overline{y} - X|\} \\ &= \{\overline{y} \in \mathbb{R}^d : |\overline{y} - x| \leq \inf_{X \in \Phi} |\overline{y} - X|\} - x \\ &= \tilde{V}(x) - x. \end{aligned}$$

□

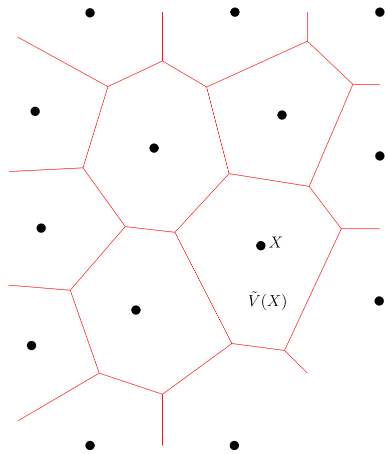

Figure 6.1: Voronoi cell

Corollary 6.1.30 implies

$$\mathbf{E}^0[|V|] = \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B |V(x)|\Phi(\mathrm{d}x)\right].$$

This can be rewritten as

$$\mathbf{E}^0[|V|] = \mathbf{E}^0[|\tilde{V}(0)|] = \frac{1}{\lambda|B|}\mathbf{E}\left[\int_B |\tilde{V}(x)|\Phi(\mathrm{d}x)\right],$$

since, on the one hand, $\tilde{V}(0) = V$ under $\mathbf{P}^0$, and on the other hand, $|V(x)| = |\tilde{V}(x)|$ in view of (6.2.3). The above equation motivates the following definition.

Definition 6.2.3. Typical cell. *Let Φ be a stationary point process on $\mathbb{R}^d$. Under the Palm probability $\mathbf{P}^0$, the virtual cell V defined by (6.2.1) is the* typical cell *of Φ.*

Remark 6.2.4. *The typical cell may be seen as a formalization of the idea randomly selected (without any bias) Voronoi cell. Note that there is no uniform distribution on a countably infinite set and hence a direct formalization of randomly, uniformly selected cell is not possible without an ergodic argument; which will developped in Chapter 8.*

Example 6.2.5. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. Observe that*

$$\begin{aligned}\mathbf{P}^0(x \in V) &= \mathbf{P}^0(\Phi(B(x,|x|)) = 0) \\ &= \mathbf{P}((\Phi + \delta_0)(B(x,|x|)) = 0) \\ &= \mathbf{P}(\Phi(B(x,|x|)) = 0) = e^{-\lambda\kappa_d|x|^d},\end{aligned}$$

where the second equality follows from Slivnyak's theorem 6.1.31(iii), and where κ_d denotes the volume of the unit-radius d-dimensional ball.

6.2.2 Inversion formula

We need the following preliminary result, which has an independent interest.

Lemma 6.2.6. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and Φ be a point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and with intensity $\lambda \in \mathbb{R}_+^*$. Then, $\mathbf{P}$-almost surely, Φ has no two distinct points equidistant from 0.*

Proof.

$$
\begin{aligned}
&\mathbf{P}\left(\{\exists X \neq Y \in \Phi : |X| = |Y|\}\right)\\
&\quad\leq \mathbf{E}[\mathrm{card}(\{X \neq Y \in \Phi : |X| = |Y|\})]\\
&\quad= \mathbf{E}\left[\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\mathbf{1}\{y\neq x, |x| = |y|\}\Phi(\mathrm{d}y)\right]\Phi(\mathrm{d}x)\right]\\
&\quad= \lambda\int_{\mathbb{R}^d}\mathbf{E}^0\left[\int_{\mathbb{R}^d}\mathbf{1}\{y\neq x, |x| = |y|\}\left(\Phi\circ\theta_{-x}\right)(\mathrm{d}y)\right]\mathrm{d}x\\
&\quad= \lambda\int_{\mathbb{R}^d}\mathbf{E}^0\left[\int_{\mathbb{R}^d}\mathbf{1}\{u\neq 0, |x| = |u+x|\}\Phi(\mathrm{d}u)\right]\mathrm{d}x\\
&\quad= \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathbf{1}\{u\neq 0, |x| = |u+x|\}\mathrm{d}x\right)\Phi(\mathrm{d}u)\right] = 0,
\end{aligned}
$$

where the third line is due to the C-L-M-M theorem 6.1.28, for the fourth line we make the change of variable $y \to u = y - x$, the fifth line follows from Fubini-Tonelli theorem and the last equality is due to the fact that, for all $u \neq 0$, the Lebesgue measure of $\{x \in \mathbb{R}^d : |x| = |u + x|\}$ is null. □

Remark 6.2.7. *Therefore $\mathbf{P}$-almost surely, the boundary of any ball contains no more than an atom of Φ. This may be generalized to all subsets of $\mathbb{R}^d$ whose Lebesgue measure is null.*

We are now ready to state the inversion formula.

Theorem 6.2.8. Palm inversion formula. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and Φ be a* simple *point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$, with intensity $\lambda \in \mathbb{R}_+^*$. Then, for all measurable functions $f : \Omega \to \mathbb{R}_+$,*

$$
\mathbf{E}[f \times \mathbf{1}\{\Phi \neq 0\}] = \lambda\mathbf{E}^0\left[\int_V f\circ\theta_x \mathrm{d}x\right], \tag{6.2.4}
$$

where V is the typical cell defined by (6.2.1). The above formula is called Palm inversion formula.

Proof. We will apply Theorem 6.1.34 with $\Phi' \equiv \ell^d$ the Lebesgue measure and

$$
g(x,\omega) = f\circ\theta_x \times \mathbf{1}\{x\in V\}, \quad x\in\mathbb{R}^d, \omega\in\Omega.
$$

By Example 6.1.24, the Palm probability of Φ' is the stationary probability $\mathbf{P}$. The left-hand side of (6.1.16) is equal to

$$
\lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d} g(x,\omega)\Phi'(\mathrm{d}x)\right] = \lambda\mathbf{E}^0\left[\int_V f\circ\theta_x\mathrm{d}x\right].
$$

Observe that for any $x \in \mathbb{R}^d, \omega \in \Omega$,

$$\begin{aligned} g(-x, \theta_x \omega) &= f \times \mathbf{1}\{-x \in V \circ \theta_x\} \\ &= f \times \mathbf{1}\{-x \in V(x)\} \\ &= f \times \mathbf{1}\left\{0 \in \tilde{V}(x)\right\} \\ &= f \times \mathbf{1}\left\{|x| \leq \inf_{X \in \Phi} |X|\right\}, \end{aligned}$$

where the second and third equalities are due to (6.2.3) and the last equality follows from (6.2.2). Thus the right-hand side of (6.1.16) equals

$$\begin{aligned} \mathbf{E}\left[\int_{\mathbb{R}^d} g(-x, \theta_x \omega) \Phi(\mathrm{d}x)\right] &= \mathbf{E}\left[f \int_{\mathbb{R}^d} \mathbf{1}\left\{|x| \leq \inf_{X \in \Phi} |X|\right\} \Phi(\mathrm{d}x)\right] \\ &= \mathbf{E}\left[f \sum_{Y \in \Phi} \mathbf{1}\left\{|Y| \leq \inf_{X \in \Phi} |X|\right\}\right] \\ &= \mathbf{E}\left[f \times \mathbf{1}\{\Phi \neq 0\}\right], \end{aligned}$$

where the last equality is due to Lemma 6.2.6. □

The following result is a generalization of the previous theorem which does not necessarily use the Voronoi cell.

Theorem 6.2.9. Ryll-Nardzewski and Slivnyak inversion formula. *Consider a stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}_{t \in \mathbb{R}^d}, \mathbf{P})$. Let Φ be a random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t \in \mathbb{R}^d}$ and with intensity $\lambda \in \mathbb{R}_+^*$. Assume that there exists a measurable function $h : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$ such that*

$$\int_{\mathbb{R}^d} h(x, \omega) \Phi(\omega, \mathrm{d}x) = \mathbf{1}\{\Phi \neq 0\}, \quad \mathbf{P}\text{-}a.s. \tag{6.2.5}$$

Then, for all measurable functions $g : \Omega \to \mathbb{R}_+$,

$$\mathbf{E}[g \times \mathbf{1}\{\Phi \neq 0\}] = \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} g(\theta_{-x}\omega) h(x, \theta_{-x}\omega) \mathrm{d}x\right]. \tag{6.2.6}$$

Proof. Applying the C-L-M-M theorem 6.1.28 to $f(x, \omega) = g(\theta_{-x}\omega) h(x, \theta_{-x}\omega)$ gives

$$\mathbf{E}\left[\int_{\mathbb{R}^d} g(\omega) h(x, \omega) \Phi(\omega, \mathrm{d}x)\right] = \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} g(\theta_{-x}\omega) h(x, \theta_{-x}\omega) \mathrm{d}x\right].$$

Due to the assumption (6.2.5), the left-hand side of the above equation equals $\mathbf{E}[g \times \mathbf{1}\{\Phi \neq 0\}]$ which completes the proof. □

Remark 6.2.10. *Theorem 6.2.8 may be proved using Theorem 6.2.9 as follows. Let*

$$h(x, \omega) := \mathbf{1}\left\{|x| \leq \inf_{X \in \Phi} |X|\right\} = \mathbf{1}\{\Phi(B(0, |x|)) = 0\}, \quad x \in \mathbb{R}^d, \omega \in \Omega,$$

where $B(y,r)$ is the open ball in $\mathbb{R}^d$ of center y and radius r. Assumption (6.2.5) holds for the above function h since by Lemma 6.2.6

$$\int_{\mathbb{R}^d} h(x,\omega)\Phi(\mathrm{d}x) = \sum_{X\in\Phi} h(X,\omega)$$
$$= \sum_{X\in\Phi} \mathbf{1}\left\{|X| = \inf_{Y\in\Phi}|Y|\right\} = \mathbf{1}\{\Phi \neq 0\},$$

where the assumption that $\Phi\left(\mathbb{R}^d\right) \neq 0$ was used to get the last equality. Moreover, note that

$$\begin{aligned} h(-x,\theta_x(\omega)) &= \mathbf{1}\{\Phi(\theta_x(\omega))(B(0,|x|)) = 0\} \\ &= \mathbf{1}\{S_x\Phi(\omega)(B(0,|x|)) = 0\} \\ &= \mathbf{1}\{\Phi(\omega)(B(x,|x|)) = 0\} = \mathbf{1}\{x \in V\}, \end{aligned}$$

where the last equality follows from the observation that, under the Palm probability, $V = \{y \in \mathbb{R}^d : \Phi(B(y,|y|)) = 0\}$. Thus, by Theorem 6.2.9

$$\mathbf{E}[f \times \mathbf{1}\{\Phi \neq 0\}] = \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d} f\circ\theta_x(\omega)h(-x,\theta_x(\omega))\mathrm{d}x\right] = \lambda\mathbf{E}^0\left[\int_V f\circ\theta_x\mathrm{d}x\right].$$

Example 6.2.11. Shifted lattice. *In the canonical stationary framework (cf. Definition 6.1.13), Equation (6.2.4) writes*

$$\mathbf{E}[f(\Phi)\times\mathbf{1}\{\Phi\neq 0\}] = \lambda\mathbf{E}^0\left[\int_V f(S_x\Phi)\,\mathrm{d}x\right] = \frac{\mathbf{P}(\Phi\neq 0)}{\mathbf{E}^0[|V|]}\mathbf{E}^0\left[\int_V f(S_x\Phi)\,\mathrm{d}x\right].$$

Assume that (i) $\mathbf{P}(\Phi \neq 0) = 1$; and (ii) under $\mathbf{P}^0$, Φ is deterministic; say $\Phi = \mu$. Then the above formula becomes

$$\mathbf{E}[f(\Phi)] = \frac{1}{|V|}\int_V f(S_x\mu)\,\mathrm{d}x,$$

which shows that, under $\mathbf{P}$, Φ may be obtained by considering a random shift of μ by x uniformly distributed in V (recall that $S_x\mu$ shifts the atoms of μ by $-x$). In particular, if μ has its atoms in the centers of some regular lattice, then under $\mathbf{P}$, the point process Φ is called the shifted lattice.

For example, $\mu = \sum_{(n,k)\in\mathbb{Z}^2}\delta_{(n,k)}$ leads to the shifted grid of Example 6.1.4. Moreover, we can contain a stationary version of the hexagonal lattice by taking $\mu = \sum_{(n,k)\in\mathbb{Z}^2}\delta_{n+ke^{i\pi/3}}$ and shifting this measure by a vector uniformly distributed in the hexagon. Lattice point processes are used to model regular structures; e.g., the hexagonal lattice is often used to model the base station locations in wireless communication networks.

Corollary 6.2.12. *Let Φ be a stationary simple point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. Then V is $\mathbf{P}^0$-almost surely bounded and*

$$\mathbf{E}^0[|V|] = \frac{\mathbf{P}(\Phi\neq 0)}{\lambda} < \infty.$$

If moreover $\mathbf{P}(\Phi \neq 0) = 1$, *then*

$$\mathbf{E}^0[|V|] = \frac{1}{\lambda}, \tag{6.2.7}$$

Proof. Applying Theorem 6.2.8 with $f \equiv 1$, we get

$$\mathbf{E}^0[|V|] = \frac{\mathbf{P}(\Phi \neq 0)}{\lambda} < \infty.$$

Then, $\mathbf{P}^0$-almost surely, $|V| < \infty$ which together with the fact that V is a polytope implies that V is bounded under $\mathbf{P}^0$. □

Remark 6.2.13. *Equation (6.2.7) can be interpreted as follows. Since* $\lambda = \mathbf{E}[\Phi([0,1]^d)]$ *is the average number of points (and hence cells) per unit volume, the inverse* $1/\lambda$ *must be the average volume of a cell. Nevertheless, observe that the two averages in this statement correspond to two different probabilities.*

6.2.3 Typical versus zero cell

We have observed in Remark 6.2.4 that the typical cell may be seen as a formalization of the idea of randomly uniformly selected Voronoi cell. Another way of selecting some cell of the Voronoi tessellation consists in taking the one covering some given fixed location; say the origin. This is formalized as follows.

Definition 6.2.14. Zero cell. *Let* Φ *be a stationary point process on* $\mathbb{R}^d$. *For each* $\omega \in \Omega$ *such that* $\Phi(\omega) \neq 0$, *let*

$$X^*(\omega) = \arg \min_{X \in \Phi(\omega)} |X|$$

be the closest point of Φ *to the origin* 0. *This point* $X^*(\omega)$ *is* $\mathbf{P}$*-a.s. unique in view of Lemma 6.2.6. The set*

$$V^* = V \circ \theta_{X^*} \tag{6.2.8}$$

is called the zero cell *of* Φ. *More precisely,*

$$V^*(\omega) = V\left(\theta_{X^*(\omega)}(\omega)\right).$$

Note that the typical cell is the Voronoi cell of 0 under $\mathbf{P}^0$, whereas the zero cell is the Voronoi cell of X^* under $\mathbf{P}$. Figure 6.2 illustrates the typical cell under $\mathbf{P}^0$ and the zero cell under $\mathbf{P}$.

Lemma 6.2.15. *Let* Φ *be a stationary point process on* $\mathbb{R}^d$ *with intensity* $\lambda \in \mathbb{R}_+^*$ *such that* $\mathbf{P}(\Phi \neq 0) = 1$. *Then, for any* $x \in \mathring{V}$, *where* $\mathring{V}$ *denotes the interior of* V,

$$X^* \circ \theta_x = -x, \quad \mathbf{P}^0\text{-almost surely.} \tag{6.2.9}$$

This equation may be interpreted as follows: θ_x consists of translating the points of Φ by $-x$; and among these points the closest one to 0 is $-x$.

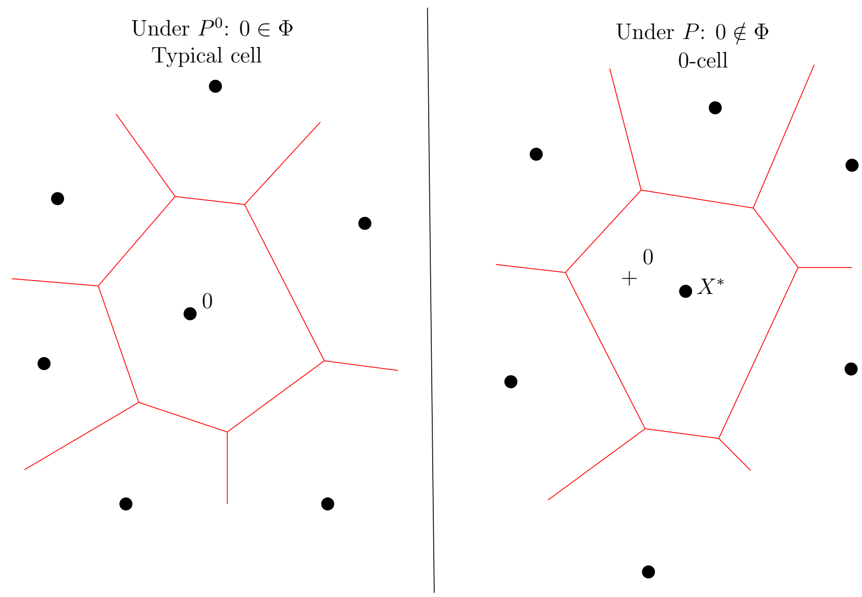

Figure 6.2: Illustration of the typical cell under $\mathbf{P}^0$ and the zero cell under $\mathbf{P}$

Proof. Let $\Omega_0 = \{\omega \in \Omega : 0 \in \Phi(\omega)\}$. Recall from Theorem 6.1.31(i) that $\mathbf{P}^0(\Omega_0) = 1$. Moreover, for each $\omega \in \Omega_0$,

$$\begin{aligned} x \in \mathring{V} &\Rightarrow |x| < \inf_{X \in \Phi \setminus \{0\}} |x - X| \\ &\Rightarrow |-x| < \inf_{X \in \Phi \setminus \{0\}} |X - x| \\ &\Rightarrow |-x| < \inf_{Y \in \Phi \circ \theta_x \setminus \{-x\}} |Y| \\ &\Rightarrow -x = X^* \circ \theta_x. \end{aligned}$$

□

Recall that the zero cell is the cell covering the origin. One can expect that this way of sampling introduces a bias towards cells of larger volume. This is illustrated by the following famous Feller paradox.

Corollary 6.2.16. Feller's paradox. *Let Φ be a stationary and simple point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$ such that $\mathbf{P}(\Phi \neq 0) = 1$. Then,*

$$\mathbf{E}^0[|V|] \leq \mathbf{E}[|V^*|],$$

where V is the typical cell and V^* is the zero cell.

Proof. Applying Theorem 6.2.8 to $f \equiv \frac{1}{|V^*(\omega)|}$ we get

$$\mathbf{E}\left[\frac{1}{|V^*|}\right] = \lambda \mathbf{E}^0\left[\int_V \frac{1}{|V^* \circ \theta_x|} \mathrm{d}x\right] = \lambda \mathbf{E}^0\left[\int_{\mathring{V}} \frac{1}{|V^* \circ \theta_x|} \mathrm{d}x\right].$$

Using the definition of the zero cell in Equation (6.2.8), we deduce that, under $\mathbf{P}^0$, for $x \in \mathring{V}$,

$$\begin{aligned} V^* \circ \theta_x &= (V \circ \theta_{X^*}) \circ \theta_x \\ &= V \circ (\theta_{X^* \circ \theta_x} \circ \theta_x) \\ &= V \circ \theta_{X^* \circ \theta_x + x} \\ &= V \circ \theta_{-x+x} = V, \end{aligned} \tag{6.2.10}$$

where for the fourth equality we use (6.2.9). Thus

$$\mathbf{E}\left[\frac{1}{|V^*|}\right] = \lambda = \frac{1}{\mathbf{E}^0[|V|]}, \tag{6.2.11}$$

where the second equality is due to (6.2.7). Jensen's inequality allows one to conclude:

$$\mathbf{E}\left[\frac{1}{|V^*|}\right] \geq \frac{1}{\mathbf{E}[|V^*|]} \Rightarrow \mathbf{E}^0[|V|] \leq \mathbf{E}[|V^*|].$$

□

The relation (6.2.11) may be generalized as follows.

Corollary 6.2.17. *Let Φ be a stationary and simple point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. Then the following results hold true.*

(i) For any $\mathbb{R}_+$-valued function g which is measurable on the set of closed sets of $\mathbb{R}^d$ (an algebra on this set will be introduced later), we have

$$\mathbf{E}\left[g\left(V^*\right) \times \mathbf{1}\left\{\Phi \neq 0\right\}\right] = \frac{\mathbf{P}\left(\Phi \neq 0\right)}{\mathbf{E}^0\left[|V|\right]} \mathbf{E}^0\left[|V| \, g\left(V\right)\right]. \tag{6.2.12}$$

(ii) Assume that $\mathbf{P}\left(\Phi \neq 0\right) = 1$. Then on the σ-algebra generated by V^*, $\mathbf{P}$ is absolutely-continuous w.r.t. $\mathbf{P}^0$ with

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{P}^0} = \frac{|V^*|}{\mathbf{E}^0\left[|V^*|\right]}.$$

Proof. Indeed, applying Theorem 6.2.8 to $f \equiv g\left(V^*\right)$ we get

$$\begin{aligned} \mathbf{E}\left[g\left(V^*\right) \times \mathbf{1}\left\{\Phi \neq 0\right\}\right] &= \lambda \mathbf{E}^0\left[\int_V g\left(V^* \circ \theta_x\right) \mathrm{d}x\right] \\ &= \lambda \mathbf{E}^0\left[\int_V g\left(V\right) \mathrm{d}x\right] \\ &= \lambda \mathbf{E}^0\left[|V| \, g\left(V\right)\right] = \frac{\mathbf{P}\left(\Phi \neq 0\right)}{\mathbf{E}^0\left[|V|\right]} \mathbf{E}^0\left[|V| \, g\left(V\right)\right], \end{aligned}$$

where the second equality follows from (6.2.10) and the last one is due to (6.2.7).
(ii) Observe that, under $\mathbf{P}^0$, $X^* = 0$ a.s. so that $V = V^*$. Hence the last relation (6.2.12) can be rewritten as

$$\mathbf{E}\left[g\left(V^*\right)\right] = \frac{\mathbf{E}^0\left[\left|V^*\right| g\left(V^*\right)\right]}{\mathbf{E}^0\left[\left|V^*\right|\right]} = \mathbf{E}^0\left[\frac{\left|V^*\right|}{\mathbf{E}^0\left[\left|V^*\right|\right]} g\left(V^*\right)\right],$$

from which the announced result follows. $\square$

Example 6.2.18. Cell sizes for Poisson on $\mathbb{R}$. *Consider a homogeneous Poisson point process Φ on $\mathbb{R}$ with intensity $\lambda \in \mathbb{R}_+^*$, and such that $\mathbf{P}(\Phi \neq 0) = 1$. Let $\{T_n\}_{n\in\mathbb{Z}}$ be its points in the increasing order with the usual convention $T_0 \leq 0 < T_1$. By Slivnyak's theorem 6.1.31(iii), $\mathbf{P}_\Phi^0 = \mathbf{P}_{\Phi+\delta_0}$. Then under $\mathbf{P}^0$, T_1 and $|T_{-1}|$ are independent exponential random variables with intensity λ, thus*

$$\mathbf{E}^0\left[|V|\right] = \mathbf{E}^0\left[\frac{T_1 - T_{-1}}{2}\right] = \frac{1}{2}\left(\mathbf{E}^0\left[T_1\right] + \mathbf{E}^0\left[|T_{-1}|\right]\right) = \frac{1}{\lambda},$$

which is a particular case of (6.2.7). Similarly,

$$\mathbf{E}^0\left[|V|^2\right] = \mathbf{E}^0\left[\frac{\left(T_1 - T_{-1}\right)^2}{2}\right] = \frac{3}{2\lambda^2},$$

thus, by (6.2.12)

$$\mathbf{E}\left[|V^*|\right] = \frac{\mathbf{E}^0\left[|V|^2\right]}{\mathbf{E}^0\left[|V|\right]} = \frac{3}{2\lambda} = \frac{3}{2}\mathbf{E}^0\left[|V|\right].$$

Example 6.2.19. Selected Voronoi cells. *Let Φ be a stationary simple point process on $\mathbb{R}^d$, with intensity $\lambda \in \mathbb{R}_+^*$, and such that $\mathbf{P}(\Phi \neq 0) = 1$. Let V be the virtual cell defined by (6.2.1) and $\tilde{V}(X)$ be the Voronoi cell associated to X defined by (6.2.2). Let $\{Y(t)\}_{t\in\mathbb{R}^d}$ be a stochastic process with values in $\{0,1\}$ compatible with $\{\theta_t\}_{t\in\mathbb{R}^d}$, and*

$$Z = \bigcup_{X\in\Phi:Y(X)=1} \tilde{V}(X),$$

(that is the union of the Voronoi cells whose atoms X satisfy the property $Y(X) = 1$). Then $\mathbf{P}(0 \in Z)$, which is called the volume fraction *of Z, equals*

$$\mathbf{P}(0 \in Z) = \frac{\mathbf{E}^0\left[|V| \mathbf{1}\{Y(0) = 1\}\right]}{\mathbf{E}^0\left[|V|\right]}.$$

Indeed, applying the inverse formula (6.2.4) to $f = \mathbf{1}\{0 \in Z\}$ and noting that

$$Z \circ \theta_x = \bigcup_{X\in\Phi\circ\theta_x:Y(X)=1} \tilde{V}(X) = \bigcup_{X\in\Phi:Y(X)=1} \left(\tilde{V}(X) - x\right) = Z - x$$

and

$$f \circ \theta_x = \mathbf{1}\{0 \in Z \circ \theta_x\} = \mathbf{1}\{0 \in Z - x\} = \mathbf{1}\{x \in Z\},$$

we get

$$\begin{aligned}
\mathbf{P}(0 \in Z) &= \mathbf{E}[f] \\
&= \lambda \mathbf{E}^0 \left[\int_V f \circ \theta_x \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 \left[\int_V \mathbf{1}\{x \in Z\} \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 \left[\int_V \mathbf{1}\{Y(0) = 1\} \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 [|V| \mathbf{1}\{Y(0) = 1\}] = \frac{\mathbf{E}^0 [|V| \mathbf{1}\{Y(0) = 1\}]}{\mathbf{E}^0[|V|]}.
\end{aligned}$$

where the last equality is due to (6.2.7).

6.2.4 Particular case of the line

In the particular case of the real line (i.e., $d = 1$), the inversion formula (6.2.4) reads

$$\mathbf{E}[f \times \mathbf{1}\{\Phi \neq 0\}] = \lambda \mathbf{E}^0 \left[\int_{-\frac{T_{-1}}{2}}^{\frac{T_1}{2}} f \circ \theta_x \mathrm{d}x \right],$$

where $\{T_n\}_{n \in \mathbb{Z}}$ are points of Φ in the increasing order with the usual convention $T_0 \leq 0 < T_1$. The points T_0 and T_1 are called called *backward* and *forward* recurrence times respectively.

We give now an alternative inversion formula which may be more useful in some cases than the above one.

Corollary 6.2.20. *Let Φ be a simple stationary point process on $\mathbb{R}$ with intensity $\lambda \in \mathbb{R}_+^*$. Then for all measurable functions $f : \Omega \to \mathbb{R}_+$,*

$$\mathbf{E}[f \times \mathbf{1}\{\Phi \neq 0\}] = \lambda \mathbf{E}^0 \left[\int_0^{T_1} f \circ \theta_x \mathrm{d}x \right],$$

where T_1 is the forward recurrence time of Φ.

Proof. As usual on $\mathbb{R}$, we enumerate the points of Φ in the increasing order in such a way that $T_0 \leq 0 < T_1$. Applying Theorem 6.2.9 with

$$h(x, \omega) := \mathbf{1}\{\Phi \neq 0, x \in [T_0, 0)\},$$

we get

$$\begin{aligned}
\mathbf{E}[f \times \mathbf{1}\{\Phi \neq 0\}] &= \lambda \mathbf{E}^0 \left[\int_{\mathbb{R}} f(\theta_{-x}\omega) \mathbf{1}\{x \in [T_0 \circ \theta_{-x}, 0)\} \, \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 \left[\int_{-\infty}^{0} f(\theta_{-x}\omega) \mathbf{1}\{x \geq T_0 \circ \theta_{-x}\} \, \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 \left[\int_{0}^{\infty} f(\theta_{x}\omega) \mathbf{1}\{-x \geq T_0 \circ \theta_{x}\} \, \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 \left[\int_{0}^{\infty} f(\theta_{x}\omega) \mathbf{1}\{-x = T_0 \circ \theta_{x}\} \, \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 \left[\int_{0}^{\infty} f(\theta_{x}\omega) \mathbf{1}\{T_1 - x > 0\} \, \mathrm{d}x \right],
\end{aligned}$$

where the fourth equality follows from the fact that $T_0 \circ \theta_x \geq -x$ and the last one is due to the fact that θ_x translates the point of Φ by $-x$. □

Corollary 6.2.21. Backward and forward recurrence times. *Let Φ be a simple stationary point process on $\mathbb{R}$ with intensity $\lambda \in \mathbb{R}_+^*$, and such that $\mathbf{P}(\Phi \neq 0) = 1$. Let T_0 and T_1 be the backward and forward recurrence times of Φ respectively. Then*

$$\mathbf{P}(-T_0 > s, T_1 > t) = \lambda \int_{t+s}^{\infty} [1 - F_0(x)] \, \mathrm{d}x, \tag{6.2.13}$$

where $F_0(x) = \mathbf{P}^0(T_1 \leq x)$. In particular, $\mathbf{P}$-almost surely, Φ has no atom at the origin.

Proof. We aim to determine the probability distribution of (T_0, T_1) under $\mathbf{P}$. Applying Corollary 6.2.20 with

$$f(\omega) = \mathbf{1}\{-T_0 > s, T_1 > t\},$$

we get

$$\begin{aligned}
\mathbf{P}(-T_0 > s, T_1 > t) &= \mathbf{E}[\mathbf{1}\{-T_0 > s, T_1 > t\}] \\
&= \lambda \mathbf{E}^0 \left[\int_0^{T_1} \mathbf{1}\{-T_0 \circ \theta_x > s, T_1 \circ \theta_x > t\} \, \mathrm{d}x \right] \\
&= \lambda \mathbf{E}^0 \left[\int_0^{T_1} \mathbf{1}\{x > s, T_1 - x > t\} \, \mathrm{d}x \right] \\
&= \lambda \int_0^{\infty} \mathbf{P}^0 \left((T_1 - t - s)^+ > x \right) \mathrm{d}x \\
&= \lambda \int_0^{\infty} \mathbf{P}^0 (T_1 > x + t + s) \, \mathrm{d}x \\
&= \lambda \int_{t+s}^{\infty} \mathbf{P}^0 (T_1 > x) \, \mathrm{d}x = \lambda \int_{t+s}^{\infty} [1 - F_0(x)] \, \mathrm{d}x.
\end{aligned}$$

Taking $s = t = 0$, we get

$$\mathbf{P}\left(T_0 < 0, T_1 > 0\right) = \lambda \int_0^\infty \mathbf{P}^0\left(T_1 > x\right) \mathrm{d}x = \lambda \mathbf{E}^0\left[T_1\right] = 1,$$

where the last equality follows from (6.2.7). □

Taking $s = 0$ in Equation (6.2.13), we get

$$\mathbf{P}\left(T_1 > t\right) = \lambda \int_t^\infty \left[1 - F_0\left(x\right)\right] \mathrm{d}x$$

and taking $t = 0$, we get

$$\mathbf{P}\left(-T_0 > s\right) = \lambda \int_s^\infty \left[1 - F_0\left(x\right)\right] \mathrm{d}x.$$

Thus T_1 and $-T_0$ are identically distributed under $\mathbf{P}$. Moreover, applying Corollary 6.2.17(i), we get

$$\mathbf{P}\left(-T_0 + T_1 \leq r\right) = \lambda \mathbf{E}^0\left[\mathbf{1}\left\{T_1 \leq r\right\} T_1\right].$$

6.2.5 Renewal processes

Definition 6.2.22. *Consider a stationary simple point process Φ on $\mathbb{R}$ with intensity $\lambda \in \mathbb{R}_+^*$ and such that $\mathbf{P}\left(\Phi \neq 0\right) = 1$. Let $\left\{T_n\right\}_{n\in\mathbb{Z}}$ be its points in the increasing order with the usual convention $T_0 \leq 0 < T_1$. The point process Φ is called a* renewal process *if, under $\mathbf{P}^0$, the sequence $\left\{S_n\right\}_{n\in\mathbb{Z}}$ defined by*

$$S_n = T_{n+1} - T_n, \quad n \in \mathbb{Z} \tag{6.2.14}$$

(called inter-events*) is i.i.d.*

A typical realization of a renewal process under the Palm and stationary probabilities is illustrated in Figure 6.3.

Proposition 6.2.23. *Let Φ be a renewal process on $\mathbb{R}$ with point $\left\{T_n\right\}_{n\in\mathbb{Z}}$ in the increasing order such that $T_0 \leq 0 < T_1$ and let S_n be defined by (6.2.14). Then the following results hold.*

(i) The sequence $S^* = \left\{S_n\right\}_{n\in\mathbb{Z}^*}$ has the same probability distribution under $\mathbf{P}$ and $\mathbf{P}^0$.

(ii) S^* and $\left(T_0, T_1\right)$ are independent under $\mathbf{P}$.

(iii) Under $\mathbf{P}$, the sequence $\left\{S_n\right\}_{n\in\mathbb{Z}^*}$ is i.i.d. with values in $\mathbb{R}_+^*$ and the probability distribution of $\left(T_0, T_1\right)$ is given by (6.2.13), where $F_0\left(x\right) = \mathbf{P}^0\left(T_1 \leq x\right),\ x \in \mathbb{R}_+$.

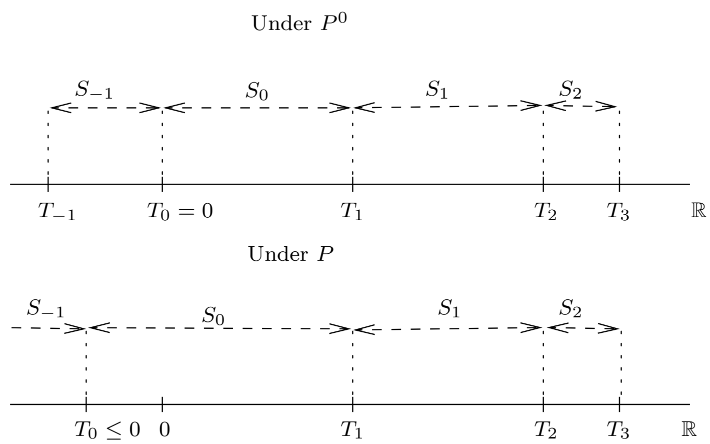

Figure 6.3: Renewal process viewed under Palm and stationary probabilities respectively

Proof. (i) Note that for all $x \in [0, T_1)$, $S^* \circ \theta_x = S^*$. Thus, by Theorem 6.2.8, for all measurable functions $f : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}_+$,

$$
\begin{aligned}
\mathbf{E}\left[f\left(S^*\right)\right] &= \lambda \mathbf{E}^0\left[\int_0^{T_1} f\left(S^* \circ \theta_x\right) \mathrm{d}x\right] \\
&= \lambda \mathbf{E}^0\left[\int_0^{T_1} f\left(S^*\right) \mathrm{d}x\right] \\
&= \lambda \mathbf{E}^0\left[T_1 f\left(S^*\right)\right] \\
&= \lambda \mathbf{E}^0\left[T_1\right] \mathbf{E}^0\left[f\left(S^*\right)\right] = \mathbf{E}^0\left[f\left(S^*\right)\right].
\end{aligned}
$$

Therefore S^* has the same probability distribution under $\mathbf{P}$ and $\mathbf{P}^0$. (ii) For all $x \in [0, T_1)$, $T_0 \circ \theta_x = T_0 - x$ and $T_1 \circ \theta_x = T_1 - x$. Then, again by Theorem 6.2.8, for all measurable functions $g : \mathbb{R}^2 \to \mathbb{R}_+$,

$$\begin{aligned}
\mathbf{E}\left[g\left(T_0, T_1\right) f\left(S^*\right)\right] &= \lambda \mathbf{E}^0\left[\int_0^{T_1} g\left(T_0 \circ \theta_x, T_1 \circ \theta_x\right) f\left(S^* \circ \theta_x\right) \mathrm{d}x\right] \\
&= \lambda \mathbf{E}^0\left[\int_0^{T_1} g\left(T_0 - x, T_1 - x\right) f\left(S^*\right) \mathrm{d}x\right] \\
&= \lambda \mathbf{E}^0\left[\left(\int_0^{T_1} g\left(-x, S_1 - x\right) \mathrm{d}x\right) f\left(S^*\right)\right] \\
&= \lambda \mathbf{E}^0\left[\left(\int_0^{T_1} g\left(-x, S_1 - x\right) \mathrm{d}x\right)\right] \mathbf{E}^0\left[f\left(S^*\right)\right] \\
&= \lambda \mathbf{E}^0\left[\left(\int_0^{T_1} g\left(T_0 \circ \theta_x, T_1 \circ \theta_x\right) \mathrm{d}x\right)\right] \mathbf{E}^0\left[f\left(S^*\right)\right] \\
&= \mathbf{E}\left[g\left(T_0, T_1\right)\right] \mathbf{E}^0\left[f\left(S^*\right)\right] = \mathbf{E}\left[g\left(T_0, T_1\right)\right] \mathbf{E}\left[f\left(S^*\right)\right],
\end{aligned}$$

where the fourth equality is due to the independence of S^* and S_0 under $\mathbf{P}^0$, the sixth equality follows from Theorem 6.2.8, and the last equality follows from the first part of the proposition. Thus S^* and (T_0, T_1) are independent under $\mathbf{P}$.
(iii) Since the a renewal process is simple by definition, it follows that $S_n \neq 0$, and thus S_n takes its values in $\mathbb{R}_+^*$. Item (i) and Corollary 6.2.21 imply the stated results. □

6.2.6 Direct and inverse construction of Palm theory

Let $(\Omega, \mathcal{A})$ be a measurable space, let $\{\theta_t\}_{t\in\mathbb{R}^d}$ be a measurable flow on $(\Omega, \mathcal{A})$ and.let Φ be a simple point process on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$.

Direct construction: from stationary to Palm probability

Let $\mathbf{P}$ be a probability defined on $(\Omega, \mathcal{A})$. Assume that $\mathbf{P}$ is invariant with respect to the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$, then $\mathbf{P}$ is called *stationary probability*. If the intensity λ of Φ satisfies $0 < \lambda < \infty$, then, by Proposition 6.1.20(iii), there is a unique probability $\mathbf{P}^0$ on $(\Omega, \mathcal{A})$, called the *Palm probability of* Φ defined by (6.1.10). This Palm probability is *not* compatible with the flow. We have already proved that it satisfies the following properties:

(C1) $\mathbf{P}^0\{\Omega_0\} = 1$, where $\Omega_0 = \{\omega \in \Omega : 0 \in \Phi\}$; cf. Theorem 6.1.31(i).

(C2) $0 < \mathbf{E}^0[|V|] < \infty$, where $V = \{y \in \mathbb{R}^d : |y| \leq \min_{z\in\Phi} |y - z|\}$; cf. Corollary 6.2.12.

(C3) $\mathbf{P}^0$ is *unimodular* with respect to Φ in the sense that it satisfies (6.1.17) for any nonnegative measurable function g on $\mathbb{R}^d \times \Omega$; cf. Corollary 6.1.35.

(C4) For any bijective point map h related to Φ, $\mathbf{P}^0\theta_h^{-1} = \mathbf{P}^0$; cf. *Mecke's invariance theorem* (Theorem 6.1.42).

Inverse construction: from Palm to stationary probability

The following result extends the Slivnyak inverse construction in the line of [8, §1.3.5] to the multidimensional case.

Proposition 6.2.24. Inverse construction: from Palm to stationary probability. *Let $(\Omega, \mathcal{A})$ be a measurable space, let $\{\theta_t\}_{t\in\mathbb{R}^d}$ be a measurable flow on $(\Omega, \mathcal{A})$, and let Φ be a simple point process on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Let $\mathbf{P}^0$ be a probability on $(\Omega, \mathcal{A})$ satisfying conditions C1, C2, and one of the two conditions: C3 (unimodularity) or C4 (Mecke's invariance) above. Then, there exists a probability $\mathbf{P}$ on $(\Omega, \mathcal{A})$ invariant with respect to the flow, such that $\mathbf{P}^0$ is the Palm probability of Φ in the stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}, \mathbf{P})$; i.e., (6.1.10) holds true. Moreover $\mathbf{P}$ is unique on $\{\omega \in \Omega : \Phi(\omega) \neq 0\}$ and can be expressed using the inverse formula (6.2.4).*

Proof. Mecke [69] and Neveu [79, Proposition II.11 p.325] proved the result with the unimodularity assumption. Heveling and Last [46, Proof of Theorem 4.1] have shown that the Mecke's invariance (C3) implies unimodularity (C4). □

Remark 6.2.25. Bibliographic notes. *An extension of the above result to random measures is presented in [79, Proposition II.11 p.325]; it relies on the Ryll-Nardzewski and Slivnyak inversion formula (6.2.6).*

6.3 Further properties of Palm probabilities

6.3.1 Independence

Lemma 6.3.1. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework. Let Φ and Φ' be two* independent *random measures on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Assume that Φ' has finite and non-nul intensity and let $\mathbf{P}^{0'}$ be its Palm probability. Then the distributions of Φ under $\mathbf{P}^{0'}$ and under $\mathbf{P}$ are identical. Moreover Φ and Φ' are independent under $\mathbf{P}^{0'}$.*

Proof. Let λ' be the intensity of Φ'. For all measurable functions $f : \mathbb{R}^d \to \bar{\mathbb{R}}_+$,

$$\begin{aligned}
&\mathbf{E}^{0'}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right)\right] \\
&= \frac{1}{\lambda'}\mathbf{E}\left[\int_{(0,1]^d} \exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi\circ\theta_y(\mathrm{d}x)\right)\Phi'(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda'}\mathbf{E}\left[\int_{(0,1]^d} \exp\left(-\int_{\mathbb{R}^d} S_{-y}f(x)\,\Phi(\mathrm{d}x)\right)\Phi'(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda'}\int_{(0,1]^d} \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} S_{-y}f(x)\,\Phi(\mathrm{d}x)\right)\right] M_{\Phi'}(\mathrm{d}y) \\
&= \int_{(0,1]^d} \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi\circ\theta_y(\mathrm{d}x)\right)\right]\mathrm{d}y \\
&= \int_{(0,1]^d} \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right)\right]\mathrm{d}y \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right)\right],
\end{aligned}$$

where the first equality follows from the very definition of Palm, the third one is due to the independence of Φ and Φ' and Proposition 1.4.2, and the fifth one from stationarity. By Corollary 1.2.2, the distribution of a random measure is characterized by its Laplace transform. Then the distributions of Φ under $\mathbf{P}^{0'}$ and under $\mathbf{P}$ are identical. On the other hand, for all measurable functions $f, g : \mathbb{R}^d \to \bar{\mathbb{R}}_+$,

$$\begin{aligned}
&\mathbf{E}^{0'}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right)\exp\left(-\int_{\mathbb{R}^d} g(x)\,\Phi'(\mathrm{d}x)\right)\right] \\
&= \frac{1}{\lambda'}\mathbf{E}\left[\int_{(0,1]^d} \exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi\circ\theta_y(\mathrm{d}x)\right)\exp\left(-\int_{\mathbb{R}^d} g(x)\,\Phi'\circ\theta_y(\mathrm{d}x)\right)\Phi'(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda'}\mathbf{E}\left[\int_{(0,1]^d} \exp\left(-\int_{\mathbb{R}^d} S_{-y}f(x)\,\Phi(\mathrm{d}x)\right)\exp\left(-\int_{\mathbb{R}^d} S_{-y}g(x)\,\Phi'(\mathrm{d}x)\right)\Phi'(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda'}\int_{(0,1]^d} \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} S_{-y}f(x)\,\Phi(\mathrm{d}x)\right)\right]\mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} S_{-y}g(x)\,\Phi'(\mathrm{d}x)\right)\Phi'(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda'}\int_{(0,1]^d} \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi\circ\theta_y(\mathrm{d}x)\right)\right]\mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} g(x)\,\Phi'\circ\theta_y(\mathrm{d}x)\right)\Phi'(\mathrm{d}y)\right] \\
&= \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right)\right]\mathbf{E}^{0'}\left[\exp\left(-\int_{\mathbb{R}^d} g(x)\,\Phi'(\mathrm{d}x)\right)\right] \\
&= \mathbf{E}^{0'}\left[\exp\left(-\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right)\right]\mathbf{E}^{0'}\left[\exp\left(-\int_{\mathbb{R}^d} g(x)\,\Phi'(\mathrm{d}x)\right)\right],
\end{aligned}$$

where the third equality is due to Proposition 1.4.2. This shows the independence of Φ and Φ' under $\mathbf{P}^{0'}$. □

Lemma 6.3.2. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework. Let Φ be a random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ with finite and non-null intensity and Palm probability $\mathbf{P}^0$. Let $\{Z(t)\}_{t\in\mathbb{R}^d}$ be a stochastic process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and* independent *of Φ. Then the distributions of $\{Z(t)\}_{t\in\mathbb{R}^d}$ under $\mathbf{P}^0$ and under $\mathbf{P}$ are identical. Moreover $\{Z(t)\}_{t\in\mathbb{R}^d}$ and Φ are independent under $\mathbf{P}^0$.*

Proof. Let λ be the intensity of Φ. The proof follows in the same lines as the proof of Lemma 6.3.1. We give only the details for the equality of the distributions of $\{Z(t)\}_{t\in\mathbb{R}^d}$ under $\mathbf{P}^0$ and under $\mathbf{P}$. For all $t_1, \ldots, t_k \in \mathbb{R}^d$ and $a_1, \ldots, a_k \in \mathbb{R}_+$,

$$\begin{aligned}
\mathbf{E}^0\left[\exp\left(-\sum_{i=1}^k a_i Z(t_i)\right)\right] &= \frac{1}{\lambda}\mathbf{E}\left[\int_{(0,1]^d} \exp\left(-\sum_{i=1}^k a_i Z\circ\theta_y(t_i)\right)\Phi(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda}\mathbf{E}\left[\int_{(0,1]^d} \exp\left(-\sum_{i=1}^k a_i Z(y+t_i)\right)\Phi(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda}\int_{(0,1]^d} \mathbf{E}\left[\exp\left(-\sum_{i=1}^k a_i Z(y+t_i)\right)\right] M_\Phi(\mathrm{d}y) \\
&= \int_{(0,1]^d} \mathbf{E}\left[\exp\left(-\sum_{i=1}^k a_i Z(t_i)\right)\right] \mathrm{d}y \\
&= \mathbf{E}\left[\exp\left(-\sum_{i=1}^k a_i Z(t_i)\right)\right],
\end{aligned}$$

where the third equality is due to Proposition 1.4.2. ☐

Corollary 6.3.3. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework. Let Φ be a random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ with finite and non-null intensity and Palm probability $\mathbf{P}^0$. Let Y be a random variable such that the stochastic process $\{Y\circ\theta_t\}_{t\in\mathbb{R}^d}$ is independent of Φ, then*

$$\mathbf{E}^0[Y] = \mathbf{E}[Y].$$

Moreover Y and Φ are independent under $\mathbf{P}^0$.

Example 6.3.4. Independence counterexample. *Note that in the above corollary it is not enough that Y and Φ are independent. Here is a counterexample. Let Φ be a stationary point process on $\mathbb{R}^d$ and let*

$$Y = \mathbf{1}\{\Phi(0) \geq 1\}.$$

Since $\mathbf{P}$-almost surely $Y = 0$, then Y and Φ are independent. Nevertheless,

$$\mathbf{E}^0[Y] \neq \mathbf{E}[Y].$$

Indeed, by Lemma 6.1.19,

$$\mathbf{E}\,[Y] = \mathbf{P}\,(\Phi\,(0) \geq 1) = 0,$$

whereas by Theorem 6.1.31(i),

$$\mathbf{E}^0\,[Y] = \mathbf{P}^0\,(\Phi\,(\{0\}) \geq 1) = 1.$$

6.3.2 Superposition

Let $\Phi_1, \ldots, \Phi_n$ be random measures on $\mathbb{R}^d$ defined on a common probability space, and let

$$\Phi = \sum_{k=1}^{n} \Phi_k$$

be their sum. The term *superposition* is also used, particularly so for point processes (see §2.2.1).

In this subsection, we focus on the case where all these random measures are defined on a stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}, \mathbf{P})$ and compatible.

A simple particular case is that where these random measures are independent. In this case, a natural stationary framework for this collection of random measure is $(\Omega, \mathcal{A}, \mathbf{P})$, the product of the canonical probability spaces associated to $\Phi_1, \ldots, \Phi_n$; that is $\Omega = \bar{\mathbb{M}}\left(\mathbb{R}^d\right)^n$, with the associated product σ-algebra $\mathcal{A}$, and $\mathbf{P} = \mathbf{P}_{\Phi_1} \otimes \cdots \otimes \mathbf{P}_{\Phi_n}$. We may consider Φ as a random measure on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$; that is, for $\omega = (\omega_1, \ldots, \omega_n) \in \bar{\mathbb{M}}\left(\mathbb{R}^d\right)^n$,

$$\Phi\,(\omega) = \sum_{k=1}^{n} \omega_k.$$

Assume each of the random measures $\Phi_1, \ldots, \Phi_n$ is stationary. Let λ_k denote the intensity of Φ_k and let $\mathbf{P}^0_{\Phi_k}$ denote its Palm probability on $\bar{\mathbb{M}}\left(\mathbb{R}^d\right)$. Define the flow θ_t on $\Omega = \bar{\mathbb{M}}\left(\mathbb{R}^d\right)^n$ by

$$\theta_t \omega = (S_t \omega_1, \ldots, S_t \omega_n)\,, \quad \omega \in \bar{\mathbb{M}}\left(\mathbb{R}^d\right)^n.$$

It is immediate that θ_t preserves $\mathbf{P}$. We hence have a stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}, \mathbf{P})$ on which $\Phi(\omega) = \sum_{i=1}^n \omega_i$ and $\Phi_k(\omega) = \omega_k$, for all $k = 1, \ldots, n$, are compatible random measures. Let $\mathbf{P}^0$ denote the Palm probability of Φ, and $\mathbf{P}^0_k$ that of Φ_k, $k = 1, \ldots, n$. Note that the last two Palm probabilities are defined on Ω.

Proposition 6.3.5. *Let $\Phi_1, \ldots, \Phi_n$ be compatible random measures defined on the stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}, \mathbf{P})$. Assume that their intensities $\lambda_1, \ldots, \lambda_n$ are in $\mathbb{R}_+^*$. Then the Palm probability of $\Phi = \sum_{k=1}^n \Phi_k$ is given by*

$$\mathbf{P}^0 = \sum_{k=1}^{n} \frac{\lambda_k}{\lambda} \mathbf{P}^0_k, \tag{6.3.1}$$

where $\lambda = \sum_{k=1}^{n} \lambda_k$ and $\mathbf{P}_k^0$ is the Palm probability of Φ_k, $k = 1, \ldots, n$. In the particular case where the measures $\Phi_1, \ldots, \Phi_n$ are independent and the stationary framework is the product space discussed above,

$$\mathbf{P}_k^0 = \left(\bigotimes_{j=1}^{k-1} \mathbf{P}_{\Phi_j} \right) \otimes \mathbf{P}_{\Phi_k}^0 \otimes \left(\bigotimes_{j=k+1}^{n} \mathbf{P}_{\Phi_j} \right). \tag{6.3.2}$$

Proof. Let $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$ with $|B| > 0$. For any $A \in \mathcal{A}$,

$$\begin{aligned}
\mathbf{P}^0(A) &= \frac{1}{\lambda |B|} \mathbf{E}\left[\int_B \mathbf{1}\{\theta_x \omega \in A\} \Phi(\mathrm{d}x) \right] \\
&= \frac{1}{\lambda |B|} \sum_k \mathbf{E}\left[\int_B \mathbf{1}\{\theta_x \omega \in A\} \Phi_k(\mathrm{d}x) \right] \\
&= \frac{1}{\lambda |B|} \sum_k \lambda_k |B| \mathbf{P}_k^0(A) = \sum_{k=1}^{n} \frac{\lambda_k}{\lambda} \mathbf{P}_k^0(A),
\end{aligned}$$

which proves (6.3.1). Assume now that $\Phi_1, \ldots, \Phi_n$ are independent and the framework as as proposed. Then, for all $A_1, \ldots, A_n \in \bar{\mathcal{M}}$,

$$\mathbf{P}_k^0\left(\prod_{j=1}^{n} A_j \right) = \mathbf{P}_k^0(\Phi_1 \in A_1, \ldots, \Phi_n \in A_n) = \left[\prod_{j=1, j\neq k}^{n} \mathbf{P}_{\Phi_j}(A_j) \right] \mathbf{P}_{\Phi_k}^0(A_k),$$

where the second equality follows from Lemma 6.3.1. This proves (6.3.2). □

Example 6.3.6. *Let $\Phi_1, \ldots, \Phi_n$ be independent stationary point processes on $\mathbb{R}$ with respective intensities $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+^*$. Let T_1^k be the first positive point of Φ_k, and let F_k and F_k^0 be the cumulative distribution functions of T_1^k under $\mathbf{P}_{\Phi_k}$ and $\mathbf{P}_{\Phi_k}^0$ respectively. Then the cumulative distribution function F^0 of the first positive point T_1 of the superposition $\Phi = \sum_{k=1}^{n} \Phi_k$ under its Palm probability equals*

$$\begin{aligned}
F^0(x) &= 1 - \mathbf{P}^0(T_1 > x) \\
&= 1 - \sum_{k=1}^{n} \frac{\lambda_k}{\lambda} \mathbf{P}_k^0(T_1 > x) \\
&= 1 - \sum_{k=1}^{n} \frac{\lambda_k}{\lambda} \mathbf{P}_k^0\left(T_1^j > x, \forall j \in \{1, \ldots, n\} \right) \\
&= 1 - \sum_{k=1}^{n} \frac{\lambda_k}{\lambda} \mathbf{P}_k^0\left(T_1^k > x \right) \prod_{j=1, j\neq k}^{n} \mathbf{P}\left(T_1^j > x \right) \\
&= 1 - \sum_{k=1}^{n} \left[\frac{\lambda_k}{\lambda} \left(1 - F_k^0(x)\right) \prod_{j=1, j\neq k}^{n} \left(1 - F_j(x)\right) \right].
\end{aligned}$$

where the second equality follows from (6.3.1) and the fourth equality is due to Lemma 6.3.1.

6.3.3 Neveu's exchange formula

Theorem 6.3.7. Neveu's exchange formula. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and Φ, Φ' be two point processes compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and with respective intensities $\lambda, \lambda' \in \mathbb{R}_+^*$ and respective Palm probabilities $\mathbf{P}^0$ and $\mathbf{P}^{0'}$. Assume that*

$$\arg\min_{X\in\Phi} |X| \text{ exists and is unique } \mathbf{P}^{0'}\text{-almost surely.} \tag{6.3.3}$$

Then, for all measurable functions $f : \Omega \to \bar{\mathbb{R}}_+$,

$$\lambda'\mathbf{E}^{0\prime}[f] = \lambda\mathbf{E}^0\left[\int_V f\circ\theta_y \Phi'(\mathrm{d}y)\right], \tag{6.3.4}$$

where $\mathbf{E}^0$ and $\mathbf{E}^{0\prime}$ are the expectations with respect to the Palm probabilities of Φ and Φ', respectively, and V is the virtual cell of Φ defined by (6.2.1).

Proof. Applying the mass transport theorem 6.1.34 with

$$g(y,\omega) = \mathbf{1}\{y \in V(\omega)\} f\circ\theta_y(\omega),$$

we get

$$\begin{aligned}
\lambda\mathbf{E}^0\left[\int_V f\circ\theta_y\Phi'(\mathrm{d}y)\right] &= \lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d} g(-x,\theta_x\omega)\Phi(\mathrm{d}x)\right]\\
&= \lambda'\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d} \mathbf{1}\{-x\in V(\theta_x\omega)\} f(\omega)\Phi(\mathrm{d}x)\right]\\
&= \lambda'\mathbf{E}^{0\prime}\left[f(\omega)\int_{\mathbb{R}^d} \mathbf{1}\{-x\in V(\theta_x\omega)\}\Phi(\mathrm{d}x)\right].
\end{aligned}$$

We will now show that $\mathbf{P}^{0'}$- almost surely, $\int_{\mathbb{R}^d}\mathbf{1}\{-x\in V(\theta_x\omega)\}\Phi(\mathrm{d}x) = 1$, which will complete the proof. Indeed,

$$-x\in V(\theta_x\omega) \Leftrightarrow |x| \le \inf_{X\in\Phi\circ\theta_x} |-x-X| = \inf_{X\in\Phi}|X|. \tag{6.3.5}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d}\mathbf{1}\{-x\in V(\theta_x\omega)\}\Phi(\mathrm{d}x) &= \int_{\mathbb{R}^d}\mathbf{1}\left\{|x|\le\inf_{X\in\Phi}|X|\right\}\Phi(\mathrm{d}x)\\
&= \sum_{Y\in\Phi}\mathbf{1}\left\{|Y| = \inf_{X\in\Phi}|X|\right\},
\end{aligned}$$

which, $\mathbf{P}^{0'}$-almost surely, equals 1 by the assumption that $\arg\min_{X\in\Phi}|X|$ exists and is unique under $\mathbf{P}^{0'}$. □

Example 6.3.8. *Applying Theorem 6.3.7 with $f\equiv 1$, we get the mean number of points of Φ' in the typical cell of Φ*

$$\mathbf{E}^0[\Phi'(V)] = \frac{\lambda'}{\lambda}, \tag{6.3.6}$$

which may be interpreted by recalling that λ' is the mean number of points of Φ' per surface unit and that the mean surface of the the typical cell of Φ is $\frac{1}{\lambda}$.

The following example shows that Condition (6.3.3) is crucial for Neveu's exchange formula to hold.

Example 6.3.9. Doubled Poisson point process. *Let Φ' be a Poisson point process and $\Phi = 2\Phi'$, with intensities λ' and $\lambda = 2\lambda'$, espectively. Then $\mathbf{E}^0[\Phi'(V)] = 1 \neq \frac{\lambda'}{\lambda}$; i.e., the equality (6.3.6) does not hold. Indeed, Condition (6.3.3) fails for this particular example.*

Example 6.3.10. *Applying Theorem 6.3.7 to $f = \min_{X\in\Phi}|X|$, we get*

$$\begin{aligned}\lambda'\mathbf{E}^{0\prime}\left[\min_{X\in\Phi}|X|\right] &= \lambda\mathbf{E}^0\left[\int_V \min_{X\in\Phi\circ\theta_y}|X|\Phi'(\mathrm{d}y)\right]\\ &= \lambda\mathbf{E}^0\left[\int_V \min_{X\in\Phi}|X-y|\Phi'(\mathrm{d}y)\right] = \lambda\mathbf{E}^0\left[\int_V |y|\Phi'(\mathrm{d}y)\right],\end{aligned}$$

since under $\mathbf{P}^0$, $0\in\Phi$ and therefore, for all $y\in V$, $\min_{X\in\Phi}|X-y| = |y|$. Assume that $\lambda' = \lambda$, the above equation says that the mean minimal modulus of points of Φ under $\mathbf{P}^{0\prime}$ equals the mean sum *of modulus of points of Φ' lying in the typical cell of Φ under $\mathbf{P}^0$ as illustrated in Figure 6.4.*

Example 6.3.11. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}^2$ of intensity $\lambda\in\mathbb{R}_+^*$. Let $\Phi' = \ell^2$ be the Lebesgue measure. The Palm probability of the latter equals $\mathbf{P}^{0'} = \mathbf{P}$, by Example 6.1.24. Then Condition (6.3.3) holds true. Thus applying Theorem 6.3.7 to $f = |X^*|^\beta$ for some constant β (with $X^* = \arg\min_{X\in\Phi}|X|$), we get*

$$\mathbf{E}^0\left[\int_V |y|^\beta \mathrm{d}y\right] = \frac{1}{\lambda}\mathbf{E}\left[|X^*|^\beta\right],$$

where we use (6.2.9). Since Φ is Poisson $\mathbf{P}(|X^*|>r) = e^{-\lambda\pi r^2}$. Then

$$\begin{aligned}\mathbf{E}\left[|X^*|^\beta\right] &= \int_0^\infty \mathbf{P}\left(|X^*|^\beta > r\right)\mathrm{d}r\\ &= \int_0^\infty e^{-\lambda\pi r^{2/\beta}}\mathrm{d}r = \frac{\Gamma(1+\beta/2)}{(\lambda\pi)^{\beta/2}}.\end{aligned}\tag{6.3.7}$$

Thus

$$\mathbf{E}^0\left[\int_V |y|^\beta \mathrm{d}y\right] = \frac{1}{\lambda}\frac{\Gamma(1+\beta/2)}{(\lambda\pi)^{\beta/2}}.$$

Taking in particular $\beta = 1$, and using the fact that $\Gamma(3/2) = \sqrt{\pi}/2$, we get

$$\mathbf{E}^0\left[\int_V |y|\mathrm{d}y\right] = \frac{1}{2\lambda^{\frac{3}{2}}}.$$

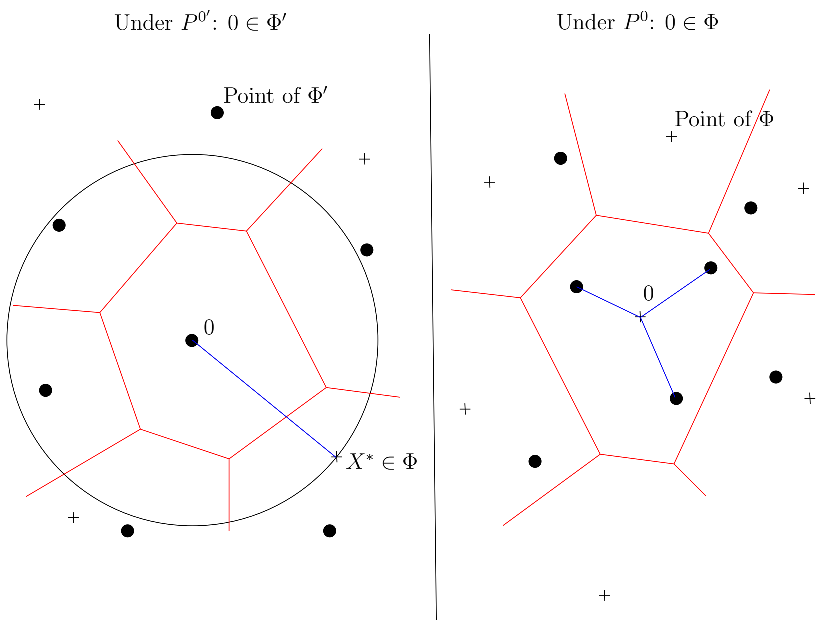

Figure 6.4: Mean minimal modulus of points of Φ under $P^{0\prime}$ equals the mean *sum* of modulus of points of Φ' lying in the typical cell of Φ under P^0

Corollary 6.3.12. *Under the conditions of Theorem 6.3.7, for all measurable functions* $g : \mathbb{R}^d \times \mathbb{M}\left(\mathbb{R}^d\right) \to \bar{\mathbb{R}}_+$,

$$\lambda' \mathbf{E}^{0\prime}\left[g\left(-X^*, S_{X^*}\Phi\right)\right] = \lambda \mathbf{E}^0\left[\int_V g\left(y, \Phi\right) \Phi'\left(\mathrm{d}y\right)\right], \tag{6.3.8}$$

where $X^* = \arg\min_{X \in \Phi} |X|$.

Proof. We apply the mass transport theorem 6.1.34 to the function

$$\tilde{g}\left(y, \omega\right) = \mathbf{1}\left\{y \in V\left(\omega\right)\right\} g\left(y, \Phi\left(\omega\right)\right).$$

Noting that

$$\tilde{g}\left(-x, \theta_x \omega\right) = \mathbf{1}\left\{-x \in V\left(\theta_x \omega\right)\right\} g\left(-x, \Phi\left(\theta_x \omega\right)\right),$$

we get

$$\begin{aligned}
\lambda \mathbf{E}^0 \left[\int_V g(y,\Phi)\,\Phi'(\mathrm{d}y) \right] &= \lambda' \mathbf{E}^{0'} \left[\int_{\mathbb{R}^d} \tilde{g}(-x, \theta_x \omega)\,\Phi(\mathrm{d}x) \right] \\
&= \lambda' \mathbf{E}^{0'} \left[\int_{\mathbb{R}^d} \mathbf{1}\{-x \in V(\theta_x \omega)\}\, g(-x, \Phi(\theta_x \omega))\,\Phi(\mathrm{d}x) \right] \\
&= \lambda' \mathbf{E}^{0'} \left[\int_{\mathbb{R}^d} 1 \left\{ |x| \le \min_{X \in \Phi} |X| \right\} g(-x, \Phi(\theta_x \omega))\,\Phi(\mathrm{d}x) \right] \\
&= \lambda' \mathbf{E}^{0'} \left[\int_{\mathbb{R}^d} 1\{x = X^*\}\, g(-x, \Phi(\theta_x \omega))\,\Phi(\mathrm{d}x) \right] \\
&= \lambda' \mathbf{E}^{0'} \left[g(-X^*, S_{X^*}\Phi) \right],
\end{aligned}$$

where we use (6.3.5) for the third equality the fact $X^* = \arg\min_{X \in \Phi} |X|$ is $\mathbf{P}^{0'}$-almost surely unique (as assumed in (6.3.3)) for the fourth one. □

Example 6.3.13. Wireless network. *The locations of base stations are represented by a simple stationary point process Φ on $\mathbb{R}^2$ with intensity $\lambda \in \mathbb{R}_+^*$ such that $X^* = \arg\min_{X \in \Phi} |X|$ exists and is unique $\mathbf{P}$-almost surely. The users locations are represented by a random measure Φ' on $\mathbb{R}^2$ with intensity $\lambda' \in \mathbb{R}_+^*$ which is jointly stationary with Φ. Assume that Φ and Φ' are independent and let $\mathbf{P}^0$ and $\mathbf{P}^{0'}$ be their respective Palm probabilities. By Lemma 6.3.1 the distributions of Φ under $\mathbf{P}^{0'}$ and under $\mathbf{P}$ are identical. Then condition (6.3.3) holds true.*

Let $g(y,\mu)$ be some nonnegative measurable function of the user location $y \in \mathbb{R}^2$ and the base station locations $\mu \in \mathbb{M}(\mathbb{R}^2)$ satisfying (6.3.13). Typically $g(y,\mu)$ denotes some quantity of interest *(for example propagation loss or signal to interference ratio) for a user located at $y \in \mathbb{R}^2$ with respect to the base station locations given by μ.*

Denoting $V = \{y \in \mathbb{R}^d : |y| \le \min_{X \in \Phi} |y - X|\}$ and applying Corollary 6.3.12, we get

$$\begin{aligned}
\mathbf{E}^0 \left[\int_V g(y,\Phi)\,\Phi'(\mathrm{d}y) \right] &= \frac{\lambda'}{\lambda} \mathbf{E}^{0'} \left[g(-X^*, S_{X^*}\Phi) \right] \\
&= \frac{\lambda'}{\lambda} \mathbf{E} \left[g(-X^*, S_{X^*}\Phi) \right] = \frac{\lambda'}{\lambda} \mathbf{E} \left[g(0,\Phi) \right].
\end{aligned}$$

where the second equality follows from Lemma 6.3.1 and the last equality follows from (6.3.13).

Assume now that Φ is Poisson. In this case, $\mathbf{P}(|X^*| > r) = e^{-\lambda \pi r^2}$.

Propagation loss*. Consider the case when the function g denotes the* propagation loss *between a user located at $y \in \mathbb{R}^2$ and his nearest base station; that is*

$$g(y,\Phi) = |y - X|^\beta, \quad y \in \tilde{V}(X), X \in \Phi,$$

for some given $\beta \in \mathbb{R}_+^*$. Then

$$\mathbf{E}^0 \left[\int_V |y|^\beta \,\Phi'(\mathrm{d}y) \right] = \frac{\lambda'}{\lambda} \mathbf{E} \left[|X^*|^\beta \right] = \frac{\lambda'}{\lambda} \frac{\Gamma(1+\beta/2)}{(\lambda \pi)^{\beta/2}}.$$

where the second equality follows in the same lines as in (6.3.7).

noindent ***Interference to signal ratio***. *Consider the case when the function g denotes the* interference to signal ratio*; that is*

$$g(y,\Phi) = |y-X|^{\beta} \sum_{Z\in\Phi\backslash\{X\}} \frac{1}{|y-Z|^{\beta}}, \quad y \in \tilde{V}(X), X\in\Phi,$$

for some given $\beta > 2$. Note that

$$g(0,\Phi) = |X^*|^{\beta} \sum_{Z\in\Phi\backslash\{X^*\}} \frac{1}{|Z|^{\beta}}.$$

On the other hand, conditionally to X^*, the point process $\Phi\backslash\{X^*\}$ is Poisson with intensity measure

$$\lambda(\mathrm{d}x) = \lambda\mathbf{1}\{|x| > |X^*|\}\,\mathrm{d}x.$$

This follows from the strong Markov property of Poisson point processes; cf. Theorem 12.1.3 below. Then, by the Campbell averaging theorem 1.2.5,

$$\begin{aligned}\mathbf{E}\left[\sum_{Z\in\Phi\backslash\{X^*\}} \frac{1}{|Z|^{\beta}}\middle| X^*\right] &= \int \frac{1}{|x|^{\beta}}\lambda\mathbf{1}\{|x|>|X^*|\}\,\mathrm{d}x\\ &= 2\pi\lambda\int_{|X^*|}^{\infty}\frac{1}{u^{\beta}}u\mathrm{d}u = \frac{2\pi\lambda}{\beta-2}|X^*|^{2-\beta}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{E}[g(0,\Phi)] &= \mathbf{E}[\mathbf{E}[g(0,\Phi)|\,X^*]] \qquad (6.3.9)\\ &= \mathbf{E}[\mathbf{E}[g(0,\Phi)|\,X^*]]\\ &= \mathbf{E}\left[|X^*|^{\beta}\,2\,|X^*|^{2-\beta}\right]\\ &= \frac{2\pi\lambda}{\beta-2}\mathbf{E}\left[|X^*|^{2}\right] = \frac{2}{\beta-2}.\end{aligned}$$

Therefore,

$$\mathbf{E}^0\left[\int_V g(y,\Phi)\,\Phi'(\mathrm{d}y)\right] = \frac{\lambda'}{\lambda}\frac{2}{\beta-2}.$$

noindent ***Total received power***. *It follows from (6.3.9) that, almost surely, $g(0,\Phi) < \infty$ and since $0 < |X^*|^{\beta} < \infty$ by (2.6.3), we deduce that the total received power is finite; that is*

$$\sum_{Z\in\Phi}\frac{1}{|Z|^{\beta}} < \infty. \qquad (6.3.10)$$

Corollary 6.3.14. *Let Φ be a simple stationary point process on $\mathbb{R}^2$ with intensity $\lambda \in \mathbb{R}_+^*$ such that $X^* = \arg\min_{X\in\Phi} |X|$ exists and is unique $\mathbf{P}$-almost surely. Let $g : \mathbb{R}^2 \times \mathbb{M}\left(\mathbb{R}^2\right) \to \mathbb{R}_+$ be some measurable function, then*

$$\begin{aligned}\mathbf{E}\left[g\left(-X^*, S_{X^*}\Phi\right)\right] &= \lambda \mathbf{E}^0\left[\int_V g\left(y, \Phi\right) \mathrm{d}y\right] \qquad (6.3.11)\\ &= \frac{1}{|B|}\mathbf{E}\left[\sum_{X\in\Phi\cap B}\int_{\tilde{V}(X)} g\left(y-X, S_X\Phi\right) \mathrm{d}y\right],\end{aligned}$$

where $\tilde{V}(X)$ is the Voronoi cell associated to X and $|B|$ denotes the Lebesgue measure of B.

Proof. Let $\Phi' = \ell^2$ be the Lebesgue measure on $\mathbb{R}^2$. Its Palm probability is $\mathbf{P}$ by Example 6.1.24. Then the first announced equality follows from Corollary 6.3.12. Applying Corollary 6.1.30 for $h(\omega) = \int_V g(y, \Phi)\,\mathrm{d}y$, we get

$$\mathbf{E}^0\left[\int_V g\left(y, \Phi\right) \mathrm{d}x\right] = \frac{1}{\lambda |B|}\mathbf{E}\left[\sum_{X\in\Phi\cap B} h\left(\theta_X\omega\right) \mathrm{d}y\right].$$

Using (6.2.3) we get

$$\begin{aligned}h\left(\theta_X\omega\right) &= \int_{V(\theta_X\omega)} g\left(y, \Phi\left(\theta_X\omega\right)\right) \mathrm{d}y\\ &= \int_{\tilde{V}(X)-X} g\left(y, \Phi\left(\theta_X\omega\right)\right) \mathrm{d}y\\ &= \int_{\tilde{V}(X)} g\left(y-X, \Phi\left(\theta_X\omega\right)\right) \mathrm{d}y. \qquad (6.3.12)\end{aligned}$$

Combining the above two equations proves the second announced equality. □

Corollary 6.3.15. *Besides the assumptions of Corollary 6.3.14, assume that $g(y,\mu)$ remains unchanged when y and μ are shifted; that is*

$$g\left(y, \mu\right) = g\left(y - x, S_x\mu\right), \quad \forall x, y \in \mathbb{R}^2, \mu \in \mathbb{M}\left(\mathbb{R}^2\right). \qquad (6.3.13)$$

Then

$$\begin{aligned}\mathbf{E}\left[g\left(0, \Phi\right)\right] &= \lambda \mathbf{E}^0\left[\int_V g\left(y, \Phi\right) \mathrm{d}y\right] \qquad (6.3.14)\\ &= \frac{1}{|B|}\mathbf{E}\left[\sum_{X\in\Phi\cap B}\int_{\tilde{V}(X)} g\left(y, \Phi\right) \mathrm{d}y\right].\end{aligned}$$

Example 6.3.16. Cell load versus SINR. *Let $P > 0$ be the power transmitted by each base station and $N > 0$ be the noise power. The* signal to interference and noise ratio *(SINR) is defined by*

$$\mathrm{SINR}\left(y, \Phi\right) = \frac{P/\ell\left(|y - X|\right)}{N + P\sum_{Z\in\Phi\setminus\{X\}} 1/\ell\left(|y - Z|\right)}, \quad y \in \tilde{V}\left(X\right), X \in \Phi.$$

The Shannon capacity at location $y \in \mathbb{R}^2$ is defined by

$$R(y,\Phi) = \log_2(1+\mathrm{SINR}(y,\Phi)).$$

The cell load *of base station $X \in \Phi$ is defined by*

$$\varrho(X) = \int_{\tilde{V}(X)} \frac{1}{R(\mathrm{SINR}(y,\Phi))} \mathrm{d}y.$$

Applying Corollary 6.3.15 we get

$$\mathbf{E}^0[\varrho(0)] = \frac{1}{\lambda}\mathbf{E}\left[\frac{1}{R(\mathrm{SINR}(0,\Phi))}\right].$$

6.3.4 Alternative version of Neveu's exchange theorem

Lemma 6.3.17. *Consider the setting of Theorem 6.3.7 and assume moreover that Φ is* simple. *Then Condition (6.3.3) is equivalent to*

$$\mathbf{E}^0[\Phi'(\partial V)] = 0, \tag{6.3.15}$$

where $V = \{y \in \mathbb{R}^d : |y| \le \min_{X\in\Phi}|y-X|\}$ and ∂V is the boundary of V.

Proof. We will apply the mass transport formula (6.1.18) with

$$m(x,y,\omega) = \mathbf{1}\left\{y \in \Phi', x = \arg\min_{X\in\Phi}|y-X| \text{ and this } \arg\min \text{ is not unique}\right\}.$$

Observe that

$$m(0,y,\omega) = \mathbf{1}\left\{y \in \Phi', 0 = \arg\min_{X\in\Phi}|y-X| \text{ and this } \arg\min \text{ is not unique}\right\}.$$

Moreover, $Z = \arg\min_{X\in\Phi}|y-X|$ is equivalent to $\forall X \in \Phi, |y-X| \ge |y-Z|$; i.e., $y \in \tilde{V}(Z)$ defined by (6.2.2). Thus

$$\begin{aligned} m(0,y,\omega) &= \mathbf{1}\left\{y \in \Phi', y \in \tilde{V}(0) \cap \tilde{V}(Z) \text{ with } Z \in \Phi\backslash\{0\}\right\} \\ &= \mathbf{1}\left\{y \in \Phi', y \in \partial\tilde{V}(0)\right\}, \end{aligned}$$

where the second equality holds $\mathbf{P}$-almost surely due to the assumption that Φ is simple. It holds also $\mathbf{P}^0$-almost surely due to Corollary 6.1.29. Then the left-hand side of (6.1.18) equals

$$\lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d} m(0,y,\omega)\Phi'(\mathrm{d}y)\right] = \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d}\mathbf{1}\{y \in \partial V\}\Phi'(\mathrm{d}y)\right] = \lambda\mathbf{E}^0[\Phi'(\partial V)].$$

Then (6.1.18) implies

$$\begin{aligned} \lambda\mathbf{E}^0[\Phi'(\partial V)] &= \lambda'\mathbf{E}^{0'}\left[\int_{\mathbb{R}^d} m(x,0,\omega)\Phi(\mathrm{d}x)\right] \\ &= \lambda'\mathbf{E}^{0'}\left[\int_{\mathbb{R}^d}\mathbf{1}\left\{x = \arg\min_{X\in\Phi}|X| \text{ and this } \arg\min \text{ is not unique}\right\}\Phi(\mathrm{d}x)\right]. \end{aligned}$$

The announced result then follows. $\square$

The following example shows that the condition "Φ is simple" in the above lemma is crucial.

Example 6.3.18. *Observe that the processes Φ and Φ' of Example 6.3.9 satisfy Condition (6.3.15) but the point process Φ is not simple. This explains that the equality (6.3.6) derived from Neveu's exchange formula does not hold.*

We deduce from of the above lemma that the Neveu exchange formula (6.3.4) holds true with an alternative sufficient condition.

Theorem 6.3.19. Alternative Neveu's exchange theorem. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and Φ, Φ' be two point processes compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and with respective intensities $\lambda, \lambda' \in \mathbb{R}_+^*$. Let $V = \{y \in \mathbb{R}^d : |y| \leq \min_{X\in\Phi} |y - X|\}$ and assume that Φ is* simple *and satisfies Condition (6.3.15). Then, Neveu's exchange formula (6.3.4) holds true for any measurable function $f : \Omega \to \mathbb{R}_+$.*

Proof. This follows from Theorem 6.3.7 and Lemma 6.3.17. ☐

Corollary 6.3.20. *Let $\Phi_1, \ldots, \Phi_n$ be stationary* simple *point processes on $\mathbb{R}^d$ with respective intensities $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+^*$. Assume that $\Phi = \Phi_1 + \ldots + \Phi_n$ is simple and let $\lambda = \sum_{k=1}^n \lambda_k$. Then, for all measurable functions $f : \Omega \to \mathbb{R}_+$, and all $k \in \{1, \ldots, n\}$,*

$$\lambda_k \mathbf{E}_k^0 [f] = \lambda \mathbf{E}^0 [f \times \mathbf{1} \{0 \in \Phi_k\}],$$

where $\mathbf{E}_k^0$ is the expectation with respect to the Palm probability associated to Φ_k and $\mathbf{E}^0$ that associated to Φ.

Proof. It is enough to show the announced result for $n = 2$ and $k = 1$. We aim to apply Theorem 6.3.19 for the point processes Φ_k and Φ. We first check Condition (6.3.15). Let V be the Voronoi cell associated to Φ. Since Φ is simple, then $\Phi_k(\partial V) = 0$. Thus $\mathbf{E}^0[\Phi_k(\partial V)] = 0$; i.e., Condition (6.3.15) holds true. Therefore,

$$\lambda_k \mathbf{E}_k^0 [f] = \lambda \mathbf{E}^0 \left[\int_V f \circ \theta_x \Phi_k (\mathrm{d}x)\right] = \lambda \mathbf{E}^0 \left[\sum_{X\in V\cap\Phi_k} f \circ \theta_X\right].$$

Under the Palm probability $\mathbf{P}^0$ associated to Φ, this point process has an atom at 0. Since Φ is simple, it has a single atom within its Voronoi cell V (which is indeed the atom at the origin 0). Thus

$$\lambda_k \mathbf{E}_k^0 [f] = \lambda \mathbf{E}^0 [f \times \mathbf{1} \{0 \in \Phi_k\}].$$

☐

Let Φ be a stationary simple point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. We denote the Voronoi cell of each $x \in \mathbb{R}^d$ with respect to the point process Φ as

$$\tilde{V}(x, \Phi) = \left\{y \in \mathbb{R}^2 : |y - x| \leq \inf_{Z\in\Phi} |y - Z|\right\}.$$

Corollary 6.3.21. *Let Φ_1, Φ_2 be stationary* simple *point processes on $\mathbb{R}^d$ with respective intensities $\lambda_1, \lambda_2 \in \mathbb{R}_+^*$ such that $\Phi_1 + \Phi_2$ is simple. Then for all measurable functions $f : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$,*

$$\lambda_2 \mathbf{E}_2^0 \left[\int_{\tilde{V}(0,\Phi_2)} f(x,\omega) \, \Phi_1(\mathrm{d}x) \right] = \lambda_1 \mathbf{E}_1^0 \left[\int_{\mathbb{R}^d} \mathbf{1}\left\{ 0 \in \tilde{V}(x, \Phi_2) \right\} f(-x, \theta_x \omega) \, \Phi_2(\mathrm{d}x) \right],$$

where $\mathbf{E}_k^0$ is the expectation with respect to the Palm probability associated to Φ_k.

Proof. Let

$$g(x,\omega) = \mathbf{1}\left\{ 0 \in \Phi_2, x \in \Phi_1, x \in \tilde{V}(0, \Phi_2) \right\} f(x,\omega).$$

Observe that

$$\begin{aligned} g(-x, \theta_x \omega) &= \mathbf{1}\left\{ 0 \in S_x \Phi_2, -x \in S_x \Phi_1, -x \in \tilde{V}(0, S_x \Phi_2) \right\} f(-x, \theta_x \omega) \\ &= \mathbf{1}\left\{ x \in \Phi_2, 0 \in \Phi_1, 0 \in \tilde{V}(x, \Phi_2) \right\} f(-x, \theta_x \omega). \end{aligned}$$

Let $\Phi = \Phi_1 + \Phi_2$ and $\lambda = \lambda_1 + \lambda_2$. By Corollary 6.3.20

$$\begin{aligned} &\lambda_2 \mathbf{E}_2^0 \left[\int_{\tilde{V}(0,\Phi_2)} f(x,\omega) \, \Phi_1(\mathrm{d}x) \right] \\ &\quad = \lambda \mathbf{E}^0 \left[\mathbf{1}\{0 \in \Phi_2\} \int_{\tilde{V}(0,\Phi_2)} f(x,\omega) \, \Phi_1(\mathrm{d}x) \right] \\ &\quad = \lambda \mathbf{E}^0 \left[\int_{\mathbb{R}^d} g(x,\omega) \, \Phi(\mathrm{d}x) \right] \\ &\quad = \lambda \mathbf{E}^0 \left[\int_{\mathbb{R}^d} g(-x, \theta_x \omega) \, \Phi(\mathrm{d}x) \right] \\ &\quad = \lambda \mathbf{E}^0 \left[\int_{\mathbb{R}^d} \mathbf{1}\left\{ x \in \Phi_2, 0 \in \Phi_1, 0 \in \tilde{V}(x, \Phi_2) \right\} f(-x, \theta_x \omega) \, \Phi(\mathrm{d}x) \right] \\ &\quad = \lambda \mathbf{E}^0 \left[\mathbf{1}\{0 \in \Phi_1\} \int_{\mathbb{R}^d} \mathbf{1}\left\{ 0 \in \tilde{V}(x, \Phi_2) \right\} f(-x, \theta_x \omega) \, \Phi_2(\mathrm{d}x) \right] \\ &\quad = \lambda_1 \mathbf{E}_1^0 \left[\int_{\mathbb{R}^d} \mathbf{1}\left\{ 0 \in \tilde{V}(x, \Phi_2) \right\} f(-x, \theta_x \omega) \, \Phi_2(\mathrm{d}x) \right], \end{aligned}$$

where the third equality is due to (6.1.16) and the last one follows again from Corollary 6.3.20. □

6.3.5 The Holroyd-Peres representation of Palm probability

This section answers the question whether one can define a "*typical point*" in the stationary configuration of the process, as we did under the Palm probability in Theorem 6.1.31(i).

Let Φ be a stationary simple point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. Define an *allocation rule* to be a measurable map κ from Ω to $\mathbb{R}^d$ such that $\kappa(\omega)$ is an atom of $\Phi(\omega)$ for all $\omega \in \Omega$. The associated *allocation* is the map defined on $\mathbb{R}^d \times \Omega$ by

$$K(x,\omega) = x + \kappa(\theta_x(\omega)), \quad \forall x \in \mathbb{R}^d.$$

Observe that $K(x,\omega)$ is an atom of $\Phi(\omega)$ for all $x \in \mathbb{R}^d, \omega \in \Omega$. An allocation rule is said to be *balanced* if its allocation satisfies the property

$$|\{x : K(x,\omega) = 0\}| = \frac{1}{\lambda}, \quad \mathbf{P}^0\text{-a.s.},$$

where $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}^d$.

The following theorem gives a representation of the Palm probability as a shifted version of the stationary probability:

Theorem 6.3.22. Holroyd-Peres representation of Palm probability. *Let Φ be a stationary simple point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. For all balanced allocation rules κ, for all measurable $f : \Omega \to \mathbb{R}_+$,*

$$\mathbf{E}[f \circ \theta_\kappa] = \mathbf{E}^0[f].$$

Proof. It follows from the mass transport theorem 6.1.34 and Example 6.1.24 that for all measurable functions $g : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$,

$$\mathbf{E}\left[\int_{\mathbb{R}^d} g(x,\omega)\Phi(\mathrm{d}x)\right] = \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} g(-x,\theta_x(\omega))\mathrm{d}x\right]. \tag{6.3.16}$$

Consider now the function $g(x,\omega) = \mathbf{1}_{\{x=\kappa(\omega)\}} f \circ \theta_x$. Then, since κ maps Ω onto the support of $\Phi(\omega)$, we have

$$\begin{aligned}
\mathbf{E}[f \circ \theta_\kappa] &= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}_{\{x=\kappa(\omega)\}} f \circ \theta_x(\omega)\Phi(\mathrm{d}x)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} g(x,\omega)\Phi(\mathrm{d}x)\right] \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} g(-x,\theta_x\omega)\mathrm{d}x\right] \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} \mathbf{1}_{\{-x=\kappa(\theta_x\omega)\}} f \circ \theta_{-x}(\theta_x\omega)\right]\mathrm{d}x \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} \mathbf{1}_{\{0=x+\kappa(\theta_x\omega)\}} f(\omega)\mathrm{d}x\right] = \mathbf{E}^0[f(\omega)],
\end{aligned}$$

where we used (6.3.16) in the third equality and the relation $|\{x : x + \kappa(\theta_x\omega) = 0\}| = \frac{1}{\lambda}$ $\mathbf{P}^0$-a.s. in the last equality. □

Examples of balanced allocations based on the stable marriage theorem and existing for all ergodic point processes (ergodicity of point processes will defined in Chapter 8) are discussed in [47].

6.3.6 Reduced second moment measure

Lemma 6.3.23. *Let Φ be a stationary point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$, let T be the mapping defined on $\mathbb{R}^d \times \mathbb{R}^d$ by $T(x,y) = (x, y-x)$, and let $\beta^{(2)} = M_{\Phi^{(2)}} \circ T^{-1}$ be the image of $M_{\Phi^{(2)}}$ by T. Then*

$$\beta^{(2)}(A \times B) = \lambda |A| \mathcal{K}(B), \quad A, B \in \mathcal{B}\left(\mathbb{R}^d\right), \tag{6.3.17}$$

where, for $B \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\mathcal{K}(B) := \mathbf{E}^0\left[\Phi^!(B)\right], \quad \text{with } \Phi^! := \Phi - \delta_0. \tag{6.3.18}$$

$\mathcal{K}$ is called the reduced second moment measure *of Φ.*

Proof. Letting $\Phi = \sum_{n \in \mathbb{Z}} \delta_{X_n}$, we deduce from (14.E.1 that

$$M_{\Phi^{(2)}}(C) = \mathbf{E}\left[\sum_{n \neq m \in \mathbb{Z}} \mathbf{1}_C(X_n, X_m)\right], \quad C \in \mathcal{B}\left(\mathbb{R}^d \times \mathbb{R}^d\right).$$

Then, for all $A, B \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\begin{aligned}
\beta^{(2)}(A \times B) &= M_{\Phi^{(2)}} \circ T^{-1}(A \times B) \\
&= \mathbf{E}\left[\sum_{n \neq m \in \mathbb{Z}} \mathbf{1}_{T^{-1}(A \times B)}(X_n, X_m)\right] \\
&= \mathbf{E}\left[\sum_{n \neq m \in \mathbb{Z}} \mathbf{1}_{A \times B}(T(X_n, X_m))\right] \\
&= \mathbf{E}\left[\sum_{n \neq m \in \mathbb{Z}} \mathbf{1}_A(X_n) \mathbf{1}_B(X_m - X_n)\right] \qquad (6.3.19)\\
&= \mathbf{E}\left[\sum_{n \in \mathbb{Z}} \mathbf{1}_A(X_n) \left(\sum_{m \in \mathbb{Z} \setminus \{n\}} \mathbf{1}_B(X_m - X_n)\right)\right] \\
&= \mathbf{E}\left[\sum_{n \in \mathbb{Z}} \mathbf{1}_A(X_n) \left(\Phi \circ \theta_{X_n}\right)^!(B)\right] \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}_A(x) \left(\Phi \circ \theta_x\right)^!(B) \Phi(\mathrm{d}x)\right] \\
&= \lambda \int_{\mathbb{R}^d} \mathbf{E}^0\left[\mathbf{1}_A(x) \Phi^!(B)\right] \mathrm{d}x = \lambda |A| \mathbf{E}^0\left[\Phi^!(B)\right],
\end{aligned}$$

where the last but one equality is due to the C-L-M-M theorem 6.1.28. □

Theorem 6.3.24. *Let Φ be a stationary point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$, and let $\mathcal{K}$ be the reduced second moment measure of Φ defined by (6.3.18). Then*

$$M_{\Phi^{(2)}}(A \times B) = \lambda \int_A \mathcal{K}(B - x)\,\mathrm{d}x, \quad A, B \in \mathcal{B}(\mathbb{R}^d). \tag{6.3.20}$$

Proof. Using the notation in Lemma 6.3.23, we get

$$\begin{aligned} M_{\Phi^{(2)}}(A \times B) &= \beta^{(2)}(T(A \times B)) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{A \times B}(x, y)\,\beta^{(2)} \circ T(\mathrm{d}x \times \mathrm{d}y). \end{aligned}$$

Making the change of variable $(u, v) = T(x, y) = (x, y - x)$; i.e., $(x, y) = (u, u + v)$, the above integral writes

$$\begin{aligned} M_{\Phi^{(2)}}(A \times B) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{A \times B}(u, u + v)\,\beta^{(2)}(\mathrm{d}u \times \mathrm{d}v) \\ &= \lambda \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{A \times B}(u, u + v)\,\mathcal{K}(\mathrm{d}v)\,\mathrm{d}u \\ &= \lambda \int_A \left(\int_{\mathbb{R}^d} \mathbf{1}_B(u + v)\,\mathcal{K}(\mathrm{d}v)\right)\mathrm{d}u = \lambda \int_A \mathcal{K}(B - x)\,\mathrm{d}x, \end{aligned}$$

where the second equality is due to Lemma 6.3.23. □

Example 6.3.25. Poisson point process. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$, then, by Slivnyak's theorem 6.1.31(iii), its reduced second moment measure equals*

$$\mathcal{K}(B) = \mathbf{E}^0\left[\Phi^!(B)\right] = \mathbf{E}[\Phi(B)] = \lambda |B|.$$

Observe that Equation (6.3.20) implies $M_{\Phi^{(2)}}(A \times B) = \lambda^2 \int_A |B - x|\,\mathrm{d}x = \lambda^2 |A|\,|B|$ which is consistent with Proposition 2.3.25.

6.4 Exercises

6.4.1 For Section 6.1

Exercise 6.4.1. *Let Φ and Φ' be two independent homogeneous Poisson point processes on $\mathbb{R}$ with respective intensities $\lambda, \lambda' \in \mathbb{R}_+^*$. Calculate the probability that between two consecutive points of Φ there are n points of Φ'.*

Solution 6.4.1. *Let $\mathbf{P}^0$ be the Palm probability associated to Φ, and T_1 be the smallest positive point of Φ. The desired probability is*

$$\mathbf{P}^0(\Phi'([0, T_1]) = n).$$

By Lemma 6.3.1, under $\mathbf{P}^0$, Φ' is a homogeneous Poisson point processes on $\mathbb{R}$ with intensity λ' and is independent of T_1. Then

$$\begin{aligned}\mathbf{P}^0\left(\Phi'\left([0,T_1]\right)=n\right) &= \mathbf{E}^0\left[\mathbf{E}^0\left[\mathbf{1}\left\{\Phi'\left([0,T_1]\right)=n\right\}|T_1\right]\right]\\ &= \mathbf{E}^0\left[e^{-\lambda' T_1}\frac{\left(\lambda' T_1\right)^n}{n!}\right]\\ &= \lambda\frac{\lambda'^n}{n!}\int_0^\infty e^{-\left(\lambda+\lambda'\right)x}x^n \mathrm{d}x\\ &= \left(\frac{\lambda'}{\lambda+\lambda'}\right)^n\frac{\lambda}{\lambda+\lambda'},\end{aligned}$$

where the third equality is due to the fact that under $\mathbf{P}^0$, T_1 is exponentially distributed with parameter λ. Indeed,

$$\mathbf{P}^0\left(T_1>t\right)=\mathbf{P}^0\left(\Phi\left(0,t\right]=0\right)=\mathbf{P}\left(\Phi\left(0,t\right]=0\right)=e^{-\lambda t},$$

where the second equality is due to Slivnyak's theorem 6.1.31(iii).

Exercise 6.4.2. Palm-Khinchin equations. *Let $\Phi=\sum_{n\in\mathbb{Z}}\delta_{T_n}$ be a stationary point process on $\mathbb{R}$ with intensity $\lambda\in\mathbb{R}_+^*$. Let*

$$\Phi_t=\Phi\left((0,t)\right),\quad t\in\mathbb{R}_+.$$

Show that

1.
$$\mathbf{P}\left(\Phi_t>k\right)=\lambda\int_0^t\mathbf{P}^0\left(\Phi_s=k\right)\mathrm{d}s,\quad t\in\mathbb{R}_+,k\in\mathbb{N}.$$

   Hint: $\mathbf{1}\left\{\Phi_t>k\right\}=\sum_{n\in\mathbb{Z}}\mathbf{1}\left\{T_n\in(0,t),\Phi\left((T_n,t)\right)=k\right\}$. The functions $t\mapsto\mathbf{P}^0\left(\Phi_t=k\right)$ are called Palm functions.

2.
$$1-\mathbf{E}\left[z^{\Phi_t}\right]=(1-z)\,\lambda\int_0^t\mathbf{E}^0\left[z^{\Phi_s}\right]\mathrm{d}s,\quad t\in\mathbb{R}_+,z\in(0,1). \tag{6.4.1}$$

3.
$$\mathbf{E}\left[\Phi_t^{(k)}\right]=\lambda\int_0^t\mathbf{E}^0\left[\Phi_s^{(k-1)}\right]\mathrm{d}s,\quad t\in\mathbb{R}_+,k\in\mathbb{N}^*.$$

Solution 6.4.2.

1. *Note that*

$$\mathbf{1}\left\{\Phi_t>k\right\}=\sum_{n\in\mathbb{Z}}\mathbf{1}\left\{T_n\in(0,t),\Phi\left((T_n,t)\right)=k\right\},$$

where the point T_n realizing the indicator function is the $(k+1)$-th point of Φ counting backward from t. Applying Theorem 6.1.28 with

$$f(x,\omega) = \mathbf{1}\{x \in (0,t), \Phi((0,t-x)) = k\},$$

we get

$$\begin{aligned}
\mathbf{P}(\Phi_t > k) &= \mathbf{E}\left[\int_{\mathbb{R}^d} f(x,\theta_x\omega)\Phi(\mathrm{d}x)\right] \\
&= \lambda \int_{\mathbb{R}^d} \mathbf{E}^0[f(x,\omega)]\mathrm{d}x \\
&= \lambda \int_0^t \mathbf{P}^0(\Phi((0,t-x)) = k)\,\mathrm{d}x \\
&= \lambda \int_0^t \mathbf{P}^0(\Phi((0,s)) = k)\,\mathrm{d}s = \lambda \int_0^t p_k(s)\,\mathrm{d}s.
\end{aligned}$$

2. *For any nonnegative random variable X we have*

$$\begin{aligned}
\mathbf{E}\left[z^X\right] &= \mathbf{E}\left[1 + \int_0^X t z^{t-1}\mathrm{d}t\right] \\
&= 1 + \mathbf{E}\left[\int_0^\infty t z^{t-1}\mathbf{1}\{t \le X\}\,\mathrm{d}t\right] \\
&= 1 + \int_0^\infty t z^{t-1}\mathbf{P}(X \ge t)\,\mathrm{d}t.
\end{aligned}$$

In particular, if X is integer-valued, then

$$\begin{aligned}
\mathbf{E}\left[z^X\right] &= 1 + \sum_{k=0}^\infty \int_k^{k+1} t z^{t-1}\mathbf{P}(X \ge t)\,\mathrm{d}t \\
&= 1 + \sum_{k=0}^\infty \mathbf{P}(X \ge k+1)\left(z^{k+1} - z^k\right) \\
&= 1 + (z-1)\sum_{k=0}^\infty \mathbf{P}(X > k)\,z^k.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{E}\left[z^{\Phi_t}\right] &= 1 + (z-1)\sum_{k=0}^\infty \mathbf{P}(\Phi_t > k)\,z^k \\
&= 1 + (z-1)\lambda \int_0^t \sum_{k=0}^\infty \mathbf{P}^0(\Phi_s = k)\,z^k\mathrm{d}s \\
&= 1 + (z-1)\lambda \int_0^t \mathbf{E}^0\left[z^{\Phi_s}\right]\mathrm{d}s.
\end{aligned}$$

3. *Differentiating (6.4.1) then letting $z \uparrow 1$ we get the announced result by using (13.A.15)*

$$\mathbf{E}\left[\Phi_t^{(k)}\right] = \lambda \int_0^t \mathbf{E}^0\left[\Phi_s^{(k-1)}\right] \mathrm{d}s, \quad t \in \mathbb{R}_+, k \in \mathbb{N}^*,$$

(which may be checked for a Poisson point process, since in this case $\mathbf{E}\left[\Phi_t^{(k)}\right] = (\lambda t)^k$ *and by Slivnyak's theorem* $\mathbf{E}^0\left[\Phi_s^{(k-1)}\right] = (\lambda s)^{k-1}$*).*

Exercise 6.4.3. *Let Φ be a homogeneous Poisson point process of intensity λ on $\mathbb{R}^2$ defined on the stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^2}, \mathbf{P})$. Let $\ell(x) = e^{-|x|^2}$. We define the random variables*

$$I = \sum_{X\in\Phi} \ell(X), \quad R^* = \inf_{X\in\Phi} |X|, \quad I^* = \sum_{X\in\Phi} \ell(X) 1\{|X| > 2R^*\}$$

and the stochastic processes

$$I(x) = I \circ \theta_x, \quad R^*(x) = R^* \circ \theta_x, \quad x \in \mathbb{R}^2.$$

1. *Calculate $\mathbf{E}[I]$, $\operatorname{var}[I]$ and $\mathbf{E}[I^*]$ with respect to $\mathbf{P}$.*
2. *Calculate $\mathcal{L}_I(z) = \mathbf{E}\left[e^{-zI}\right]$ for $z \in \mathbb{R}_+$.*
3. *Calculate $\mathbf{E}^0\left[\int_{\mathbb{R}^2} I(x) \mathbf{1}\{|x| \leq R^*(x)\} \mathrm{d}x\right]$ where $\mathbf{E}^0$ is the expectation with respect to the Palm probability associated to Φ.*
4. *Let T be a random variable exponentially distributed with mean 1, independent of Φ. Let $Z = \left\{x \in \mathbb{R}^2 : I(x) \geq T\right\}$. Calculate $\mathbf{E}\left[\int_{(0,1]^2} \mathbf{1}\{I(x) \geq T\} \mathrm{d}x\right]$.*

Solution 6.4.3. *Note that*

$$I = \int_{\mathbb{R}^2} \ell(x) \Phi(\mathrm{d}x) = \Phi(\ell)$$

is a shot-noise.

1. *a) By Campbell averaging formula given in theorem 1.2.5*

$$\begin{aligned}
\mathbf{E}[I] &= \mathbf{E}\left[\int_{\mathbb{R}^2} \ell(x) \Phi(\mathrm{d}x)\right] \\
&= \int_{\mathbb{R}^2} \ell(x) \lambda \mathrm{d}x \\
&= \lambda \int_0^\infty \int_0^{2\pi} \ell(r) r \mathrm{d}r \mathrm{d}\theta \\
&= \lambda 2\pi \int_0^\infty e^{-r^2} r \mathrm{d}r = \lambda\pi.
\end{aligned}$$

b) We deduce from Proposition 2.4.6 that

$$\begin{aligned}\operatorname{var}[I] &= \operatorname{cov}(\Phi(\ell),\Phi(\ell))\\ &= \int_{\mathbb{R}^2} \ell(x)^2 \lambda \mathrm{d}x\\ &= \lambda \int_0^\infty \int_0^{2\pi} \ell(r)^2 r \mathrm{d}r\mathrm{d}\theta\\ &= \lambda 2\pi \int_0^\infty e^{-2r^2} r\mathrm{d}r = \frac{\lambda\pi}{2}.\end{aligned}$$

c) Let $f(x,\mu) = \ell(x)\mathbf{1}\{|x| > 2\inf_{X\in\mu}|X|\}$ for $x \in \mathbb{R}^2, \mu \in \mathbb{M}(\mathbb{R}^2)$, then by C-L-M theorem 3.1.9

$$\begin{aligned}\mathbf{E}[I^*] &= \mathbf{E}\left[\int_{\mathbb{R}^2} f(x,\Phi)\Phi(\mathrm{d}x)\right]\\ &= \lambda \int_{\mathbb{R}^2\times\mathbb{M}(\mathbb{R}^2)} f(x,\mu)\mathbf{P}_\Phi^x(\mathrm{d}\mu)\mathrm{d}x\\ &= \lambda \int_{\mathbb{R}^2\times\mathbb{M}(\mathbb{R}^2)} f(x,\mu)\mathbf{P}_{\Phi+\delta_x}(\mathrm{d}\mu)\mathrm{d}x\\ &= \lambda \int_{\mathbb{R}^2\times\mathbb{M}(\mathbb{R}^2)} f(x,\mu+\delta_x)\mathbf{P}_{\Phi}(\mathrm{d}\mu)\mathrm{d}x\\ &= \lambda \int_{\mathbb{R}^2\times\mathbb{M}(\mathbb{R}^2)} \ell(x)1\left\{|x| > 2\inf_{X\in\mu+\delta_x}|X|\right\}\mathbf{P}_{\Phi}(\mathrm{d}\mu)\mathrm{d}x\\ &= \lambda \int_{\mathbb{R}^2\times\mathbb{M}(\mathbb{R}^2)} \ell(x)1\left\{|x| > 2\inf_{X\in\mu}|X|\right\}\mathbf{P}_{\Phi}(\mathrm{d}\mu)\mathrm{d}x\\ &= \lambda \int_{\mathbb{R}^2} \ell(x)\mathbf{P}\left(|x| > 2\inf_{X\in\Phi}|X|\right)\mathrm{d}x,\end{aligned}$$

where the third equality follows from Slivnyak's theorem 3.2.4. Note that

$$\mathbf{P}\left(|x| > 2\inf_{X\in\Phi}|X|\right) = \mathbf{P}\left(\Phi\left(B\left(0,\frac{|x|}{2}\right)\right) \geq 1\right) = 1 - e^{-\lambda\pi\frac{|x|^2}{4}}.$$

Then

$$\begin{aligned}\mathbf{E}[I^*] &= \mathbf{E}[I] - \lambda\int_{\mathbb{R}^2} e^{-\left(1+\frac{\lambda\pi}{4}\right)|x|^2}\mathrm{d}x\\ &= \lambda\pi - \lambda 2\pi \int_0^\infty e^{-\left(1+\frac{\lambda\pi}{4}\right)r^2} r\mathrm{d}r\\ &= \lambda\pi - \lambda\pi\left[-\frac{e^{-\left(1+\frac{\lambda\pi}{4}\right)r^2}}{\left(1+\frac{\lambda\pi}{4}\right)}\right]_0^\infty\\ &= \lambda\pi - \frac{4\lambda\pi}{4+\lambda\pi} = \frac{(\lambda\pi)^2}{4+\lambda\pi}.\end{aligned}$$

2. *Since I is a Poisson shot-noise, its Laplace transform follows from Proposition 2.1.4:*

$$\begin{aligned}
\mathbf{E}\left[e^{-zI}\right] &= \mathbf{E}\left[e^{-z\int_{\mathbb{R}^2}\ell(x)\Phi(\mathrm{d}x)}\right] \\
&= \exp\left(-\lambda\int_{\mathbb{R}^2}(1-e^{-z\ell(x)})\mathrm{d}x\right) \\
&= \exp\left(-\lambda 2\pi\int_0^\infty(1-e^{-ze^{-r^2}})r\mathrm{d}r\right) \\
&= \exp\left(-\lambda\pi\int_0^z\frac{1-e^{-s}}{s}\mathrm{d}s\right),
\end{aligned}$$

where for the fourth equality we make the change of variable $s = ze^{-r^2}$.

3. *Using the fact that $I(x) = I\circ\theta_x$ and $R^*(x) = R^*\circ\theta_x$, we get*

$$\begin{aligned}
\mathbf{E}^0\left[\int_{\mathbb{R}^2} I(x)\mathbf{1}\{|x|\le R^*(x)\}\,\mathrm{d}x\right] &= \mathbf{E}^0\left[\int_{\mathbb{R}^2} I\circ\theta_x\mathbf{1}\{|x|\le R^*\circ\theta_x\}\,\mathrm{d}x\right] \\
&= \mathbf{E}^0\left[\int_{\mathbb{R}^2} I\circ\theta_{-y}\mathbf{1}\{|y|\le R^*\circ\theta_{-y}\}\,\mathrm{d}y\right] \\
&= \int_{\mathbb{R}^2}\mathbf{E}^0\left[I\circ\theta_{-y}\mathbf{1}\{|y|\le R^*\circ\theta_{-y}\}\right]\mathrm{d}y,
\end{aligned}$$

where the second equality follows from the change of variable $y = -x$. Letting $f(y,\omega) = I\circ\theta_{-y}\mathbf{1}\{|y|\le R^*\circ\theta_{-y}\}$, and applying the C-L-M-M theorem 6.1.28 we get

$$\begin{aligned}
\int_{\mathbb{R}^2}\mathbf{E}^0\left[I\circ\theta_{-y}\mathbf{1}\{|y|\le R^*\circ\theta_{-y}\}\right]\mathrm{d}y &= \frac{1}{\lambda}\mathbf{E}\left[\int_{\mathbb{R}^2} I\mathbf{1}\{|y|\le R^*\}\,\Phi(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda}\mathbf{E}\left[I\int_{\mathbb{R}^2}\mathbf{1}\{|y|\le R^*\}\,\Phi(\mathrm{d}y)\right] \\
&= \frac{1}{\lambda}\mathbf{E}[I] = \pi,
\end{aligned}$$

where the third equality follows from Lemma 6.2.6.

4. *We have*

$$\begin{aligned}
\mathbf{E}\left[\int_{(0,1]^2} \mathbf{1}\{I(x) \geq T\} \mathrm{d}x\right] &= \mathbf{E}\left[\mathbf{E}\left[\int_{(0,1]^2} \mathbf{1}\{I(x) \geq T\} \mathrm{d}x \,\middle|\, \Phi\right]\right] \\
&= \mathbf{E}\left[\int_{(0,1]^2} \mathbf{P}\left(\mathbf{1}\{I(x) \geq T\} | \Phi\right) \mathrm{d}x\right] \\
&= \mathbf{E}\left[\int_{(0,1]^2} \left(1 - e^{-I(x)}\right) \mathrm{d}x\right] \\
&= 1 - \int_{(0,1]^2} \mathbf{E}\left[e^{-I \circ \theta_x}\right] \mathrm{d}x \\
&= 1 - \int_{(0,1]^2} \mathbf{E}\left[e^{-I}\right] \mathrm{d}x \\
&= 1 - \mathbf{E}\left[e^{-I}\right] = 1 - \exp\left(-\lambda\pi \int_0^1 \frac{1 - e^{-s}}{s} \mathrm{d}s\right).
\end{aligned}$$

Exercise 6.4.4. Slivnyak's theorem for homogeneous Poisson. *The present exercise proposes a more straightforward proof of Slivnyak's theorem 6.1.31(iii). Let Φ be a stationary point process on $\mathbb{R}^d$ defined on the stationary framework $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{R}^d}, \mathbf{P})$. Let $\lambda \in \mathbb{R}_+^*$ denote its intensity, $\mathbf{P}^0$ its Palm probability and $\mathcal{L}_\Phi$ its Laplace transform.*

1. *Show that, for all measurable $f, g : \mathbb{R}^d \to \mathbb{R}_+$ such that g is M_Φ-integrable,*

$$\frac{\partial}{\partial t} \mathcal{L}_\Phi (f + tg)\Big|_{t=0} = -\lambda \int_{\mathbb{R}^d} g(x) \mathbf{E}^0 \left[e^{-\int_{\mathbb{R}^d} f(x+y)\Phi(\mathrm{d}y)}\right] \mathrm{d}x. \qquad (6.4.2)$$

2. *Deduce that if Φ is a homogeneous Poisson point process with intensity $\lambda \in \mathbb{R}_+^*$ on $\mathbb{R}^d$, then, for all $A \in \mathcal{M}\left(\mathbb{R}^d\right)$*

$$\mathbf{P}^0(\Phi \in A) = \mathbf{P}(\Phi + \delta_0 \in A) \quad \text{or, equivalently, } \mathbf{P}^0[\Phi^! \in A] = \mathbf{P}[\Phi \in A],$$

with $\Phi^! = \Phi - \delta_0$ under $\mathbf{P}^0$.

Solution 6.4.4. *1. In the same line as in the proof of Proposition 3.2.1 one shows that*

$$\frac{\partial}{\partial t} \mathcal{L}_\Phi (f + tg)\Big|_{t=0} = -\mathbf{E}\left[\Phi(g) e^{-\Phi(f)}\right]$$

Moreover,

$$\begin{aligned}
\mathbf{E}\left[\Phi(g)\, e^{-\Phi(f)}\right] &= \mathbf{E}\left[\int_{\mathbb{R}^d} g(x)\Phi(\mathrm{d}x) e^{-\int_{\mathbb{R}^d} f(y)\Phi(\mathrm{d}y)}\right] \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} g(x) e^{-\int_{\mathbb{R}^d} f(y)\Phi\circ\theta_{-x}\circ\theta_x(\mathrm{d}y)}\Phi(\mathrm{d}x)\right] \\
&= \lambda \int_{\mathbb{R}^d} g(x)\mathbf{E}^0\left[e^{-\int f(y)\Phi\circ\theta_{-x}(\mathrm{d}y)}\right]\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^d} g(x)\mathbf{E}^0\left[e^{-\int_{\mathbb{R}^d} f(x+y)\Phi(\mathrm{d}y)}\right]\mathrm{d}x.
\end{aligned}$$

where the third equality is due to the C-L-M-M theorem 6.1.28.

2. In the same line as in the proof of the direct part of Theorem 3.2.4, one shows that, for all $B \in \mathcal{B}_\mathrm{c}\left(\mathbb{R}^d\right)$,

$$\frac{\partial}{\partial t}\mathcal{L}_\Phi\left(f + t1_B\right)\bigg|_{t=0} = -\lambda \int_B \mathcal{L}_\Phi\left(f\right) e^{-f(x)}\mathrm{d}x,$$

which combined with (6.4.2) implies that, for Lebesgue almost all x

$$\mathbf{E}^0\left[e^{-\int_{\mathbb{R}^d} f(x+y)\Phi(\mathrm{d}y)}\right] = e^{-f(x)}\mathcal{L}_\Phi(f),$$

or, equivalently,

$$\mathbf{E}^0\left[e^{-\int f(x+y)\Phi^!(dy)}\right] = \mathcal{L}_\Phi(f).$$

When denoting by $\{X_k\}$ the points of Φ, this gives

$$\mathbf{E}^0\left[e^{-\sum_{k\neq 0} f(X_k+x)}\right] = \mathcal{L}_\Phi(f).$$

Hence, thanks to the characterization of Poisson point processes by their Laplace transform, there exists $x \in \mathbb{R}^d$ s.t. $\{x + X_k\}_{k\neq 0}$ is a homogeneous Poisson point process with intensity λ under $\mathbf{P}^0$. Therefore $\{X_k\}_{k\neq 0}$ is a homogeneous Poisson point process with intensity λ under $\mathbf{P}^0$.

Exercise 6.4.5. Mutual closest neighbors. *Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$. Show that the Palm probability that the atom 0 has a mutual closest neighbor (MCN) in Φ equals*

$$\mathbf{P}^0\left(0 \text{ has a MCN in } \Phi\right) = \lambda \int_{\mathbb{R}^d} e^{-\lambda \ell^d(B(-x,|x|)\cup B(0,|x|))}\mathrm{d}x,$$

with ℓ^d the Lebesgue measure on $\mathbb{R}^d$. (Hint: give first a representation of the form:

$$\mathbf{1}\left\{0 \text{ has a MCN in } \Phi\right\} = \int_{\mathbb{R}^d} f(\omega, x)\Phi^!(\omega, \mathrm{d}x), \quad \mathbf{P}^0\text{-almost surely,}$$

for some function f, with $\Phi^! = \Phi - \delta_0$.)

Solution 6.4.5. *Since 0 has at most one MCN, we have*

$$\begin{aligned}\mathbf{1}\{0 \textit{ has a MCN in } \Phi\} &= \sum_{n\in\mathbb{Z}^*} \mathbf{1}\{0 \textit{ and } X_n \textit{ are MCN in } \Phi\} \\ &= \sum_{n\in\mathbb{Z}^*} \mathbf{1}\big\{\Phi(B(0,|X_n|)) = 1, \Phi(\bar{B}(0,|X_n|)) = 2, \\ &\qquad\qquad \Phi(B(X_n,|X_n|)) = 1, \Phi(\bar{B}(X_n,|X_n|)) = 2\big\} \\ &= \int_{\mathbb{R}^d} \mathbf{1}\big\{\Phi^!(B(0,|x|)) = 0, \Phi^!(\bar{B}(0,|x|)) = 1, \\ &\qquad\qquad \Phi^!(\bar{B}(x,|x|)) = 1\big\}\, \Phi^!(\mathrm{d}x),\end{aligned}$$

with $B(x,r)$ (resp. $\bar{B}(x,r)$) the open (resp. closed) ball of center x and radius r and $\Phi^! = \Phi - \delta_0$. Hence

$$\begin{aligned}&\mathbf{P}^0\,(0 \textit{ has a MCN in } \Phi) \\ &\quad= \mathbf{E}^0\left[\int_{\mathbb{R}^d} \mathbf{1}\big\{\Phi^!(B(0,|x|)) = 0, \Phi^!(\bar{B}(0,|x|)) = 1, \Phi^!(\bar{B}(x,|x|)) = 1\big\}\, \Phi^!(\mathrm{d}x)\right] \\ &\quad= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\big\{\Phi(B(0,|x|)) = 0, \Phi(\bar{B}(0,|x|)) = 1, \Phi(\bar{B}(x,|x|)) = 1\big\}\, \Phi(\mathrm{d}x)\right],\end{aligned}$$

when making use of Slivnyak's theorem 6.1.31 (iii). We now rewrite

$$\Phi(B(0,|x|)) = \Phi(B(-x,|x|)) \circ \theta_x, \quad \Phi(B(x,|x|)) = \Phi(B(0,|x|)) \circ \theta_x,$$

so as to apply the Campbell-Little-Mecke-Matthes theorem 6.1.28 which gives:

$$\begin{aligned}&\mathbf{P}^0\,(0 \textit{ has a MCN in } \Phi) \\ &\quad= \lambda \int_{\mathbb{R}^d} \mathbf{P}^0\left(\Phi(B(-x,|x|)) = 0, \Phi(\bar{B}(-x,|x|)) = 1, \Phi(\bar{B}(0,|x|)) = 1\right) \mathrm{d}x \\ &\quad= \lambda \int_{\mathbb{R}^d} \mathbf{P}^0\left(\Phi^!(\bar{B}(-x,|x|)) = 0, \Phi^!(\bar{B}(0,|x|)) = 0\right) \mathrm{d}x \\ &\quad= \lambda \int_{\mathbb{R}^d} \mathbf{P}\left(\Phi(\bar{B}(-x,|x|)) = 0, \Phi(\bar{B}(0,|x|)) = 0\right) \mathrm{d}x,\end{aligned}$$

where the last relation follows from Slivnyak's theorem again. Hence the result.

6.4.2 For Section 6.2

Exercise 6.4.6. *Consider a stationary point process on $\mathbb{R}$ with intensity $\lambda \in \mathbb{R}_+^*$. Show that if $\mathbf{E}^0[(T_1 - T_0)^2] = \infty$, then $\mathbf{E}\,[T_1 - T_0] = \infty$.*

Solution 6.4.6. *Applying Corollary 6.2.17 with $g \equiv 1$ we get*

$$\mathbf{E}\,[T_1 - T_0] = \lambda \mathbf{E}^0\left[(T_1 - T_0)^2\right],$$

from which we get the announced result.

Exercise 6.4.7. *Use Slivnyak's theorem to show that the homogeneous Poisson point process on $\mathbb{R}$ with intensity $\lambda \in \mathbb{R}_+^*$ is a renewal point process.*

Solution 6.4.7. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}$ with intensity $\lambda \in \mathbb{R}_+^*$. Then $\mathbf{P}(\Phi(\mathbb{R}) = 0) = e^{-\lambda \times \infty} = 0$. Let $\{T_n\}_{n\in\mathbb{Z}}$ be its points in the increasing order with the usual convention $T_0 \leq 0 < T_1$. By Definition 6.2.22, we have to show that under $\mathbf{P}^0$, the sequence $\{S_n\}_{n\in\mathbb{Z}}$ defined by*

$$S_n = T_{n+1} - T_n$$

is i.i.d. By Slivnyak's theorem 6.1.31 (iii), under $\mathbf{P}^0$, $T_0 = 0$ and $\tilde{\Phi} = \sum_{n\in\mathbb{Z}^*} \delta_{T_n}$ is a homogeneous Poisson point process on $\mathbb{R}$ with intensity λ with the convention $T_{-1} \leq 0 < T_1$. Let $n \in \mathbb{Z}$ and $x, y_n, y_{n-1}, \ldots \in \mathbb{R}_+^*$, with $y_0 = 0$, then

$$\mathbf{P}^0 (T_{n+1} - T_n > x | T_n = y_n, T_{n-1} = y_{n-1}, \ldots) = \mathbf{P}^0 \left(\tilde{\Phi}((y, y+x] = 0) \right) = e^{-\lambda x}.$$

Since the right-hand side of the above equation does not depend on $\{y_k\}_{k\leq n}$, it follows that $T_{n+1} - T_n$ is independent of $\{T_k\}_{k\leq n}$. Then under $\mathbf{P}^0$, S_n is independent of $\{S_k\}_{k<n}$. It follows that the random variables $\{S_n\}_{n\in\mathbb{Z}}$ are mutually independent and the above equation shows that each one is exponentially distributed with parameter λ.

6.4.3 For Section 6.3

Exercise 6.4.8. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and Φ_1, Φ_2 be two point processes on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and with respective intensities $\lambda_1, \lambda_2 \in \mathbb{R}_+^*$. Assume that Φ_1 and Φ_2 are independent.*

1. *Prove that for all measurable functions $f : \mathbb{R}^d \to \bar{\mathbb{R}}_+$,*

$$\mathbf{E}_1^0 \left[\int_{\mathbb{R}^d} f(y) \Phi_2(\mathrm{d}y) \right] = \lambda_2 \int_{\mathbb{R}^d} f(x) \mathrm{d}x,$$

where $\mathbf{E}_1^0$ is the expectation with respect to the Palm probability of Φ_1.

2. *Let $V_1 = \{x \in \mathbb{R}^d : |x| \leq \inf_{X\in\Phi_1} |x - X|\}$. Prove that for all $f : \mathbb{R}^d \to \bar{\mathbb{R}}_+$,*

$$\mathbf{E}_1^0 \left[\int_{V_1} f(y) \Phi_2(\mathrm{d}y) \right] = \lambda_2 \mathbf{E}_1^0 \left[\int_{V_1} f(x) \mathrm{d}x \right]. \tag{6.4.3}$$

Solution 6.4.8. *1. We use the context and notation of Section 6.3.2 with $n = 2$. By (6.3.2)*

$$\mathbf{P}_1^0 = \mathbf{P}_{\Phi_1}^0 \otimes \mathbf{P}_{\Phi_2}.$$

Let Ω_i be the canonical space of Φ_i for $i = 1, 2$. Then

$$\begin{aligned}
\mathbf{E}_1^0\left[\int_{\mathbb{R}^d} f(y)\Phi_2(\mathrm{d}y)\right] &= \int_{\Omega_1\times\Omega_2}\left(\int_{\mathbb{R}^d} f(y)\Phi_2(\mathrm{d}y)\right)\mathbf{P}^0_{\Phi_1}(\mathrm{d}\omega_1)\,\mathbf{P}_{\Phi_2}(\mathrm{d}\omega_2)\\
&= \left[\int_{\Omega_1}\mathbf{P}^0_{\Phi_1}(\mathrm{d}\omega_1)\right]\left[\int_{\Omega_2}\left(\int_{\mathbb{R}^d} f(y)\Phi_2(\mathrm{d}y)\right)\mathbf{P}_{\Phi_2}(\mathrm{d}\omega_2)\right]\\
&= \int_{\Omega_2}\left(\int_{\mathbb{R}^d} f(y)\Phi_2(\mathrm{d}y)\right)\mathbf{P}_{\Phi_2}(\mathrm{d}\omega_2) = \lambda_2\int_{\mathbb{R}^d} f(x)\mathrm{d}x
\end{aligned}$$

where the last equality is due to Campbell averaging theorem 1.2.5.

2. By the same arguments

$$\begin{aligned}
\mathbf{E}_1^0\left[\int_{V_1} f(y)\Phi_2(\mathrm{d}y)\right] &= \int_{\Omega_1\times\Omega_2}\left(\int_{V_1(\omega_1)} f(y)\Phi_2(\mathrm{d}y)\right)\mathbf{P}^0_{\Phi_1}(\mathrm{d}\omega_1)\,\mathbf{P}_{\Phi_2}(\mathrm{d}\omega_2)\\
&= \int_{\Omega_1}\left[\int_{\Omega_2}\left(\int_{V_1(\omega_1)} f(y)\Phi_2(\mathrm{d}y)\right)\mathbf{P}_{\Phi_2}(\mathrm{d}\omega_2)\right]\mathbf{P}^0_{\Phi_1}(\mathrm{d}\omega_1)\\
&= \int_{\Omega_1}\left[\int_{\Omega_2}\left(\int_{V_1(\omega_1)} f(y)\Phi_2(\mathrm{d}y)\right)\mathbf{P}_{\Phi_2}(\mathrm{d}\omega_2)\right]\mathbf{P}^0_{\Phi_1}(\mathrm{d}\omega_1)\\
&= \int_{\Omega_1}\left[\lambda_2\int_{V_1(\omega_1)} f(y)\mathrm{d}y\right]\mathbf{P}^0_{\Phi_1}(\mathrm{d}\omega_1) = \lambda_2\mathbf{E}_1^0\left[\int_{V_1} f(x)\mathrm{d}x\right].
\end{aligned}$$

Exercise 6.4.9. Downlink power control [9]. *Consider the downlink of a wireless network with base stations distributed as a homogeneous Poisson point process $\Phi_1 = \sum_{n\in\mathbb{Z}}\delta_{X_n}$ on $\mathbb{R}^2$ with intensity $\lambda_1 \in \mathbb{R}_+^*$, and users distributed as a stationary point process $\Phi_2 = \sum_{n\in\mathbb{Z}}\delta_{Y_n}$ on $\mathbb{R}^2$ with intensity $\lambda_2 \in \mathbb{R}_+^*$. The point processes Φ_1 and Φ_2 are independent. Each user is attached to the closest base station. Let $\beta > 2$ be the path loss exponent. The attenuation of the signal power of the user (located at) Y_n, when the latter is attached to the base station (located at) X_m, is $|X_m - Y_n|^\beta$. Downlink power is controlled in such a way that each user gets the same signal power, whatever its location. To do so, for each user Y_n attached to it, base station X_m employs the transmit power $|X_m - Y_n|^\beta$ for its signalling to Y_n. The signal power received by each user is hence 1. Use the results of Exercise 6.4.8 to express the mean total downlink power of the typical base station, namely, $\mathbf{E}_1^0\left[\sum_{n\in\mathbb{Z}}\mathbf{1}\{Y_n\in V_1\}|Y_n|^\beta\right]$, in terms of the Euler Gamma function ($\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}\mathrm{d}u$) and the model parameters. Deduce that $\mathbf{E}_1^0\left[\sum_{n\in\mathbb{Z}}\mathbf{1}\{Y_n\in V_1\}|Y_n|\right] = \frac{\lambda_2}{2\lambda_1^{3/2}}$. (You may use the fact that $\Gamma(3/2) = \sqrt{\pi}/2$.)*

Solution 6.4.9. *Applying (6.4.3) with $f(x) = |x|^\beta$ we get*

$$\begin{aligned}
\mathbf{E}_1^0\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{Y_n \in V_1\}|Y_n|^\beta\right] &= \lambda_2 \mathbf{E}_1^0\left[\int_{V_1} |x|^\beta \,\mathrm{d}x\right] \\
&= \frac{\lambda_2}{\lambda_1}\mathbf{E}\left[|X^*|^\beta\right] \\
&= \frac{\lambda_2}{\lambda_1}\int_0^\infty \mathbf{P}\left(|X^*|^\beta > r\right)\mathrm{d}r \\
&= \frac{\lambda_2}{\lambda_1}\int_0^\infty e^{-\lambda\pi r^{2/\beta}}\mathrm{d}r = \frac{\lambda_2}{\lambda_1}\frac{\Gamma(1+\beta/2)}{(\lambda\pi)^{\beta/2}}.
\end{aligned}$$

where the second equality follows from (6.3.11). (Note the coherence of the above four last equalities with (6.3.7).). Taking in particular $\beta = 1$, and using the fact that $\Gamma(3/2) = \sqrt{\pi}/2$, we get

$$\mathbf{E}_1^0\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{Y_n \in V_1\}|Y_n|\right] = \frac{\lambda_2}{2\lambda_1^{3/2}}.$$

Exercise 6.4.10. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and $\Phi = \sum_{n\in\mathbb{Z}}\delta_{X_n}, \Phi' = \sum_{n\in\mathbb{Z}}\delta_{X'_n}$ be random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and with respective intensities $\lambda, \lambda' \in \mathbb{R}_+^*$. Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}_+$ be a measurable function such that $f(x) = f(-x), \forall x \in \mathbb{R}^d$.*

1. *Prove that*
$$\lambda\mathbf{E}^0\left[\sum_{n\in\mathbb{Z}} f(X'_n)\right] = \lambda'\mathbf{E}^{0\prime}\left[\sum_{n\in\mathbb{Z}} f(X_n)\right],$$
where $\mathbf{E}^0$ and $\mathbf{E}^{0\prime}$ are the expectations with respect to the Palm probabilities of Φ and Φ' respectively.

2. *Assume now that $\mathbf{E}^0\left[\sum_{n\in\mathbb{Z}} f(X'_n)\right] < \infty$. Deduce from the above equality that for all Borel sets D of $\mathbb{R}^d$ with positive and finite Lebesgue measure,*
$$\begin{aligned}
&\mathbf{E}\left[\sum_{m\in\mathbb{Z}}\mathbf{1}\{X_m \in D\}\sum_{n\in\mathbb{Z}}\mathbf{1}\{X'_n \in D^{\mathrm{c}}\}f(X'_n - X_m)\right] \\
&= \mathbf{E}\left[\sum_{n\in\mathbb{Z}}\mathbf{1}\{X'_n \in D\}\sum_{m\in\mathbb{Z}}\mathbf{1}\{X_m \in D^{\mathrm{c}}\}f(X_m - X'_n)\right],
\end{aligned}$$
with D^{c} the complement of D in $\mathbb{R}^d$.

Solution 6.4.10. *1. It is enough to apply the mass transport formula in Theorem 6.1.34 with $g(x, \omega) = f(x)$.*

2. By Corollary 6.1.30 we have

$$
\begin{aligned}
&\lambda\,|D|\,\mathbf{E}^0\left[\sum_{n\in\mathbb{Z}} f(X'_n)\right] \\
&= \mathbf{E}\left[\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\sum_{n\in\mathbb{Z}} f(X'_n)\circ\theta_{X_m}\right] \\
&= \mathbf{E}\left[\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\sum_{n\in\mathbb{Z}} f(X'_n - X_m)\right] \\
&= \mathbf{E}\left[\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D\}\, f(X'_n - X_m)\right] \\
&+ \mathbf{E}\left[\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D^{\mathrm{c}}\}\, f(X'_n - X_m)\right],
\end{aligned}
$$

which combined with Question 1, implies

$$
\begin{aligned}
\mathbf{E}&\left[\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D\}\, f(X'_n - X_m)\right] \\
&\quad+ \mathbf{E}\left[\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D^{\mathrm{c}}\}\, f(X'_n - X_m)\right] \\
&= \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D\}\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\, f(X_m - X'_n)\right] \\
&\quad+ \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D\}\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D^{\mathrm{c}}\}\, f(X_m - X'_n)\right].
\end{aligned}
$$

From the assumption, each term in the last equation is positive and finite. Thanks to the symmetry of f, we have

$$
\begin{aligned}
\mathbf{E}&\left[\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D\}\, f(X'_n - X_m)\right] \\
&= \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n\in D\}\sum_{m\in\mathbb{Z}} \mathbf{1}\{X_m\in D\}\, f(X_m - X'_n)\right]
\end{aligned}
$$

and each term of this equality is finite. The result follows when using the above two equalities.

Exercise 6.4.11. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}$, with points $\{T_k\}$ with the usual numbering convention on $\mathbb{R}$. Let h be the point-map $h(\omega) = \arg\min_{l\neq 0} |T_l(\omega)|$. Show that the shift θ_h does not preserve $\mathbf{P}^0$.*

Solution 6.4.11. *The point map is* $h = T_1$ *if* $|T_1| < |T_{-1}|$, *and* $h = T_{-1}$ *if* $|T_1| > |T_{-1}|$ *(the case with equality is of* $\mathbf{P}^0$ *probability 0 under the Poisson assumption). The law of* $\Phi \circ \theta_h$ *cannot be the Palm distribution of a Poisson point process of intensity* λ*: either on the left or the right of* 0, *the second interval is larger than the first. This is incompatible with the assumption that all intervals are i.i.d. exponential with parameter* λ.

Chapter 7

Marks in the stationary framework

7.1 Stationary marked random measures

Let $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ be a l.c.s.h. space. Marked random measures on $\mathbb{R}^d \times \mathbb{K}$ were defined in Section 2.2.6. This section is focused on the stationary case.

Recall that $\tilde{\mathbb{M}}\left(\mathbb{R}^d \times \mathbb{K}\right)$ is the space of measures $\tilde{\mu}$ on $\left(\mathbb{R}^d \times \mathbb{K}, \mathcal{B}\left(\mathbb{R}^d\right) \otimes \mathcal{B}(\mathbb{K})\right)$ such that $\tilde{\mu}(B \times \mathbb{K}) < \infty$ for all $B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right)$. For any $t \in \mathbb{R}^d$, we introduce a *shift operator* S_t on $\tilde{\mathbb{M}}\left(\mathbb{R}^d \times \mathbb{K}\right)$ by

$$S_t\tilde{\mu}(B \times K) := \tilde{\mu}((B+t) \times K), \quad \forall \tilde{\mu} \in \tilde{\mathbb{M}}\left(\mathbb{R}^d \times \mathbb{K}\right), B \in \mathcal{B}\left(\mathbb{R}^d\right), K \in \mathcal{B}(\mathbb{K}).$$

Note that for $\tilde{\mu} = \sum_{l \in \mathbb{Z}} \delta_{(x_l, z_l)} \in \tilde{\mathbb{M}}\left(\mathbb{R}^d \times \mathbb{K}\right)$, we have

$$S_t\tilde{\mu} = \sum_{k \in \mathbb{Z}} \delta_{(x_k - t, z_k)}.$$

The shift S_t also acts on functions $\tilde{f}$ defined on $\mathbb{R}^d \times \mathbb{K}$ as follows

$$S_t\tilde{f}(x, y) := \tilde{f}(x+t, y), \quad \forall x, t \in \mathbb{R}^d, y \in \mathbb{K}.$$

Definition 7.1.1. *A marked random measure $\tilde{\Phi}$ on $\mathbb{R}^d \times \mathbb{K}$, defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, is said to be* stationary *if*

$$\mathbf{P}_{S_t\tilde{\Phi}} = \mathbf{P}_{\tilde{\Phi}}, \quad \forall t \in \mathbb{R}^d. \tag{7.1.1}$$

By Corollary 1.3.4, a marked random measure $\tilde{\Phi}$ is stationary if and only if, for all $t \in \mathbb{R}^d, B_1, \ldots, B_n \in \mathcal{B}\left(\mathbb{R}^d\right)$, $K_1, \ldots, K_n \in \mathcal{B}(\mathbb{K})$,

$$\left(S_t\tilde{\Phi}(B_1 \times K_1), \ldots, S_t\tilde{\Phi}(B_n \times K_n)\right) \stackrel{\text{dist.}}{=} \left(\tilde{\Phi}(B_1 \times K_1), \ldots, \tilde{\Phi}(B_n \times K_n)\right).$$

Example 7.1.2. *Let $\tilde{\Phi}$ be an i.i.d. marked point process on $\mathbb{R}^d \times \mathbb{K}$ with ground point process Φ on $\mathbb{R}^d$ (cf. Definition 2.2.18). If Φ is stationary then so is $\tilde{\Phi}$. Indeed, writing $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z_k)}$ we get*

$$S_t\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k - t, Z_k)}.$$

Since Φ is stationary, $S_t\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k - t}$ has the same probability distribution as Φ. On the other hand, the random variables $\{Z_k\}_{k\in\mathbb{Z}}$ are i.i.d. and independent of Φ. Then $S_t\tilde{\Phi}$ has the same probability distribution as $\tilde{\Phi}$.

Definition 7.1.3. *A marked random measure $\tilde{\Phi}$ on $\mathbb{R}^d \times \mathbb{K}$ in the stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ is said to be* compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ *if*

$$\tilde{\Phi} \circ \theta_t = S_t\tilde{\Phi}, \quad \forall t \in \mathbb{R}^d.$$

Note that a marked random measure $\tilde{\Phi}$ on $\mathbb{R}^d \times \mathbb{K}$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ is stationary, since for all $t \in \mathbb{R}^d$,

$$\mathbf{P}_{S_t\tilde{\Phi}} = \mathbf{P}_{\tilde{\Phi}\circ\theta_t} = \mathbf{P} \circ \left(\tilde{\Phi} \circ \theta_t\right)^{-1} = \mathbf{P} \circ \theta_t^{-1} \circ \tilde{\Phi}^{-1} = \mathbf{P} \circ \tilde{\Phi}^{-1} = \mathbf{P}_{\tilde{\Phi}}.$$

Lemma 7.1.4. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ be a l.c.s.h. space. If a marked random measure $\tilde{\Phi}$ on $\mathbb{R}^d \times \mathbb{K}$ is compatible with the flow, then so is its ground random measure $\Phi(\cdot) = \tilde{\Phi}(\cdot \times \mathbb{K})$.*

Proof. For any $t \in \mathbb{R}^d, B \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$,

$$\begin{aligned}\Phi(\theta_t\omega, B) &= \tilde{\Phi}(\theta_t\omega, B \times \mathbb{K}) \\ &= S_t\tilde{\Phi}(\omega, B \times \mathbb{K}) \\ &= \tilde{\Phi}(\omega, (B + t) \times \mathbb{K}) \\ &= \Phi(\omega, B + t) = S_t\Phi(\omega, B).\end{aligned}$$

□

Definition 7.1.5. *Let $\tilde{\Phi}$ be a stationary marked random measure. Then the stationary framework $\left(\tilde{\mathbb{M}}(\mathbb{R}^d\times\mathbb{K}), \tilde{\mathcal{M}}(\mathbb{R}^d\times\mathbb{K}), \{S_t\}_{t\in\mathbb{R}^d}, \mathbf{P}_{\tilde{\Phi}}\right)$ is called a* canonical stationary framework *for $\tilde{\Phi}$. (Indeed, $\{S_t\}_{t\in\mathbb{R}^d}$ is a measurable flow on $(\tilde{\mathbb{M}}(\mathbb{R}^d\times\mathbb{K}), \tilde{\mathcal{M}}(\mathbb{R}^d\times\mathbb{K}))$ as may be proved following the same lines as in Lemma 6.1.7. Moreover, $\mathbf{P}_{\tilde{\Phi}}$ is invariant with respect to S_t by (7.1.1).)*

Remark 7.1.6. Marked random measure disintegration. *Let $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ be a l.c.s.h. space, $\tilde{\Phi}$ be marked random measure on $\mathbb{R}^d \times \mathbb{K}$ and Φ be its ground random measure. Recall that, by definition of marked random measures, $\tilde{\Phi}(B \times \mathbb{K}) < \infty$ for all $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$. Then $\tilde{\Phi}$ is σ-finite on $\mathbb{R}^d \times \mathbb{K}$. On the other hand, since $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ is a l.c.s.h. space, it is Polish by [56, Theorem 5.3, p.29,]. Since Φ*

is σ-finite , it follows from the measure disintegration theorem 14.D.10 that $\tilde{\Phi}$ *admits the disintegration*

$$\tilde{\Phi}\left(B\times K\right)=\int_{B}Q\left(x,K\right)\Phi\left(\mathrm{d}x\right),\quad B\in\mathcal{B}\left(\mathbb{R}^{d}\right),K\in\mathcal{B}\left(\mathbb{K}\right),$$

where $Q\left(\cdot,\cdot\right)$ *is a probability kernel from* $\mathbb{R}^d$ *to* $\mathbb{K}$*. Note that it is implicit that* Q *depends on* $\omega\in\Omega$ *but the question of the measurability of* Q *with respect to* ω *is not addressed here.*

7.1.1 Stationary marked point processes

We now give an explicit way to construct stationary marked point processes. For this, we need the following preliminary result, where $(\mathbb{K},\mathcal{K})$ is any measurable space, not necessarily l.c.s.h.

Lemma 7.1.7. Shadowing property. *Let* $(\Omega,\mathcal{A},\{\theta_t\}_{t\in\mathbb{R}^d},\mathbf{P})$ *be a stationary framework, let* $\Phi=\sum_{n\in\mathbb{Z}}\delta_{X_n}$ *be a point process on* $\mathbb{R}^d$ *and* $\{Z\left(t\right)\}_{t\in\mathbb{R}^d}$ *be a stochastic process with values in some measurable space* $(\mathbb{K},\mathcal{K})$*, both compatible with the flow. For all* n*, let* $Z_n:=Z(X_n)$*. Then*

$$Z_n=Z(0)\circ\theta_{X_n},\quad n\in\mathbb{Z}. \tag{7.1.2}$$

Moreover for each t,ω*, there exists a bijective mapping* $\mathbb{Z}\to\mathbb{Z}$*,* $k\mapsto l$*, (which depends* t,ω *and on on the enumeration of points; cf. Remark 1.6.17) such that*

$$X_k-t=X_l\circ\theta_t,\quad k\in\mathbb{Z}, \tag{7.1.3}$$

and

$$Z_k=Z_l\circ\theta_t,\quad k\in\mathbb{Z}. \tag{7.1.4}$$

Proof. For any $n\in\mathbb{Z}$,

$$Z\left(X_n\right)=Z\left(X_n\left(\omega\right),\omega\right)=Z\left(0,\theta_{X_n(\omega)}\left(\omega\right)\right),$$

where the third equality follows from (6.1.7). This proves (7.1.2). Fix some $t\in\mathbb{R}^d$ and $\omega\in\Omega$. Observe that

$$\Phi\circ\theta_t=\sum_{l\in\mathbb{Z}}\delta_{X_l\circ\theta_t}=\sum_{k\in\mathbb{Z}}\delta_{X_k-t}.$$

Then the set of atoms of $\Phi\circ\theta_t$ is

$$\left\{X_l\circ\theta_t:l\in\mathbb{Z}\right\}=\left\{X_k-t:k\in\mathbb{Z}\right\},$$

where the atoms are possibly repeated due to their multiplicity. The two sides of the above equality are two enumerations of the same countable set. This ensures the existence of a bijective map $\mathbb{Z} \to \mathbb{Z}, k \mapsto l$ such that (7.1.3) holds true. For the marks, observe that

$$\begin{aligned} Z_l \circ \theta_t &= Z(0) \circ \theta_{X_l} \circ \theta_t \\ &= Z(0) \circ \theta_{t+X_l \circ \theta_t} \\ &= Z(0) \circ \theta_{X_k} = Z_k, \end{aligned}$$

where the first and last equalities are due to (7.1.2), the second equality follows from (6.1.4) and the third equality is due to (7.1.3). □

Proposition 7.1.8. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework and $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ be a l.c.s.h. space. Then the following results hold true.*

(i) Let $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a point process on $\mathbb{R}^d$ and $\{Z(t)\}_{t\in\mathbb{R}^d}$ be a stochastic process with values in $\mathbb{K}$ both compatible with the flow. Let $Z_k := Z(X_k), k \in \mathbb{Z}$. Then

$$\tilde{\Phi} := \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z_k)}$$

is a marked point process on $\mathbb{R}^d \times \mathbb{K}$ compatible with the flow.

(ii) Conversely, let $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z_k)}$ be a marked point process on $\mathbb{R}^d \times \mathbb{K}$ with simple ground process. If $\tilde{\Phi}$ is compatible with the flow, then so is its ground process, and there exists a stochastic process $\{Z(t)\}_{t\in\mathbb{R}^d}$ and compatible with the flow such that

$$Z_k = Z(X_k), \quad k \in \mathbb{Z}.$$

Proof. (i) Note that for all $B \in \mathcal{B}_c(\mathbb{R}^d)$, and for all $L \in \mathcal{B}(\mathbb{K})$,

$$\tilde{\Phi}(B \times L) \le \tilde{\Phi}(B \times \mathbb{K}) = \Phi(B) < \infty.$$

Then $\tilde{\Phi}$ is a point process. Moreover $\tilde{\Phi}$ is compatible with the flow since

$$\tilde{\Phi} \circ \theta_t = \sum_{l\in\mathbb{Z}} \delta_{(X_l\circ\theta_t, Z_l\circ\theta_t)} = \sum_{k\in\mathbb{Z}} \delta_{(X_k - t, Z_k)} = S_t \tilde{\Phi},$$

where the second equality follows from Lemma 7.1.7. (ii) Let $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z_k)}$ be a flow-compatible marked point process on $\mathbb{R}^d \times \mathbb{K}$ with simple ground process. By Lemma 7.1.4, its ground process is also flow-compatible. Let $\{Z(x)\}_{x\in\mathbb{R}^d}$ be the stochastic process defined by

$$Z(x, \omega) = \begin{cases} z, & \text{if } (x, z) \in \tilde{\Phi}(\omega), \\ \Delta, & \text{otherwise,} \end{cases}$$

where Δ is some dummy value. $Z(x, \omega)$ is well defined since the ground process is simple. Note that, for $\tilde{z} \neq \Delta$,

$$\begin{aligned} Z(x, \theta_t\omega) = \tilde{z} &\Leftrightarrow (x, \tilde{z}) \in \tilde{\Phi}(\theta_t\omega) = S_t\tilde{\Phi}(\omega) \\ &\Leftrightarrow (x + t, \tilde{z}) \in \tilde{\Phi}(\omega) \\ &\Leftrightarrow Z(x + t, \omega) = \tilde{z}. \end{aligned}$$

Moreover,

$$\begin{aligned} Z(x, \theta_t \omega) = \Delta &\Leftrightarrow (x, \tilde{z}) \notin \tilde{\Phi}(\theta_t \omega) = S_t \tilde{\Phi}(\omega) \\ &\Leftrightarrow (x + t, \tilde{z}) \notin \tilde{\Phi}(\omega) \\ &\Leftrightarrow Z(x + t, \omega) = \Delta. \end{aligned}$$

Then $\{Z(x)\}_{x \in \mathbb{R}^d}$ satisfies (6.1.6), thus it is compatible with with the flow $\{\theta_t\}_{t \in \mathbb{R}^d}$. □

Observe that, by Equations (7.1.4) and (7.1.4), marks accompany the points when the latter are shifted by the flow; marks may be seen as *shadows* the points. Indeed, recall that θ_t translates all the points of the ground process Φ by $-t$; thus the point process $\Phi' := \Phi \circ \theta_t$ has points $X_l \circ \theta_t = X_k - t$ and marks $Z_l \circ \theta_t = Z_k$. In particular for $t = X_n$, the point process $\Phi' := \Phi \circ \theta_{X_n}$ has the points $\{X_k - X_n\}_{k \in \mathbb{Z}}$. Equation (7.1.2) says that the mark generated by the stochastic process $\{Z(t)\}_{t \in \mathbb{R}^d}$ at the point X_n of Φ equals the mark at the point $X_n - X_n = 0$ of Φ'.

The effect of the shift on the numbering of points and marks (the bijection in Lemma 7.1.7) is not explicit in general; except for the line as shown in the following example.

Example 7.1.9. Shadowing property in $\mathbb{R}$. *Let $\tilde{\Phi} = \sum_{k \in \mathbb{Z}} \delta_{(X_k, Z_k)}$ be a stationary marked point process on $\mathbb{R} \times \mathbb{K}$ with ground process $\Phi = \sum_{k \in \mathbb{Z}} \delta_{X_k}$ with the usual enumeration convention (1.6.8). Then*

$$X_k \circ \theta_{X_n} = X_{k+n} - X_n, \quad Z_k \circ \theta_{X_n} = Z_{k+n}, \quad k, n \in \mathbb{Z}. \tag{7.1.5}$$

Figure 7.1 illustrates Equation (7.1.5) (with $n = 2$).

Indeed,

$$\tilde{\Phi} \circ \theta_{X_n} = \sum_{k \in \mathbb{Z}} \delta_{(X_k \circ \theta_{X_n}, Z_k \circ \theta_{X_n})}.$$

On the other hand,

$$\tilde{\Phi} \circ \theta_{X_n} = S_{X_n}\left(\tilde{\Phi}\right) = \sum_{k \in \mathbb{Z}} \delta_{(X_k - X_n, Z_k)} = \sum_{k \in \mathbb{Z}} \delta_{(X_{k+n} - X_n, Z_{k+n})}.$$

Since $k = 0$ corresponds to an atom of the ground process of $\tilde{\Phi} \circ \theta_{X_n}$ at 0, then

$$X_0 \circ \theta_{X_n} = 0, \quad Z_0 \circ \theta_{X_n} = Z_n,$$

which shows (7.1.5) for $k = 0$. Since the atoms of the ground process of $\tilde{\Phi} \circ \theta_{X_n}$ should be enumerated in the increasing order, we get (7.1.5) for all $k \in \mathbb{Z}$.

Remark 7.1.10. *The two enumerations of points of counting measures on $\mathbb{R}^d$ described in Example 1.6.15 have the following property:*

$$\textit{If the point process } \Phi = \sum_{k \in \mathbb{Z}} \delta_{X_k} \textit{ on } \mathbb{R}^d \textit{ has an atom at } 0, \textit{ then } X_0 = 0. \tag{7.1.6}$$

This property also holds for the usual numbering convention (1.6.8) on $\mathbb{R}$.

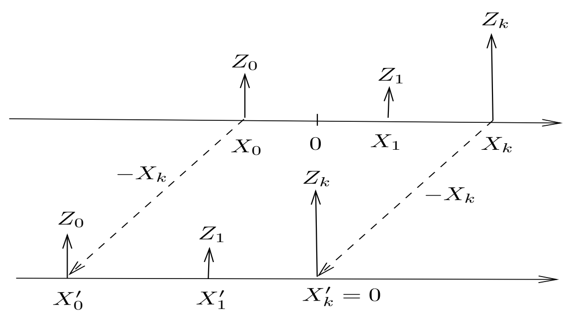

Figure 7.1: The marks may be seen as shadows of the points

Note that in addition to (7.1.2), if the atom enumeration of the ground process satisfies (7.1.6), then we also have

$$Z_k = Z_0 \circ \theta_{X_k}, \quad k \in \mathbb{Z}. \tag{7.1.7}$$

This follows from

$$\begin{aligned} Z_0 \circ \theta_{X_k} &= Z(0) \circ \theta_{X_0} \circ \theta_{X_k} \\ &= Z(0) \circ \theta_{X_k + X_0 \circ \theta_{X_k}} \\ &= Z(0) \circ \theta_{X_k} = Z_k, \end{aligned}$$

where the second equality follows from (6.1.4) and for the third equality we used the fact that $X_0 \circ \theta_{X_k} = 0$ which follows from (7.1.6). The property (7.1.7) characterizes marks in the following sense:

Lemma 7.1.11. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a point process on $\mathbb{R}^d$ compatible with the flow and $\{Z_k\}_{k\in\mathbb{Z}}$ be a sequence of random variables with values in some (non necessarily l.c.s.h) measurable space $(\mathbb{K}, \mathcal{K})$ satisfying (7.1.7). Let*

$$Z(t,\omega) := Z_0(\theta_t \omega), \quad t \in \mathbb{R}^d, \omega \in \Omega.$$

Then the stochastic process $\{Z\,(t)\}_{t\in\mathbb{R}^d}$ is compatible with the flow and

$$Z_k = Z(X_k), \quad k \in \mathbb{Z}.$$

Proof. Observe that

$$Z(t, \theta_x\omega) = Z_0(\theta_t\theta_x\omega) = Z_0(\theta_{t+x}\omega) = Z(t+x, \omega),$$

which, by Lemma 6.1.9, shows that $\{Z\,(t)\}_{t\in\mathbb{R}^d}$ is compatible with the flow. Moreover, by (7.1.7)

$$\begin{aligned} Z_k &= Z_0 \circ \theta_{X_k} \\ &= Z_0\left(\theta_{X_k(\omega)}\,(\omega)\right) \\ &= Z\left(X_k\,(\omega)\,, \omega\right) = Z(X_k). \end{aligned}$$

$\square$

Example 7.1.12. Stationary i.i.d. marked point process in $\mathbb{R}$. *Let*

$$\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k,Z_k)}$$

be a stationary i.i.d. marked point process on $\mathbb{R}\times\mathbb{K}$ as in Example 7.1.2 with simple ground process $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ having intensity $\lambda \in \mathbb{R}_+^*$ and Palm probability $\mathbf{P}^0$. Then, under $\mathbf{P}^0$, $\{Z_k\}_{k\in\mathbb{Z}}$ is an i.i.d sequence independent of Φ and Z_0 has the same distribution as that under $\mathbf{P}$. (Exercise 7.4.5 will extend this result in $\mathbb{R}^d$.)

Indeed, let $\hat{Z} = \{Z_k\}_{k\in\mathbb{Z}}$ and observe that, for any $n \in \mathbb{Z}$,

$$\hat{Z}\circ\theta_{X_n} = \{Z_k\circ\theta_{X_n}\}_{k\in\mathbb{Z}} = \{Z_{k+n}\}_{k\in\mathbb{Z}}\,,$$

where the third equality is due to (7.1.5). Then under $\mathbf{P}$, $\hat{Z}\circ\theta_{X_n}$ is independent of Φ and has the same distribution as $\hat{Z}$. By the very definition of Palm probability (6.1.10), for all $L \in \mathcal{M}(\mathbb{R})$ and $U \in \mathcal{B}(\mathbb{K})^{\otimes\mathbb{Z}}$ (the product σ-algebra on $\mathbb{K}^{\mathbb{Z}}$),

$$\begin{aligned}
&\mathbf{P}^0\left(\hat{Z}\in U, \Phi\in L\right)\\
&\quad= \frac{1}{\lambda}\mathbf{E}\left[\int_{(0,1]} \mathbf{1}_{\hat{Z}\in U,\Phi\in L}\circ\theta_x\Phi(\mathrm{d}x)\right]\\
&\quad= \frac{1}{\lambda}\mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{\hat{Z}\circ\theta_{X_n}\in U, \Phi\circ\theta_{X_n}\in L\right\}\mathbf{1}\left\{X_n\in(0,1]\right\}\right]\\
&\quad= \frac{1}{\lambda}\mathbf{E}\left[\mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{\hat{Z}\circ\theta_{X_n}\in U, \Phi\circ\theta_{X_n}\in L\right\}\mathbf{1}\left\{X_n\in(0,1]\right\}\,\middle|\,\Phi\right]\right]\\
&\quad= \frac{1}{\lambda}\mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{P}\left(\hat{Z}\circ\theta_{X_n}\in U\,\middle|\,\Phi\right)\mathbf{1}\left\{\Phi\circ\theta_{X_n}\in L\right\}\mathbf{1}\left\{X_n\in(0,1]\right\}\right]\\
&\quad= \frac{1}{\lambda}\mathbf{P}\left(\hat{Z}\in U\right)\mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{\Phi\circ\theta_{X_n}\in L\right\}\mathbf{1}\left\{X_n\in(0,1]\right\}\right]\\
&\quad= \mathbf{P}\left(\hat{Z}\in U\right)\mathbf{P}^0\left(\Phi\in L\right).
\end{aligned}$$

Taking $L = \mathbb{M}(\mathbb{R})$, we get $\mathbf{P}^0\left(\hat{Z}\in U\right) = \mathbf{P}\left(\hat{Z}\in U\right)$, which together with the above equality shows that

$$\mathbf{P}^0\left(\hat{Z}\in U, \Phi\in L\right) = \mathbf{P}^0\left(\hat{Z}\in U\right)\mathbf{P}^0\left(\Phi\in L\right).$$

Example 7.1.13. Closest neighbor. *Let Φ be a simple stationary point process on $\mathbb{R}^d$ such that $\Phi(\omega,\mathbb{R}^d) = \infty$, $\forall\omega\in\Omega$; cf. Remark 6.1.27. Assume the enumeration convention (7.1.6). Consider the stochastic process defined by*

$Z(0) = \inf_{l \neq 0} |X_l|$ and $Z(t) = Z(0) \circ \theta_t$. Observe that the mark

$$\begin{aligned} Z_k &= Z(0) \circ \theta_{X_k} \\ &= \inf_{l \neq 0} |X_l \circ \theta_{X_k}| = \inf_{j \neq k} |X_j - X_k|, \end{aligned}$$

where the last equality is due to the fact that, for each l, $X_l \circ \theta_{X_k} = X_j - X_k$ for some j and $X_0 \circ \theta_{X_k} = 0$ due to the enumeration convention (7.1.6). Then Z_k is the distance to the closest neighbor of X_k in Φ.

Example 7.1.14. *For all real numbers $r \geq 0$, the compatibe stochastic process $Z(t) = \Phi(B(t,r))$ defines the marks $Z_k = \Phi(B(X_k, r))$, where $B(t,r)$ denotes the ball of center t and radius r.*

Example 7.1.15. On-off traffic source. *A source of traffic* on-off *is modelled as a stationary marked point process*

$$\tilde{\Phi} = \sum_{k \in \mathbb{Z}} \delta_{(T_k, Z_k)},$$

where T_k is the instant when the source switches on and Z_k is the duration of the on-period which is followed by an off-period of duration

$$Y_k = T_{k+1} - (T_k + Z_k)$$

assumed positive. Assume that the ground process $\Phi = \sum_{k \in \mathbb{Z}} \delta_{T_k}$ has intensity $\lambda \in \mathbb{R}_+^*$ and satisfies $\mathbf{P}(\Phi \neq 0) = 1$. Let $\mathbf{P}^0$ be the Palm probability associated to Φ. The probability that at time 0 the source has been on for a duration exceeding x equals

$$\begin{aligned} \mathbf{P}(T_0 \leq -x, T_0 + Z_0 \geq 0) &= \lambda \mathbf{E}^0 \left[\int_0^{T_1} \mathbf{1}\{T_0 \circ \theta_t \leq -x, T_0 \circ \theta_t + Z_0 \circ \theta_t \geq 0\} \, \mathrm{d}t \right] \\ &= \lambda \mathbf{E}^0 \left[\int_0^{T_1} \mathbf{1}\{-t \leq -x, -t + Z_0 \geq 0\} \, \mathrm{d}t \right] \\ &= \lambda \mathbf{E}^0 \left[\int_0^{T_1} \mathbf{1}\{x \leq t \leq Z_0\} \, \mathrm{d}t \right] \\ &= \lambda \mathbf{E}^0 \left[\int_x^{\infty} \mathbf{1}\{t \leq Z_0, t \leq T_1\} \, \mathrm{d}t \right] \\ &= \lambda \int_x^{\infty} \mathbf{P}^0 (Z_0 \geq t) \, \mathrm{d}t, \end{aligned}$$

where the first equality follows from Corollary 6.2.20 and the second equality is due to the fact that for all $t \in [0, T_1)$, $T_0 \circ \theta_t = T_0 - t$ and $Z_0 \circ \theta_t = Z_0$.

7.1.2 Extension of PASTA to $\mathbb{R}^d$

Let $\mathbb{K}$ be a l.c.s.h. space and $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(X_n,\sigma_n)}$ be an i.i.d. marked point process on $\mathbb{R}^d \times \mathbb{K}$. Assume that the ground process $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ is a homogeneous Poisson point process on $\mathbb{R}^d$ with positive intensity and let $\mathbf{P}^0$ be its Palm probability By Example 7.1.2, $\tilde{\Phi}$ is a stationary marked point process.

Observe that Slivnyak's theorem 6.1.31(iii) together with Exercise 7.4.5 imply that the distribution of $\tilde{\Phi}$ under $\mathbf{P}^0$ equals that of

$$\tilde{\Phi}' = \tilde{\Phi} + \delta_{(0,\sigma')} \tag{7.1.8}$$

under $\mathbf{P}$ where σ' has distribution G and is independent of $\tilde{\Phi}$. Equivalently, for all measurable functions $f : \mathbb{M}\left(\mathbb{R}^d \times \mathbb{K}\right) \to \mathbb{R}$, we have

$$\mathbf{E}^0\left[f\left(\tilde{\Phi}\right)\right] = \mathbf{E}\left[f\left(\tilde{\Phi}'\right)\right]. \tag{7.1.9}$$

The following proposition may be seen as an extension of the so-called PASTA (Poisson Arrivals See Time Averages) property [8, §3.1.1]. We will say that a function $f : \mathbb{M}\left(\mathbb{R}^d \times \mathbb{K}\right) \to \mathbb{R}$ does not depend on the potential atom at 0 and the associated mark, if

$$f(\tilde{\mu}) = f\left(\tilde{\mu}^!\right), \quad \tilde{\mu} \in \mathbb{M}\left(\mathbb{R}^d \times \mathbb{K}\right),$$

where $\tilde{\mu}^!$ is the measure obtained from $\tilde{\mu}$ by supressing any potential atom at 0 and the associated mark.

Proposition 7.1.16. PASTA extension on $\mathbb{R}^d$. *Let $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(X_n,\sigma_n)}$ be an i.i.d. stationary marked Poisson point process on $\mathbb{R}^d \times \mathbb{K}$ where $\mathbb{K}$ is a l.c.s.h. space. For any measurable function $f : \mathbb{M}\left(\mathbb{R}^d \times \mathbb{K}\right) \to \mathbb{R}_+$ which does not depend on the potential atom at 0 and the associated mark, we have*

$$\mathbf{E}^0\left[f\left(\tilde{\Phi}\right)\right] = \mathbf{E}\left[f\left(\tilde{\Phi}\right)\right], \tag{7.1.10}$$

where $\mathbf{E}^0[\cdot]$ is the expectation with respect of the Palm probability of the ground point process.

Proof. Let $\tilde{\Phi}'$ be given by (7.1.8). Since f does not depend on its potential atom at 0, then

$$f\left(\tilde{\Phi}'\right) = f\left(\tilde{\Phi}\right), \quad \mathbf{P}\text{-a.s.}$$

Thus

$$\mathbf{E}\left[f\left(\tilde{\Phi}'\right)\right] = \mathbf{E}\left[f\left(\tilde{\Phi}\right)\right],$$

which together with (7.1.9) implies

$$\mathbf{E}^0\left[f\left(\tilde{\Phi}\right)\right] = \mathbf{E}\left[f\left(\tilde{\Phi}\right)\right].$$

□

Example 7.1.17. PASTA for $M/GI/m/\infty$. *Consider a stable $M/GI/m/\infty$ queueing system [8, §2.1] in the stationary regime, and let $N(t)$ be the number of customers present in the system at time t. Then*

$$\mathbf{P}(N(0)=j)=\mathbf{P}^0\left(N\left(0^-\right)=j\right), \quad j\in\mathbb{N}. \tag{7.1.11}$$

Indeed, the function $f\left(\tilde{\Phi}\right)=\mathbf{1}\left\{N\left(0^-\right)=j\right\}$ does not depend on the potential atom at 0. Then by Proposition 7.1.16

$$\mathbf{P}^0\left(N\left(0^-\right)=j\right)=\mathbf{P}\left(N\left(0^-\right)=j\right)=\mathbf{P}(N(0)=j),$$

where the second equality follows from stationarity.

7.2 Marks in a general measurable space

It is sometimes useful to consider marks taking values in some measurable space $(\mathbb{K},\mathcal{K})$ which is not necessarily l.c.s.h. In this case, strictly speaking, we can no longuer consider point processes on $\mathbb{R}^d\times\mathbb{K}$. This leads to the following definition.

Definition 7.2.1. *Let $(\Omega,\mathcal{A},\{\theta_t\}_{t\in\mathbb{R}^d},\mathbf{P})$ be a stationary framework. Let $\Phi=\sum_{k\in\mathbb{Z}}\delta_{X_k}$ be a point process on $\mathbb{R}^d$ compatible with the flow, and $\{Z(t)\}_{t\in\mathbb{R}^d}$ be a stochastic process with values in some measurable space $(\mathbb{K},\mathcal{K})$ and compatible with the flow. We call*

$$Z_k:=Z(X_k)=Z(0)\circ\theta_{X_k}, \quad k\in\mathbb{Z} \tag{7.2.1}$$

the mark *associated to $X_k\in\Phi$* generated by *the stochastic process $\{Z(t)\}_{t\in\mathbb{R}^d}$ or by the random variable $Z(0)$.*

Note that the stochastic process $\{Z(t)\}_{t\in\mathbb{R}^d}$ (and hence the sequence of marks $\{Z_k\}_{k\in\mathbb{Z}}$) is fully characterized by the random variable $Z(0)$.

Observe that Lemmas 7.1.7 and Lemma 7.1.11 have been stated and proved in this more general context.

Example 7.2.2. *The flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ may be seen as a stochastic process. It generates the marks $\{\theta_{X_k}\}_{k\in\mathbb{Z}}$ called the* universal marks. *This terminology is justified by the fact that, by Equation (7.2.1), each mark of Φ is of the form $Z_k=f(\theta_{X_k})$ where $f=Z(0)$.*

Example 7.2.3. *The virtual cell V defined by (6.2.1) takes its values in the space of closed sets of $\mathbb{R}^d$ (which is a topological and hence a measurable space). It generates the marks*

$$V_k=V(X_k)=\tilde{V}(X_k)-X_k,$$

where $\tilde{V}(X)$ is the Voronoi cell of nucleus X in Φ, and where the last equality follows from (6.2.3).

Given a stochastic process $\{Z(t)\}_{t\in\mathbb{R}^d}$ compatible with the flow, then applying Corollary 6.1.30 with $g = Z(0)$ gives

$$\mathbf{E}^0[Z(0)] = \frac{1}{\lambda|B|}\mathbf{E}\left[\sum_{X\in\Phi\cap B} Z(0,\theta_X\omega)\right] = \frac{1}{\lambda|B|}\mathbf{E}\left[\sum_{X\in\Phi\cap B} Z(X)\right],$$

where the last equality follows from (6.1.7). More generally, we have the following version of the C-L-M-M theorem 6.1.28 for stationary point processes with marks.

Proposition 7.2.4. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, Φ be a point process compatible with the flow and with intensity $\lambda \in \mathbb{R}_+^*$, and $\{Z(t)\}_{t\in\mathbb{R}^d}$ be a stochastic process with values in some measurable space $(\mathbb{K}, \mathcal{K})$, compatible with the flow. Then for all measurable functions $g : \mathbb{R}^d \times \mathbb{K} \to \mathbb{R}_+$,*

$$\mathbf{E}\left[\sum_{k\in\mathbb{Z}} g(X_k, Z_k)\right] = \lambda\int_{\mathbb{R}^d} \mathbf{E}^0[g(x, Z(0))]\mathrm{d}x,$$

where $Z_k := Z(X_k)$.

Proof. It is enough to apply Theorem 6.1.28 to $f(x,\omega) = g(x, Z(0,\omega))$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} f(x,\theta_x\omega)\Phi(\mathrm{d}x) &= \sum_{k\in\mathbb{Z}} f(X_k,\theta_{X_k}\omega)\\ &= \sum_{k\in\mathbb{Z}} g(X_k, Z(0,\theta_{X_k}\omega))\\ &= \sum_{k\in\mathbb{Z}} g(X_k, Z(X_k)) = \sum_{k\in\mathbb{Z}} g(X_k, Z_k).\end{aligned}$$

□

7.2.1 Selected marks and conditioning

Let Φ be a compatible point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$, defined on the stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$. For $U \in \mathcal{A}$, define

$$\Phi_U(\omega, C) = \int_C \mathbf{1}_U(\theta_t\omega)\Phi(\omega,\mathrm{d}t), \quad C \in \mathcal{B}(\mathbb{R}^d). \tag{7.2.2}$$

Such a point process is often referred to as a (non independent) *thinning* of Φ.

Example 7.2.5. *Let $\{Z(t)\}_{t\in\mathbb{R}^d}$ be a stochastic process with values in some measurable space $(\mathbb{K}, \mathcal{K})$, compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Let $Z_k = Z(X_k)$. Let $U = \{Z(0) \in L\}$, for some $L \in \mathcal{K}$. Then*

$$\Phi_U(C) = \sum_{k\in\mathbb{Z}} \mathbf{1}_L(Z(0)\circ\theta_{X_k})\mathbf{1}_C(X_k) = \sum_{k\in\mathbb{Z}} \mathbf{1}_L(Z_k)\mathbf{1}_C(X_k).$$

The point process Φ_U is the thinning of Φ that selects the points X_k with mark Z_k falling in L.

The point process Φ_U is obviously stationary and has a finite intensity (since $\Phi_U \le \Phi$). If $\lambda_U > 0$, the Palm probability of Φ_U is given by

$$\mathbf{P}_U^0(A) = \frac{1}{\lambda_U}\mathbf{E}\left[\int_{(0,1]^d} (\mathbf{1}_A \circ \theta_s)\Phi_U(\mathrm{d}s)\right], \quad A \in \mathcal{A},$$

where λ_U is the intensity of Φ_U. But

$$\lambda_U = \mathbf{E}\left[\int_{(0,1]^d} (\mathbf{1}_U \circ \theta_t)\Phi(\mathrm{d}t)\right] = \lambda\mathbf{P}^0(U). \tag{7.2.3}$$

where $\mathbf{P}^0$ is the Palm probability of Φ. In addition, we have

$$\begin{aligned}\mathbf{E}\left[\int_{(0,1]^d} (\mathbf{1}_A \circ \theta_s)\Phi_U(\mathrm{d}s)\right] &= \mathbf{E}\left[\int_{(0,1]^d} (\mathbf{1}_A \circ \theta_s)(\mathbf{1}_U \circ \theta_s)\Phi(\mathrm{d}s)\right] \\ &= \lambda\mathbf{P}^0(A \cap U),\end{aligned}$$

which gives

$$\mathbf{P}_U^0(A) = \frac{\mathbf{P}^0(A \cap U)}{\mathbf{P}^0(U)},$$

that is,

$$\mathbf{P}_U^0(A) = \mathbf{P}^0(A \mid U). \tag{7.2.4}$$

Example 7.2.6. Superposition of point processes. *Let $\Phi_k, 1 \le k \le n$ be point processes, all defined on the same stationary framework, and all with finite and non-null intensities λ_k, respectively. Assume that their superposition $\Phi = \sum_{j\in\mathbb{Z}} \delta_{X_j}$ is simple and let $\lambda = \sum_{k=1}^n \lambda_k$ be its intensity. Let $\mathbf{P}^0$ and $\mathbf{P}_k^0$ be the Palm probabilities of Φ and $\Phi_k, 1 \le k \le n$, respectively. Observe from (6.1.10) that*

$$\mathbf{P}^0\left(\Phi_k(\{0\}) = 1\right) = \frac{1}{\lambda}\mathbf{E}\left[\sum_{j\in\mathbb{Z}} \mathbf{1}_{(0,1]^d}(X_j)\mathbf{1}_{\Phi_k(\{X_j\})=1}\right] = \frac{1}{\lambda}\mathbf{E}\left[\Phi_k\left((0,1]^d\right)\right] = \frac{\lambda_k}{\lambda}.$$

Let $U = \{\Phi_k(\{0\}) = 1\}$. Since we have $\Phi_U = \Phi_k$ (with the notation of (7.2.2)), we obtain from (7.2.4)

$$\mathbf{P}_k^0(A) = \mathbf{P}^0\left(A \mid \Phi_k(\{0\}) = 1\right) = \frac{\lambda}{\lambda_k}\mathbf{P}^0\left(A \cap \{\Phi_k(\{0\}) = 1\}\right).$$

Therefore

$$\lambda_k\mathbf{P}_k^0(A) = \lambda\mathbf{P}^0\left(A \cap \{\Phi_k(\{0\}) = 1\}\right),$$

which has already been proved in Corollary 6.3.20 with an alternative way.

7.2.2 Transformations of stationary point process based on marks

In order to simplify the notation, we will assume in the present section that the atoms enumeration satisfies (7.1.6).

Proposition 7.2.7. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework harboring a flow-compatible point process $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ with intensity $\lambda \in \mathbb{R}_+^*$ and Palm probability $\mathbf{P}^0$. Let $\{Y(t)\}_{t\in\mathbb{R}^d}$ be a flow-compatible stochastic process with values in $\mathbb{R}_+$ and $Y_k = Y(X_k)$. Then the random measure*

$$\Phi' = \sum_{k\in\mathbb{Z}} Y_k \delta_{X_k}$$

is stationary with intensity $\lambda \mathbf{E}^0[Y_0]$ and Palm probability $\mathbf{P}^{0\prime}$ characterized by

$$\mathbf{E}^{0\prime}[f] = \frac{\mathbf{E}^0[Y_0 f]}{\mathbf{E}^0[Y_0]}, \quad f \in \mathfrak{F}_+\left(\mathbb{R}^d\right),$$

where $\mathbf{E}^0$ and $\mathbf{E}^{0\prime}$ are the expectations with respect to $\mathbf{P}^0$ and $\mathbf{P}^{0\prime}$ respectively.

Proof. By Lemma 7.1.7, for each t, ω, there exists a bijective mapping $\mathbb{Z} \to \mathbb{Z}, k \mapsto l = l(k,t,\omega)$ such that

$$X_l \circ \theta_t = X_k - t,$$

and

$$Y_l \circ \theta_t = Y_k.$$

Thus

$$\begin{aligned}
\Phi' \circ \theta_t &= \sum_{l\in\mathbb{Z}} Y_l \circ \theta_t \delta_{X_l \circ \theta_t} \\
&= \sum_{k\in\mathbb{Z}} Y_k \delta_{X_k - t} = S_t \Phi',
\end{aligned}$$

which shows that Φ' is stationary. Its intensity equals

$$\begin{aligned}
\lambda' &= \mathbf{E}\left[\Phi'\left((0,1]^d\right)\right] \\
&= \mathbf{E}\left[\sum_{n\in\mathbf{Z}} Y_n \mathbf{1}\left\{X_n \in (0,1]^d\right\}\right] \\
&= \lambda \int_{\mathbb{R}} \mathbf{E}^0\left[Y_0 \mathbf{1}\left\{x \in (0,1]^d\right\}\right] \mathrm{d}x = \lambda \mathbf{E}^0[Y_0],
\end{aligned}$$

where the second equality is due to Proposition 7.2.4. By the very definition of

Palm probability

$$\begin{aligned}
\mathbf{E}^{0\prime}[f] &= \frac{1}{\lambda'}\mathbf{E}\left[\int_{(0,1]^d} f\circ\theta_x \Phi'(\mathrm{d}x)\right] \\
&= \frac{1}{\lambda'}\mathbf{E}\left[\sum_{n\in\mathbb{Z}} Y_n \mathbf{1}\left\{X_n\in(0,1]^d\right\} f\circ\theta_{X_n}\right] \\
&= \frac{\lambda}{\lambda'}\int_{\mathbb{R}^d}\mathbf{E}^0[\mathbf{1}\left\{x\in(0,1]^d\right\} Y_0 f]\mathrm{d}x \\
&= \frac{\lambda}{\lambda'}\mathbf{E}^0[Y_0 f] = \frac{\mathbf{E}^0[Y_0 f]}{\mathbf{E}^0[Y_0]},
\end{aligned}$$

where the third equality is due to Proposition 7.2.4. □

Example 7.2.8. Thinning of a stationary point process. *Consider the context of the above proposition where* $\{Y(t)\}$ *takes values in* $\{0,1\}$*, then* Φ' *is a* thinning *of* Φ.

Example 7.2.9. Displaced stationary point process. *Let* $(\Omega,\mathcal{A},\{\theta_t\}_{t\in\mathbb{R}^d},\mathbf{P})$ *be a stationary framework harboring a flow-compatible point process* $\Phi=\sum_{k\in\mathbb{Z}}\delta_{X_k}$ *with intensity* $\lambda\in\mathbb{R}_+^*$ *and a flow-compatible stochastic process* $\{Y(t)\}_{t\in\mathbb{R}^d}$ *with values in* $\mathbb{R}^d$*. Let* $Y_k=Y(X_k)$*, then*

$$\Phi'=\sum_{k\in\mathbb{Z}}\delta_{X_k+Y_k}$$

is stationary with intensity λ.

Indeed,

$$\begin{aligned}
\Phi'\circ\theta_t &= \sum_{l\in\mathbb{Z}}\delta_{X_l\circ\theta_t+Y_l\circ\theta_t} \\
&= \sum_{k\in\mathbb{Z}}\delta_{X_k+Y_k-t} = S_t\Phi'.
\end{aligned}$$

where the second equality is due to Lemma 7.1.7. This shows that Φ' *is stationary. Moreover,*

$$\begin{aligned}
\mathbf{E}\left[\Phi'\left((0,1]^d\right)\right] &= \mathbf{E}\left[\sum_{n\in\mathbf{Z}}\mathbf{1}\left\{X_n+Y_n\in(0,1]^d\right\}\right] \\
&= \lambda\int_{\mathbb{R}}\mathbf{E}^0\left[\mathbf{1}\left\{x+Y_0\in(0,1]^d\right\}\right]\mathrm{d}x \\
&= \lambda\mathbf{E}^0\left[\int_{\mathbb{R}^d}\mathbf{1}\left\{x+Y_0\in(0,1]^d\right\}\mathrm{d}x\right] = \lambda,
\end{aligned}$$

where the second equality is due to Proposition 7.2.4.

We extend the results of Example 7.2.9 by adding supplementary marks. The following result generalizes [8, Property 1.4.3] for $d \geq 2$.

Corollary 7.2.10. Displaced stationary point process with marks. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework harboring a flow-compatible point process $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ with intensity $\lambda \in \mathbb{R}_+^*$ and two flow-compatible stochastic processes $\{Y(t)\}_{t\in\mathbb{R}^d}$ and $\{Z(t)\}_{t\in\mathbb{R}^d}$ with values respectively in $\mathbb{R}^d$ and in some measurable space $(\mathbb{K}, \mathcal{K})$. Let $Y_k = Y(X_k)$ and $Z_k = Z(X_k)$. Then*

$$\tilde{\Phi}' = \sum_{k\in\mathbb{Z}} \delta_{X_k+Y_k, Z_k}$$

is a stationary marked point process on $\mathbb{R}^d \times \mathbb{K}$ with intensity λ. Let (X'_k, Z'_k) be the sequence of points of the above point process, then for any nonnegative measurable function f on $\mathbb{K}$,

$$\mathbf{E}^0 [f(Z_0)] = \mathbf{E}^{0\prime} [f(Z'_0)],$$

where $\mathbf{E}^0$ and $\mathbf{E}^{0\prime}$ are the expectations with respect to the Palm probabilities of Φ and $\Phi' = \sum_{k\in\mathbb{Z}} \delta_{X_k+Y_k}$ respectively.

Proof. Indeed,

$$\tilde{\Phi}' \circ \theta_t = \sum_{l\in\mathbb{Z}} \delta_{X_l\circ\theta_t + Y_l\circ\theta_t, Z_l\circ\theta_t} = \sum_{k\in\mathbb{Z}} \delta_{X_k+Y_k-t, Z_k} = S_t\Phi',$$

where the second equality is due to Lemma 7.1.7. This shows that Φ' is stationary. Moreover, the intensity of $\tilde{\Phi}'$ equals

$$\begin{aligned}
\mathbf{E}\left[\tilde{\Phi}'\left((0,1]^d \times \mathbb{K}\right)\right] &= \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{X_n + Y_n \in (0,1]^d\right\}\right] \\
&= \lambda \int_{\mathbb{R}} \mathbf{E}^0 \left[\mathbf{1}\left\{x + Y_0 \in (0,1]^d\right\}\right] \mathrm{d}x \\
&= \lambda \mathbf{E}^0 \left[\int_{\mathbb{R}^d} \mathbf{1}\left\{x + Y_0 \in (0,1]^d\right\} \mathrm{d}x\right] = \lambda,
\end{aligned}$$

where the second equality is due to Proposition 7.2.4. We will apply the mass transport formula (6.1.16) with

$$g(x', \omega) = \mathbf{1}\{Y(0) = x'\} f(Z(0)).$$

(Each point X_n sends to $X_n + Y_n$ a mass equal to Z_n.) The left hand side of (6.1.16) equals

$$\begin{aligned}
\mathbf{E}^0 \left[\int_{\mathbb{R}^d} g(x', \omega) \Phi'(\mathrm{d}x')\right] &= \mathbf{E}^0 \left[\sum_{n\in\mathbb{Z}} g(X_n + Y_n, \omega)\right] \\
&= \mathbf{E}^0 \left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{Y(0) = X_n + Y_n\} f(Z(0))\right] = \mathbf{E}^0 [f(Z_0)].
\end{aligned}$$

Moreover, $g(-x, \theta_x \omega) = \mathbf{1}\{Y(x) = -x\} f(Z(x))$. Then the right hand side of (6.1.16) equals

$$\begin{aligned}
\mathbf{E}^{0\prime}\left[\int_{\mathbb{R}^d} g(-x, \theta_x\omega)\Phi(\mathrm{d}x)\right] &= \mathbf{E}^{0\prime}\left[\sum_{n\in\mathbb{Z}} g(-X_n, \theta_{X_n}\omega)\right] \\
&= \mathbf{E}^{0\prime}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{Y(X_n) = -X_n\} f(Z(X_n))\right] \\
&= \mathbf{E}^{0\prime}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{X'_n = 0\} f(Z(X_n))\right] = \mathbf{E}^{0\prime}\left[f(Z'_0)\right],
\end{aligned}$$

which concludes the proof. □

7.3 Palm theory for stationary marked random measures

The present section shows how the Palm theory applies to stationary marked random measures.

Let $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ be a l.c.s.h and $\tilde{\Phi}$ be a stationary marked random measure on $\mathbb{R}^d \times \mathbb{K}$ whose associated ground random measure Φ has intensity $\lambda \in \mathbb{R}_+^*$ and Palm probability $\mathbf{P}^0$.

7.3.1 Palm distribution of the mark

The mean measure of $\tilde{\Phi}$ is by definition

$$M_{\tilde{\Phi}}(B \times K) = \mathbf{E}[\tilde{\Phi}(B \times K)], \quad \forall B \in \mathcal{B}\left(\mathbb{R}^d\right), K \in \mathcal{B}(\mathbb{K}).$$

By stationarity, we get

$$\begin{aligned}
M_{\tilde{\Phi}}(B \times K) &= \mathbf{E}[S_t\tilde{\Phi}(B \times K)] \\
&= \mathbf{E}[\tilde{\Phi}((B+t) \times K)] = M_{\Phi}((B+t) \times K).
\end{aligned}$$

Thus, for fixed $K \in \mathcal{B}(\mathbb{K})$, the measure $M_{\tilde{\Phi}}(\cdot \times K)$ is invariant under left translation. Moreover, for all $B \in \mathcal{B}_c\left(\mathbb{R}^d\right), K \in \mathcal{B}(\mathbb{K})$,

$$M_{\tilde{\Phi}}(B \times K) \leq M_{\tilde{\Phi}}(B \times \mathbb{K}) = M_{\Phi}(B) < \infty,$$

where the last inequality follows from the fact that M_{Φ} is locally finite (as the mean measure of a stationary random measure with finite intensity; cf. Section 6.1). Then it follows from the Haar theorem [80, Theorem 2] that $M_{\tilde{\Phi}}$ equals the Lebesgue measure up to a multiplicative constant; i.e.,

$$M_{\tilde{\Phi}}(B \times K) = \lambda |B| \pi(K), \quad \forall B \in \mathcal{B}\left(\mathbb{R}^d\right), K \in \mathcal{B}(\mathbb{K}), \tag{7.3.1}$$

where $\pi(\cdot)$ is a probability measure on $\mathbb{K}$ called the *Palm distribution of the mark*. This name is justified by the following example.

Example 7.3.1. Stationary marked point process. *Let* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ *be a stationary framework,* $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ *be a point process on* $\mathbb{R}^d$*, compatible with the flow, and* $\{Z(t)\}_{t\in\mathbb{R}^d}$ *be a stochastic process with values in a l.c.s.h. space* $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$*, also compatible with the flow. Then the mean measure of* $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z(X_k))}$ *is given by*

$$\begin{aligned}
M_{\tilde{\Phi}}(B \times K) &= \mathbf{E}\left[\int \mathbf{1}\{x \in B, z \in K\}\, \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] \\
&= \mathbf{E}\left[\sum_k \mathbf{1}\{X_k \in B, Z(X_k) \in K\}\right] \\
&= \lambda \int \mathbf{E}^0[\mathbf{1}\{x \in B, Z(0) \in K\}]\, \mathrm{d}x \\
&= \lambda |B|\, \mathbf{P}^0(Z(0) \in K), \qquad (7.3.2)
\end{aligned}$$

for all $B \in \mathcal{B}(\mathbb{R}^d), K \in \mathcal{B}(\mathbb{K})$*. The third equality follows from Proposition 7.2.4 with* λ *and* $\mathbf{P}^0$ *being respectively the intensity and the Palm probability of* $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$*. We see that in this particular case*

$$\pi(K) = \mathbf{P}^0(Z(0) \in K), \quad K \in \mathcal{B}(\mathbb{K}), \qquad (7.3.3)$$

which explains why it is called the Palm distribution of the mark*. Note that the authors of [31] call* $\pi(\cdot)$ *the stationary mark distribution, whereas those of [24] call it the mark distribution.*

Example 7.3.2. Universal mark. *If* $(\Omega, \mathcal{A})$ *is l.c.s.h, then the Palm distribution of the universal mark (defined in Example 7.2.2) is the Palm probability, by (7.3.3).*

Example 7.3.3. I.i.d. marks. *Let* $\mathbb{K}$ *be a l.c.s.h. space and* $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{(X_n, Z_n)}$ *be an i.i.d marked point process on* $\mathbb{R}^d \times \mathbb{K}$ *with mark distribution* F *(cf. Definition 2.2.18). Assume that the ground process* $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ *is stationary with positive intensity* λ *and let* $\mathbf{P}^0$ *be its Palm probability Then* $\tilde{\Phi}$ *is a stationary marked point process by Example 7.1.2. Moreover, the mean measure of* $\tilde{\Phi}$ *is given by by (2.2.8); that is, for* $B \in \mathcal{B}(\mathbb{R}^d), K \in \mathcal{B}(\mathbb{K})$,

$$M_{\tilde{\Phi}}(B \times K) = M_{\Phi}(B) = \lambda |B| F(K).$$

Thus the Palm distribution of the mark equals the mark distribution:

$$\pi(K) = F(K), \quad K \in \mathcal{B}(\mathbb{K}).$$

7.3.2 Palm distributions of marked random measure

Since a stationary marked random measure is itself a random measure on $\mathbb{R}^d \times \mathbb{K}$, we can consider its Palm distributions considered in Section 3.1.

Recall that by Lemma 3.1.1, the Campbell measure of $\tilde{\Phi}$ is

$$\mathcal{C}_{\tilde{\Phi}}(B \times K \times L) = \mathbf{E}\left[\int_{B\times K} \mathbf{1}\left\{\tilde{\Phi} \in L\right\} \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right], \quad B \in \mathcal{B}\left(\mathbb{R}^d\right), K \in \mathcal{B}(\mathbb{K}), L \in \tilde{\mathcal{M}}\left(\mathbb{R}^d \times \mathbb{K}\right).$$

The Palm distributions of $\tilde{\Phi}$ are defined by the disintegration in Equation (3.1.2)

$$\mathcal{C}_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z \times \mathrm{d}\mu) = \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\mu)\, M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z) = \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\mu)\, \lambda \mathrm{d}x \pi(\mathrm{d}z),$$

where the second equality is due to (7.3.1).

The C-L-M theorem 3.1.9 reads as follows: for all measurable $f : \mathbb{R}^d \times \mathbb{K} \times \tilde{\mathbb{M}}\left(\mathbb{R}^d \times \mathbb{K}\right) \to \bar{\mathbb{R}}_+$, we have

$$\mathbf{E}\left[\int_{\mathbb{R}^d\times\mathbb{K}} f\left(x, z, \tilde{\Phi}\right) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] = \int_{\mathbb{R}^d\times\mathbb{K}\times\tilde{\mathbb{M}}(\mathbb{R}^d\times\mathbb{K})} f(x, z, \mu)\, \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\mu)\, M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}z). \tag{7.3.4}$$

7.3.3 Palm probability conditional on the mark

Since the ground random measure is stationary, one may consider its Palm probability $\mathbf{P}^0$. On the other hand, one may consider the Palm distributions $\mathbf{P}_{\tilde{\Phi}}^{(x,z)}$ associated to the marked random measure. We will now define a "hybrid" Palm probability denoted by $\mathbf{P}^{(0,z)}$ and called the *Palm probability conditional on the mark*. In this regard, we need first the following extension of Proposition 6.1.20 to account for marks.

Proposition 7.3.4. Campbell-Matthes measure of a stationary marked random measure. *Let* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ *be a stationary framework,* $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ *be a l.c.s.h. and* $\tilde{\Phi}$ *be a marked random measure on* $\mathbb{R}^d \times \mathbb{K}$ *compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^d}$*. Then the following results hold true.*

(i) There exists a unique σ*-finite measure* $\mathcal{C}$ *on* $\mathbb{R}^d \times \mathbb{K} \times \Omega$ *characterized by*

$$\mathcal{C}(B\times K\times A) := \mathbf{E}\left[\int_{B\times K} \mathbf{1}\{\theta_x(\omega) \in A\}\tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right], \quad K \in \mathcal{B}(\mathbb{K}), A \in \mathcal{A}.$$

for all $B \in \mathcal{B}\left(\mathbb{R}^d\right)$ *and called the* Campbell-Matthes measure *of the stationary marked random measure* $\tilde{\Phi}$.

(ii) The measure $\mathcal{C}$ *defined above satisfies*

$$\mathcal{C}((B+t) \times K \times A) = \mathcal{C}(B \times K \times A) = C(K \times A)\,|B|,$$

for all $B \in \mathcal{B}\left(\mathbb{R}^d\right), K \in \mathcal{B}(\mathbb{K}), A \in \mathcal{A}$*, and for some set function* C *defined on* $\mathcal{B}(\mathbb{K}) \otimes \mathcal{A}$.

Proof. The proof follows the same lines as that of Proposition 6.1.20. □

Let λ be the intensity of the ground random measure, fix some $B \in \mathcal{B}(\mathbb{R}^d)$ such that $|B| \neq 0$ and let

$$\check{C}(K \times A) = \frac{1}{\lambda |B|} \mathcal{C}(B \times K \times A), \quad K \in \mathcal{B}(\mathbb{K}), A \in \mathcal{A}.$$

which, by Proposition 7.3.4(ii), does not depend on the choice of B. Following the same lines as Theorem 6.1.28, it is straightforward to show the following extension of the *Campbell-Little-Mecke-Matthes (C-L-M-M)* theorem to stationary marked random measures: For all measurable functions $f : \mathbb{R}^d \times \mathbb{K} \times \Omega \to \bar{\mathbb{R}}_+$,

$$\mathbf{E}\left[\int_{\mathbb{R}^d \times \mathbb{K}} f(x, z, \theta_x \omega) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] = \lambda \int_{\mathbb{R}^d \times \mathbb{K} \times \Omega} f(x, z, \omega) \check{C}(\mathrm{d}z \times \mathrm{d}\omega)\, \mathrm{d}x. \tag{7.3.5}$$

Corollary 7.3.5. *Consider the conditions of Proposition 7.3.4 and let $f : \mathbb{R}^d \times \mathbb{K} \times \Omega \to \bar{\mathbb{C}}$ be measurable. If either of the following conditions*

$$\mathbf{E}\left[\int_{\mathbb{R}^d \times \mathbb{K}} |f(x, z, \theta_x \omega)|\, \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] < \infty$$

or

$$\int_{\mathbb{R}^d \times \mathbb{K} \times \Omega} \mathbf{E}^0[|f(x, z, \omega)|] \check{C}(\mathrm{d}z \times \mathrm{d}\omega)\, \mathrm{d}x < \infty$$

holds, then the other one holds, and Equality (7.3.5) is true.

Proof. Similar arguments to Corollary 3.1.10. □

Note that

$$\check{C}(K \times \Omega) = \frac{1}{\lambda |B|} \mathcal{C}(B \times K \times \Omega) = \frac{1}{\lambda |B|} M_{\tilde{\Phi}}(B \times K) = \pi(K),$$

where π is the Palm distribution of the mark. Since $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ is a l.c.s.h space, it is Polish by [56, Theorem 5.3, p.29,]. If follows from the measure disintegration theorem 14.D.10 that $\check{C}$ admits a disintegration

$$\check{C}(K \times A) = \int_K \mathbf{P}^{(0,z)}(A)\, \pi(\mathrm{d}z), \quad K \in \mathcal{B}(\mathbb{K}), A \in \mathcal{A},$$

where $\mathbf{P}^{(0,\cdot)}(\cdot)$ is a probability kernel from $\mathbb{K}$ to Ω called the *Palm probability conditional on the mark*. We will denote the expectation with respect to $\mathbf{P}^{(0,z)}$ by $\mathbf{E}^{(0,z)}$. Therefore, Equality (7.3.5) reads

$$\mathbf{E}\left[\int_{\mathbb{R}^d \times \mathbb{K}} f(x, z, \theta_x \omega) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] = \lambda \int_{\mathbb{R}^d \times \mathbb{K}} \mathbf{E}^{(0,z)}[f(x, z, \omega)]\, \pi(\mathrm{d}z)\, \mathrm{d}x, \tag{7.3.6}$$

which may be seen as an extension of (6.1.11) to account for marks.

We will now give the relation between the Palm probability conditional on the mark and the Palm distributions of the marked random measure which extends (6.1.14).

Proposition 7.3.6. *Consider the conditions of Proposition 7.3.4 and let*

$$\mathbf{P}_{\tilde{\Phi}}^{(0,z)} = \mathbf{P}^{(0,z)} \circ \tilde{\Phi}^{-1}, \quad z \in \mathbb{K}.$$

Then

$$\mathbf{P}_{\tilde{\Phi}}^{(0,z)} = \mathbf{P}_{\tilde{\Phi}}^{(x,z)} \circ S_x^{-1}, \quad \text{for } M_{\tilde{\Phi}}\text{-almost all } (x,z) \in \mathbb{R}^d \times \mathbb{K}.$$

Proof. Consider a measurable nonnegative function g defined on $\tilde{\mathbb{M}}\left(\mathbb{R}^d \times \mathbb{K}\right)$. Applying (7.3.6) to the function

$$f(x,z,\omega) = h(x,z) g\left(\tilde{\Phi}(\omega)\right),$$

we get

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^d \times \mathbb{K}} h(x,z) \mathbf{E}^{(0,z)}\left[g\left(\tilde{\Phi}\right)\right] \pi(\mathrm{d}z)\,\mathrm{d}x \\
&\quad = \mathbf{E}\left[\int_{\mathbb{R}^d \times \mathbb{K}} h(x,z) g\left(\tilde{\Phi}(\theta_x \omega)\right) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] \\
&\quad = \mathbf{E}\left[\int_{\mathbb{R}^d \times \mathbb{K}} h(x,z) g\left(S_x \tilde{\Phi}\right) \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z)\right] \\
&\quad = \lambda \int_{\mathbb{R}^d \times \mathbb{K} \times \tilde{\mathbb{M}}(\mathbb{R}^d \times \mathbb{K})} h(x,z) g(S_x \mu) \mathbf{P}_{\tilde{\Phi}}^{(x,z)}(\mathrm{d}\mu) \pi(\mathrm{d}z)\,\mathrm{d}x \\
&\quad = \lambda \int_{\mathbb{R}^d \times \mathbb{K} \times \tilde{\mathbb{M}}(\mathbb{R}^d \times \mathbb{K})} h(x,z) g(\varphi) \mathbf{P}_{\tilde{\Phi}}^{(x,z)} \circ S_x^{-1}(\mathrm{d}\varphi) \pi(\mathrm{d}z)\,\mathrm{d}x,
\end{aligned}$$

where the third equality follows from (7.3.4) and the forth one from the change of variable $\varphi = S_x \mu$. Then for $M_{\tilde{\Phi}}$-almost all (x,z).

$$\int_{\tilde{\mathbb{M}}(\mathbb{R}^d \times \mathbb{K})} g(\mu) \mathbf{P}_{\tilde{\Phi}}^{(x,z)} \circ S_x^{-1}(\mathrm{d}\mu) = \mathbf{E}^{(0,z)}\left[g\left(\tilde{\Phi}\right)\right] = \int_{\tilde{\mathbb{M}}(\mathbb{R}^d \times \mathbb{K})} g(\mu) \mathbf{P}_{\tilde{\Phi}}^{(0,z)}(\mathrm{d}\mu),$$

from which the announced equality follows. □

7.4 Exercises

7.4.1 For Section 7.1.1

Exercise 7.4.1. Little's law. *We consider a system where users arrive, remain for some time, then leave. The user's* arrival *instants to the system are modelled by a stationary point process on $\mathbb{R}$, say $\Phi = \sum_{n \in \mathbb{Z}} \delta_{T_n}$, of intensity $\lambda \in \mathbb{R}_+^*$ with the usual numbering convention (1.6.8). The sojourn duration of the user arriving at T_n is modelled by a random variable denoted by $V_n \in \mathbb{R}_+^*$. Assume that the* sojourn *random variables $\{V_n\}_{n \in \mathbb{Z}}$ satisfy (7.1.7). The* number of users *in the system at time t is*

$$X(t) = \sum_{n \in \mathbb{Z}} \mathbf{1}\{T_n \leq t < T_n + V_n\}.$$

Applying the mass transport formula in Theorem 6.1.34 show that

$$\mathbf{E}[X(0)] = \lambda \mathbf{E}^0[V_0],$$

which is the famous Little's law.

Solution 7.4.1. *(Cf. [8, §3.1.2] for a proof relying on Proposition 7.2.4.) We will apply Theorem 6.1.34 with*

$$\Phi'(\omega)(\mathrm{d}y) = \mathrm{d}y$$

and

$$g(t,\omega) = \mathbf{1}\{T_0 \le t < T_0 + V_0\}.$$

Then

$$g(-x,\theta_x\omega) = \mathbf{1}\{-x \in [T_0(\theta_x\omega), T_0(\theta_x\omega) + V_0(\theta_x\omega))\}$$

and

$$\begin{aligned} g(-T_n,\theta_{T_n}\omega) &= \mathbf{1}\{-T_n \in [T_0(\theta_{T_n}\omega), T_0(\theta_{T_n}\omega) + V_0(\theta_{T_n}\omega))\} \\ &= \mathbf{1}\{-T_n \in [0, V_n)\} \\ &= \mathbf{1}\{T_n \le 0 < T_n + V_n\}, \end{aligned}$$

where the second equality follows from the numbering convention (1.6.8) and from the shadowing property (7.1.7). Thus

$$\begin{aligned} \mathbf{E}^{0\prime}\left[\int_{\mathbb{R}} g(-x,\theta_x\omega)\Phi(\mathrm{d}x)\right] &= \mathbf{E}^{0\prime}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{T_n \le 0 < T_n + V_n\}\right] \\ &= \mathbf{E}^{0\prime}[X(0)] = \mathbf{E}[X(0)], \end{aligned}$$

where the last equality follows from Example 6.1.24. On the other hand

$$\mathbf{E}^0\left[\int_{\mathbb{R}} g(y,\omega)\Phi'(\mathrm{d}y)\right] = \mathbf{E}^0\left[\int_{\mathbb{R}} \mathbf{1}\{y \in [T_0, T_0 + V_0)\}\,\mathrm{d}y\right] = \mathbf{E}^0[V_0].$$

If follows from Theorem 6.1.34 that

$$\mathbf{E}[X(0)] = \lambda \mathbf{E}^0[V_0],$$

We may interpret this application of the mass transport formula as follows. Each point T_n of Φ sends to 'a point' $\mathrm{d}y$ of Φ' within $[T_n, T_n + V_n)$ a mass equal to 1. The 'mass' received by 'a point' $\mathrm{d}y$ of Φ' from all the points of Φ is precisely the number of users in the system at time y.

Exercise 7.4.2. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ with intensity $\lambda \in \mathbb{R}_+^*$, and $\{R(t)\}_{t\in\mathbb{R}^d}$ be an $\mathbb{R}_+$-valued stochastic process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Let M denote the number of balls of center $X_n \in \Phi$ and radius $R(X_n)$ that cover the origin; that is $M = \sum_{n\in\mathbb{Z}} \mathbf{1}\{0 \in B(X_n, R(X_n))\}$.*

1. *Show that* $\mathbf{E}[M] = \lambda \mathbf{E}^0[|B(0, R(0))|]$.

2. *Comment on the connections with Little's law in queuing theory.*

Solution 7.4.2. *1. Let* $f(x, r) = \mathbf{1}\{0 \in B(x, r)\} = \mathbf{1}\{x \in B(0, r)\}$. *By Proposition 7.2.4*

$$
\begin{aligned}
\mathbf{E}[M] &= \mathbf{E}\left[\sum_{n\in\mathbb{Z}} f(X_n, R(X_n))\right] \\
&= \lambda \int_{\mathbb{R}^d} \mathbf{E}^0[f(x, R(0))]\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^d} \mathbf{E}^0[\mathbf{1}\{x \in B(0, R(0))\}]\mathrm{d}x \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} \mathbf{1}\{x \in B(0, R(0))\}\,\mathrm{d}x\right] = \lambda \mathbf{E}^0\left[|B(0, R(0))|\right].
\end{aligned}
$$

2. Little's law states that the long-term average number of customers in a stable system L *is equal to the long-term average effective arrival rate* λ, *multiplied by the Palm-average time a customer spends in the system,* W, *i.e.,* $L = \lambda W$. *Furthermore, this relationship is not influenced by the arrival process distribution, the service distribution, or the service order. Similarly,* $\mathbf{E}[M]$ *in our model denotes the average number of* $X_n \in \Phi$ *with a spatial arrival rate of* λ *that covers the origin with a disc* $B(X_n, R(X_n))$. *Hence, the product* $\mathbf{E}[M] = \lambda \mathbf{E}^0[|B(0, R(0))|]$ *in our model also gives the average arrival rate times the Palm-average covered area.*

Exercise 7.4.3. Users aggregate in hot spots. *Cluster point processes allow one to analyze the effect of users aggregation in hot spots. The main objective is to evaluate the interference seen at the center of a hot spot or at a typical point of such a cluster point process.*

Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^2}, \mathbf{P})$ *be a stationary framework. Let* $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ *be a point process compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^2}$ *describing the hot spot 'centers'. We assume* Φ *has a finite and positive intensity* λ. *Let* $\mathbf{P}^0$ *be the Palm probability of* Φ. *We assume that* Φ *admits a collection of marks* $\{\Phi_k\}_{k\in\mathbb{Z}}$ *compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^2}$, *where each* Φ_k *is a finite point process on* $\mathbb{R}^2$ *with points* $\{Y_j^k\}_j$. *The points* $\{X_k + Y_j^k\}_j$ *form the* k*-th hot spot. Assume the enumeration convention (7.1.6).*

1. *Let* Φ' *be the point process* $\Phi' = \sum_{k\in\mathbb{Z}} \sum_j \delta_{X_k+Y_j^k}$. *Show that* Φ' *is compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^2}$.

2. *Assume that, under* $\mathbf{P}^0$, Φ_0 *has a finite mean measure* Λ*; i.e.,* $\Lambda(\mathbb{R}^2) < \infty$. *Show that the intensity of* Φ' *equals*

$$\lambda' = \lambda\Lambda(\mathbb{R}^2).$$

3. *Let* $g : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ *be a measurable function and let* $I = \sum_{Z\in\Phi'} g(|Z|)$. *Compute* $\mathbf{E}[I]$ *as an integral w.r.t. Lebesgue measure.*

4. *Below, we focus on the special case where Φ is an independently marked Poisson point process with i.i.d. marks $\{\Phi_k\}_{k\in\mathbb{Z}}$ being each a Poisson point process on $\mathbb{R}^2$ of intensity measure Λ. Compute the Laplace functional of Φ'.*

5. *Compute the Laplace transform of I under $\mathbf{P}$.*

6. *Compute the mean value and the Laplace transform of I under $\mathbf{P}^0$ (i.e., seen at a typical hot spot center). How do these quantities compare to the corresponding quantities under $\mathbf{P}$?*

7. *Assume that Λ admits a density α with respect to Lebesgue measure. Use the fact that for all positive functions f,*

$$\mathbf{E}\left[\sum_{Z_p\neq Z_q\in\Phi'} f(Z_p,Z_q)\right] = \mathbf{E}\left[\sum_{X_k\in\Phi}\sum_{l\neq m} f(X_k+Y_l^k, X_k+Y_m^k)\right] + \mathbf{E}\left[\sum_{X_k\neq X_j\in\Phi}\sum_{l,m} f(X_k+Y_l^k, X_j+Y_m^j)\right]$$

to compute the density $\rho^{(2)}(\cdot,\cdot)$ of the factorial moment measure of order 2 of Φ'.

8. *Let $\mathbf{P}^{!0'}$ denote the reduced Palm probability of Φ'. Use the density $\rho^{(2)}$ evaluated in 7 to compute $\mathbf{E}^{!0'}[I]$ (i.e., the mean interference seen at a typical point of Φ'). How does this compare to the corresponding quantity under $\mathbf{P}$?*

Solution 7.4.3. *1. For $t\in\mathbb{R}^2$, consider*

$$\Phi'\circ\theta_t = \sum_{k\in\mathbb{Z}}\sum_j \delta_{X_k\circ\theta_t + Y_j^k\circ\theta_t}.$$

For any $k\in\mathbb{Z}$, there exists $l\in\mathbb{Z}$ such that $X_k\circ\theta_t = X_l - t$. Moreover, the mapping $\mathbb{Z}\to\mathbb{Z}; k\mapsto l$ is bijective and by the shadowing property of compatible marks

$$\Phi_k\circ\theta_t = \Phi_l.$$

Thus

$$\Phi'\circ\theta_t = \sum_{l\in\mathbb{Z}}\sum_j \delta_{X_l - t + Y_j^l} = S_t\Phi',$$

which shows that Φ' is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^2}$.

2. For $B \in \mathcal{B}_c\left(\mathbb{R}^2\right)$,

$$\begin{aligned}
M_{\Phi'}(B) &= \mathbf{E}\left[\Phi'(B)\right] \\
&= \mathbf{E}\left[\sum_{k\in\mathbb{Z}} \Phi_k\left(B - X_k\right)\right] \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^2} \Phi_0\left(B - y\right) \mathrm{d}y\right] \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathbf{1}\left\{x \in B\right\} \Phi_0\left(\mathrm{d}x - y\right) \mathrm{d}y\right] \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathbf{1}\left\{z + y \in B\right\} \Phi_0\left(\mathrm{d}z\right) \mathrm{d}y\right] \\
&= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^2} |B|\, \Phi_0\left(\mathrm{d}z\right)\right] \\
&= \lambda \mathbf{E}^0[\Phi_0(\mathbb{R}^2)]\, |B| = \lambda\Lambda(\mathbb{R}^2)\, |B|\,,
\end{aligned}$$

where the third equality is due to Proposition 7.2.4, and the fifth one follows by the change of variable $z = x - y$.

3. By Campbell's averaging formula 1.2.2

$$\mathbf{E}\left[I\right] = \mathbf{E}\left[\sum_{Z\in\Phi'} g(|Z|)\right] = \lambda' \int_{\mathbb{R}^2} g\left(|z|\right) \mathrm{d}z = \lambda\Lambda(\mathbb{R}^2) \int_{\mathbb{R}^2} g\left(|z|\right) \mathrm{d}z.$$

4. It follows from Example 2.3.18 that Φ' *is a well defined point process. Moreover, its Laplace transform is given by (2.3.14), for* $f \in \mathfrak{F}_+\left(\mathbb{R}^d\right)$,

$$\mathcal{L}_{\Phi'}\left(f\right) = \mathcal{L}_{\Phi}\left(\bar{f}\right) = \exp\left(-\lambda \int_{\mathbb{R}^2} (1 - e^{-\bar{f}(x)}) \mathrm{d}x\right),$$

where for $x \in \mathbb{R}^d$,

$$\begin{aligned}
\bar{f}\left(x\right) &= -\log \mathcal{L}_{\Phi_1}\left(S_x f\right) \\
&= -\log\left(\exp\left[-\int_{\mathbb{R}^2} (1 - e^{-S_x f(y)}) \Lambda\left(\mathrm{d}y\right)\right]\right) \\
&= \int_{\mathbb{R}^2} \left[1 - e^{-f(x+y)}\right] \Lambda\left(\mathrm{d}y\right).
\end{aligned}$$

Thus

$$\mathcal{L}_{\Phi'}\left(f\right) = \exp\left(-\lambda \int_{\mathbb{R}^2} \left[1 - e^{-\int_{\mathbb{R}^2}(1 - e^{-f(x+y)})\Lambda(\mathrm{d}y)}\right] \mathrm{d}x\right).$$

5. In particular, for all $s \in \mathbb{R}_+$,

$$\mathbf{E}\left[e^{-sI}\right] = \mathcal{L}_{\Phi'}\left(sg\left(|\cdot|\right)\right) = \exp\left(-\lambda \int_{\mathbb{R}^2} \left[1 - e^{-\int_{\mathbb{R}^2}(1 - e^{-sg(|x+y|)})\Lambda(\mathrm{d}y)}\right] \mathrm{d}x\right).$$

6. *Consider, for all $s \in \mathbb{R}_+$,*

$$\begin{aligned}
\mathbf{E}^0\left[e^{-sI}\right] &= \mathbf{E}^0\left[e^{-s\sum_{k\in\mathbb{Z}}\sum_j g\left(\left|X_k+Y_j^k\right|\right)}\right] \\
&= \mathbf{E}^0\left[e^{-s\sum_{k\in\mathbb{Z}^*}\sum_j g\left(\left|X_k+Y_j^k\right|\right)}e^{-s\sum_j g\left(\left|Y_j^k\right|\right)}\right] \\
&= \mathbf{E}^0\left[e^{-s\sum_{k\in\mathbb{Z}^*}\sum_j g\left(\left|X_k+Y_j^k\right|\right)}\right]\mathbf{E}^0\left[e^{-s\sum_j sg\left(\left|Y_j^k\right|\right)}\right] \\
&= \mathbf{E}\left[e^{-sI}\right]\mathbf{E}^0\left[e^{-s\sum_j sg\left(\left|Y_j^k\right|\right)}\right] \\
&= \mathbf{E}\left[e^{-sI}\right]\exp\left(-\int_{\mathbb{R}^2}(1-e^{-sg(|x|)})\Lambda\left(\mathrm{d}x\right)\right),
\end{aligned}$$

where the fourth equality is due to Slivnyak's theorem 6.1.31 (iii). Then

$$\begin{aligned}
\mathbf{E}^0\left[I\right] &= \frac{\partial}{\partial s}\mathbf{E}^0\left[e^{-sI}\right]\Big|_{s=0} \\
&= \mathbf{E}\left[I\right]+\int_{\mathbb{R}^2} g\left(|x|\right)\Lambda\left(\mathrm{d}x\right) \\
&= \lambda\Lambda(\mathbb{R}^2)\int_{\mathbb{R}^2} g\left(|z|\right)\mathrm{d}z+\int_{\mathbb{R}^2} g\left(|x|\right)\Lambda\left(\mathrm{d}x\right).
\end{aligned}$$

7. *Let $h\left(X_k,\Phi_k\right) := \sum_{l\neq m} f(X_k+Y_l^k, X_k+Y_m^k)$, then*

$$\begin{aligned}
&\mathbf{E}\left[\sum_{X_k\in\Phi}\sum_{l\neq m} f(X_k+Y_l^k, X_k+Y_m^k)\right] \\
&\qquad= \mathbf{E}\left[\sum_{k\in\mathbb{Z}} h\left(X_k,\Phi_k\right)\right] \\
&\qquad= \lambda\int_{\mathbb{R}^2}\mathbf{E}^0[h(y,\Phi_0)]\mathrm{d}y \\
&\qquad= \lambda\int_{\mathbb{R}^2}\mathbf{E}[h(y,\Phi_0)]\mathrm{d}y \\
&\qquad= \lambda\int_{\mathbb{R}^2}\mathbf{E}\left[\sum_{l\neq m} f(y+Y_l^0, y+Y_m^0)\right]\mathrm{d}y \\
&\qquad= \lambda\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2\times\mathbb{R}^2} f(y+u,y+v)M_{\Phi^{(2)}}\left(\mathrm{d}u\times\mathrm{d}v\right)\right]\mathrm{d}y \\
&\qquad= \lambda\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2\times\mathbb{R}^2} f(y+u,y+v)\alpha\left(u\right)\alpha\left(v\right)\mathrm{d}u\mathrm{d}v\right]\mathrm{d}y \\
&\qquad= \lambda\int_{\mathbb{R}^2\times\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\alpha\left(s-y\right)\alpha\left(t-y\right)\mathrm{d}y\right] f(s,t)\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

where the second equality follows from Proposition 7.2.4, the third one for the fact that the marks $\{\Phi_k\}_{k\in\mathbb{Z}}$ are independent from Φ, the sixth one from the

Campbell averaging formula 1.2.2 applied to $\Phi^{(2)}$, the sixth one from (2.3.18), and the last one by the change of variable $u \to s = y+u, v \to t = y+v$.

Let $\hat{\Phi} := \sum_{k\in\mathbb{Z}} \delta_{X_k,\Phi_k}$ and $\varphi\left(\left(X_k,\Phi_k\right),\left(X_j,\Phi_j\right)\right) := \sum_{l,m} f(X_k+Y_l^k, X_j+Y_m^j)$, then

$$
\begin{aligned}
&\mathbf{E}\left[\sum_{X_k\neq X_j\in\Phi}\sum_{l,m} f(X_k+Y_l^k, X_j+Y_m^j)\right]\\
&\quad=\mathbf{E}\left[\sum_{k\neq j\in\mathbb{Z}}\varphi\left(\left(X_k,\Phi_k\right),\left(X_j,\Phi_j\right)\right)\right]\\
&\quad=\mathbf{E}\left[\sum_{T\neq S\in\hat{\Phi}}\varphi\left(T,S\right)\right]\\
&\quad=\int\varphi\left(t,s\right)M_{\hat{\Phi}^{(2)}}\left(\mathrm{d}s\times\mathrm{d}t\right)\\
&\quad=\lambda^2\int\varphi\left(\left(y,\mu\right),\left(z,\nu\right)\right)\mathrm{d}y\mathrm{d}z\mathbf{P}_{\Phi_1}\left(\mathrm{d}\mu\right)\mathbf{P}_{\Phi_1}\left(\mathrm{d}\nu\right)\\
&\quad=\lambda^2\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathbf{E}\left[\sum_{l,m} f(y+Y_l^1, z+Y_m^2)\right]\mathrm{d}y\mathrm{d}z\\
&\quad=\lambda^2\int_{\mathbb{R}^2\times\mathbb{R}^2}\left[\int_{\mathbb{R}^2\times\mathbb{R}^2} f(y+u,z+v)\alpha\left(u\right)\alpha\left(v\right)\mathrm{d}u\mathrm{d}v\right]\mathrm{d}y\mathrm{d}z\\
&\quad=\lambda^2\int_{\mathbb{R}^2\times\mathbb{R}^2} f(s,t)\left[\int_{\mathbb{R}^2\times\mathbb{R}^2}\alpha\left(s-y\right)\alpha\left(t-z\right)\mathrm{d}y\mathrm{d}z\right]\mathrm{d}s\mathrm{d}t\\
&\quad=\lambda^2\Lambda(\mathbb{R}^2)^2\int_{\mathbb{R}^2\times\mathbb{R}^2} f(s,t)\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

where the third and sixth equalities are due to Campbell averaging formula, and the last but one by the change of variable $u \to s = y+u, v \to t = z+v$. Therefore, the density of the factorial moment measure of order 2 of Φ' equals

$$
\rho^{(2)}\left(s,t\right)=\lambda\left[\int_{\mathbb{R}^2}\alpha\left(s-y\right)\alpha\left(t-y\right)\mathrm{d}y\right]+\lambda^2\Lambda(\mathbb{R}^2)^2.
$$

8. *Let $\Phi' := \sum_{p\in\mathbb{Z}} \delta_{Z_p}$. Consider*

$$\begin{aligned}
\mathbf{E}^{!0'}[I] &= \mathbf{E}^{!0'}\left[\sum_{p\in\mathbb{Z}} g(|Z_p|)\right] \\
&= \mathbf{E}^{0'}\left[\sum_{p\in\mathbb{Z}^*} g(|Z_p|)\right] \\
&= \frac{1}{\lambda'|B|}\mathbf{E}\left[\sum_{Z_p\in B}\sum_{q\neq p} g(|Z_p - Z_q|)\right] \\
&= \frac{1}{\lambda'|B|}\mathbf{E}\left[\sum_{Z_p\neq Z_q\in\Phi'} \mathbf{1}\{Z_p\in B\}\, g(|Z_p - Z_q|)\right] \\
&= \frac{1}{\lambda'|B|}\mathbf{E}\left[\sum_{Z_p\neq Z_q\in\Phi'} f(Z_p, Z_q)\right] = \frac{1}{\lambda'|B|}\int \mathbf{1}\{s\in B\}\, g(|s-t|)\rho^{(2)}(s,t)\,\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

where for the fifth equality we introduce $f(s,t) := \mathbf{1}\{s\in B\}\, g(|s-t|)$ and the last equality is due to Campbell averaging formula.

Exercise 7.4.4. Matérn I hard-core point process. *Let $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda \in \mathbb{R}_+^*$ and let $h > 0$.*

1. *Show that the point process*

$$\Phi_1 = \sum_{k\in\mathbb{Z}} \delta_{X_k} \mathbf{1}\{\Phi(B(X_k, h)) = 1\}$$

is stationary with intensity parameter $\lambda_1 = \lambda e^{-\lambda\pi h^2}$ (cf. Example 3.4.1).

2. *Letting $\mathcal{K}$ be the reduced second moment measure of Φ_1 defined by (6.3.18), show that*

$$\mathcal{K}(B(0,r)) = \lambda e^{-\lambda\pi h^2}\int_{\mathbb{R}^d} \mathbf{1}_{\{h\le|z|<r\}} e^{-\lambda V_z(h)}\mathrm{d}z + e^{-\lambda\pi h^2} - 1,$$

where

$$V_h(x) = |B(0,h)\backslash B(x,h)|.$$

Solution 7.4.4. *1. Let $Z(t) = \Phi(B(t,h))$ and observe that*

$$\begin{aligned}
Z(0)\circ\theta_t &= \Phi\circ\theta_t(B(0,h)) \\
&= S_t\Phi(B(0,h)) \\
&= \Phi(B(t,h)) = Z(t).
\end{aligned}$$

Then the stochastic process $\{Z(t)\}_{t\in\mathbb{R}^d}$ is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Thus $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z(X_k))}$ is a stationary marked point process. Hence $\Phi_1 =$

$\sum_{k\in\mathbb{Z}} \delta_{X_k}, \mathbf{1}\{Z(X_k)=1\}$ *is stationary. Its intensity equals* $\lambda_1 = \lambda e^{-\lambda\pi h^2}$ *by Example 3.4.1.*

2. The reduced second moment measure of Φ_1 *is given by (6.3.18); that is*

$$\mathcal{K}(B(0,r)) = \mathbf{E}^0[\Phi_1(B(0,r))] - 1 = \mathbf{E}^0[g(\Phi)] - 1,$$

where $\mathbf{E}^0$ *be the expectation with respect to the Palm probability associated to* Φ_1 *and*

$$g(\Phi) := \Phi_1(B(0,r)) = \sum_{X\in\Phi} f(X,\Phi),$$

where

$$f(x,\Phi) := \mathbf{1}_{B(0,r)}(x)\,\mathbf{1}_{\{\Phi(B(x,h))=1\}}.$$

Applying Corollary 6.1.30 we get, for $B\in\mathcal{B}(\mathbb{R}^d)$ *such that* $0<|B|<\infty$,

$$\begin{aligned}\mathbf{E}^0[g(\Phi)] &= \frac{1}{\lambda|B|}\mathbf{E}\left[\sum_{X_m\in\Phi_1\cap B} g(\Phi\circ\theta_{X_m})\right]\\ &= \frac{1}{\lambda|B|}\mathbf{E}\left[\sum_{X_m\in\Phi\cap B} g(\Phi\circ\theta_{X_m})\,\mathbf{1}_{\{\Phi(B(X_m,h))=1\}}\right].\end{aligned}$$

Observe that

$$\begin{aligned}g(\Phi\circ\theta_x) &= \sum_{X\in\Phi\circ\theta_x} f(X,\Phi\circ\theta_x)\\ &= \sum_{X\in\Phi} f(X-x,\Phi\circ\theta_x)\\ &= \sum_{X\in\Phi} \mathbf{1}_{B(0,r)}(X-x)\,\mathbf{1}_{\{\Phi\circ\theta_x(B(X-x,h))=1\}}\\ &= \sum_{X\in\Phi} \mathbf{1}_{B(x,r)}(X)\,\mathbf{1}_{\{\Phi(B(X,h))=1\}}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}^0[g(\Phi)] &= \frac{1}{\lambda|B|}\mathbf{E}\left[\sum_{X_m\in\Phi}\sum_{X_k\in\Phi} \mathbf{1}_B(X_m)\,\mathbf{1}_{B(X_m,r)}(X_k)\,\mathbf{1}_{\{\Phi(B(X_k,h))=\Phi(B(X_m,h))=1\}}\right]\\ &= \frac{1}{\lambda|B|}\mathbf{E}\left[\sum_{X_m,X_k\in\Phi}\psi(X_m,X_k,\Phi)\right],\end{aligned}$$

where

$$\psi(x,y,\Phi) := \mathbf{1}_B(x)\,\mathbf{1}_{\{|x-y|<r\}}\mathbf{1}_{\{\Phi(B(x,h))=\Phi(B(y,h))=1\}}.$$

Applying Equation (3.5.2) we get

$$\mathbf{E}\left[\sum_{X_m,X_k\in\Phi}\psi(X_m,X_k,\Phi)\right] = \lambda^2\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{E}[\psi(x,y,\Phi+\delta_x+\delta_y)]\,\mathrm{d}x\mathrm{d}y + \lambda\int_{\mathbb{R}^d}\mathbf{E}[\psi(x,x,\Phi+\delta_x)]\,\mathrm{d}x.$$

Note that

$$\psi(x,x,\Phi+\delta_x)=\mathbf{1}_B(x)\,\mathbf{1}_{\{\Phi(B(x,h))=0\}},$$

then

$$\int_{\mathbb{R}^d}\mathbf{E}\left[\psi(x,x,\Phi+\delta_x)\right]\mathrm{d}x=|B|\,e^{-\lambda\pi h^2}.$$

On the other hand,

$$\begin{aligned}\psi(x,y,\Phi+\delta_x+\delta_y)&=\mathbf{1}_B(x)\,\mathbf{1}_{\{|x-y|<r\}}\mathbf{1}_{\left\{\Phi(B(x,h))=\Phi(B(y,h))=\mathbf{1}_{\{|x-y|<h\}}=0\right\}}\\&=\mathbf{1}_B(x)\,\mathbf{1}_{\{h\leq|x-y|<r\}}\mathbf{1}_{\{\Phi(B(x,h))=\Phi(B(y,h))=0\}}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}\left[\psi(x,y,\Phi+\delta_x+\delta_y)\right]&=\mathbf{1}_B(x)\,\mathbf{1}_{\{h\leq|x-y|<r\}}\mathbf{P}\left(\Phi\left(B\left(x,h\right)\right)=\Phi\left(B\left(y,h\right)\right)=0\right)\\&=\mathbf{1}_B(x)\,\mathbf{1}_{\{h\leq|x-y|<r\}}\mathbf{P}\left(\Phi\left(B\left(0,h\right)\right)=\Phi\left(B\left(y-x,h\right)\right)=0\right)\\&=\mathbf{1}_B(x)\,\varphi\left(y-x\right),\end{aligned}$$

where

$$\begin{aligned}\varphi(z)&:=\mathbf{1}_{\{h\leq|z|<r\}}\mathbf{P}\left(\Phi\left(B\left(0,h\right)\right)=\Phi\left(B\left(z,h\right)\right)=0\right)\\&=\mathbf{1}_{\{h\leq|z|<r\}}\mathbf{P}\left(\Phi\left(B\left(z,h\right)\right)=\Phi\left(B\left(0,h\right)\setminus B\left(z,h\right)\right)=0\right)\\&=\mathbf{1}_{\{h\leq|z|<r\}}e^{-\lambda\pi h^2}e^{-\lambda V_z(h)}.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{E}\left[\psi(x,y,\Phi+\delta_x+\delta_y)\right]\mathrm{d}x\mathrm{d}y&=\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_B(x)\,\varphi\left(y-x\right)\mathrm{d}x\mathrm{d}y\\&=\int_B\left(\int_{\mathbb{R}^d}\varphi\left(y-x\right)\mathrm{d}y\right)\mathrm{d}x\\&=\int_B\left(\int_{\mathbb{R}^d}\varphi\left(z\right)\mathrm{d}z\right)\mathrm{d}x\\&=|B|\,e^{-\lambda\pi h^2}\int_{\mathbb{R}^d}\mathbf{1}_{\{h\leq|z|<r\}}e^{-\lambda V_z(h)}\mathrm{d}z.\end{aligned}$$

Therefore,

$$\mathcal{K}\left(B\left(0,r\right)\right)=\lambda e^{-\lambda\pi h^2}\int_{\mathbb{R}^d}\mathbf{1}_{\{h\leq|z|<r\}}e^{-\lambda V_z(h)}\mathrm{d}z+e^{-\lambda\pi h^2}-1.$$

When $h\to 0$, $\mathcal{K}\left(B\left(0,r\right)\right)\to\lambda\pi r^2$ which is the reduced second moment measure of the Poisson point process Φ.

Exercise 7.4.5. Stationary i.i.d. marked point process on $\mathbb{R}^d$. *(Extension of Example 7.1.12 to $\mathbb{R}^d$.) Let $\tilde{\Phi}=\sum_{k\in\mathbb{Z}}\delta_{(X_k,Z_k)}$ be a stationary i.i.d. marked point process on $\mathbb{R}^d\times\mathbb{K}$ as in Example 7.1.2 with simple ground process $\Phi=\sum_{k\in\mathbb{Z}}\delta_{X_k}$ having intensity $\lambda\in\mathbb{R}_+^*$ and Palm probability $\mathbf{P}^0$. Show that, under $\mathbf{P}^0$, $\{Z_k\}_{k\in\mathbb{Z}}$ is an i.i.d sequence independent of Φ and Z_0 has the same distribution as that under $\mathbf{P}$.*

Solution 7.4.5. *By Proposition 7.1.8, there exists a stochastic process* $\{Z(t)\}_{t\in\mathbb{R}^d}$ *such that* $Z_k = Z(X_k)$. *Let* $\hat{Z} = \{Z_k\}_{k\in\mathbb{Z}}$ *and observe that*

$$\begin{aligned} Z_k \circ \theta_{X_n} &= Z(0) \circ \theta_{X_k} \circ \theta_{X_n} \\ &= Z(0) \circ \theta_{X_n + X_k \circ \theta_{X_n}} \\ &= Z(X_n + X_k \circ \theta_{X_n}) \end{aligned}$$

where the first equality is due to (7.1.2) and the second equality follows from (6.1.4).

Fix some $n \in \mathbb{Z}$. *In the same lines as Lemma 7.1.7, it may be proved that, for each* $k \in \mathbb{Z}$, *there exists some* $l = l(k, \Phi) \in \mathbb{Z}$ *such that*

$$X_k \circ \theta_{X_n} = X_l - X_n.$$

The above two equalities imply

$$Z_k \circ \theta_{X_n} = Z\left(X_{l(k,\Phi)}\right) = Z_{l(k,\Phi)}.$$

Thus

$$\hat{Z} \circ \theta_{X_n} = \{Z_k \circ \theta_{X_n}\}_{k\in\mathbb{Z}} = \left\{Z_{l(k,\Phi)}\right\}_{k\in\mathbb{Z}}.$$

Observe that $\hat{Z} \circ \theta_{X_n}$ *is a random permutation of* $\hat{Z}$ *where the permutation* $\mathbb{Z} \to \mathbb{Z}, k \mapsto l(k,\Phi)$ *is independent of* $\hat{Z}$ *under* $\mathbf{P}$. *Therefore, conditionally to* Φ, $\hat{Z} \circ \theta_{X_n} \overset{\text{dist}}{=} \hat{Z}$ *under* $\mathbf{P}$. *Then the proof continues in the same lines as in Example 7.1.12.*

Exercise 7.4.6. *Let* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ *be a stationary framework,* $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ *be a point process on* $\mathbb{R}^d$ *compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^d}$. *Let*

$$\tilde{V}_n = \{y \in \mathbb{R}^d : |y - X_n| \le |y - X_k|, \forall k \in \mathbb{Z}\}$$

and

$$V_n = \tilde{V}_n - X_n, \quad n \in \mathbb{Z}.$$

Assume the enumeration property (7.1.6). Show that $V_n = V_0 \circ \theta_{X_n}$.

Solution 7.4.6. *Note that*

$$\begin{aligned} V_n &= \{y - X_n \in \mathbb{R}^d : |y - X_n| \le |y - X_k|, \forall k \in \mathbb{Z}\} \\ &= \{y \in \mathbb{R}^d : |y| \le |y + X_n - X_k|, \forall k \in \mathbb{Z}\}. \end{aligned}$$

Then

$$V_0 := \{y \in \mathbb{R}^d : |y| \le |y + X_0 - X_k|, \forall k \in \mathbb{Z}\}, \quad x \in \mathbb{R}^d.$$

Property (7.1.6), implies that $X_0 \circ \theta_{X_n} = 0$. *Moreover,* $\forall k \in \mathbb{Z}$, $X_k \circ \theta_{X_n} = X_m - X_n$ *for some* $m \in \mathbb{Z}$. *Then*

$$V_0 \circ \theta_{X_n} = \{y \in \mathbb{R}^d : |y| \le |y - (X_m - X_n)|, \forall m \in \mathbb{Z}\} = V_n.$$

Exercise 7.4.7. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a point process on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Assume the enumeration property (7.1.6). Among the following sequences of marks of $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$, which ones satisfies (7.1.7)? In each case, justify your answer.*

1. *Sequence $\{a_n\}_{n\in\mathbb{Z}}$, :*

$$a_n = \Phi(B(X_n, r)), \quad n \in \mathbb{Z},$$

where $r \in \mathbb{R}_+^*$ is a fixed parameter.

2. *Sequence $\{b_n\}_{n\in\mathbb{Z}}$:*

$$b_n = \Phi(B(2X_n, r)), \quad n \in \mathbb{Z}.$$

3. *Sequence $\{c_n\}_{n\in\mathbb{Z}}$:*

$$c_n = \inf_{k\neq n} |X_n - X_k|, \quad n \in \mathbb{Z}.$$

4. *Sequence $\{d_n\}_{n\in\mathbb{Z}}$:*

$$d_n = \inf_{q\in\mathbb{Z}^d} |X_n - q|, \quad n \in \mathbb{Z}.$$

Solution 7.4.7. *Property (7.1.6), implies that $X_0 \circ \theta_{X_n} = 0$.*

1. *We have to check that $a_n = a_0 \circ \theta_{X_n}$. Since $a_0 = \Phi(B(X_0, r))$, then*

$$\begin{aligned} a_0 \circ \theta_{X_n} &= \Phi(B(X_0 \circ \theta_{X_n}, r) + X_n) \\ &= \Phi(B(0, r) + X_n) = \Phi(B(X_n, r)) = a_n. \end{aligned}$$

2. *Consider*

$$\begin{aligned} b_0 \circ \theta_{X_n} &= \Phi(B(2X_0 \circ \theta_{X_n}, r) + X_n) \\ &= \Phi(B(0, r) + X_n) \\ &= \Phi(B(X_n, r)) \neq b_n. \end{aligned}$$

3. *Recall that, $\forall k \in \mathbb{Z}$, $X_k \circ \theta_{X_n} = X_m - X_n$ for some $m \in \mathbb{Z}$. Then*

$$\begin{aligned} c_0 \circ \theta_{X_n} &= \inf_{k\neq 0} |X_0 \circ \theta_{X_n} - X_k \circ \theta_{X_n}| \\ &= \inf_{m\neq n} |0 - (X_m - X_n)| = c_n. \end{aligned}$$

4. *Consider*

$$\begin{aligned} d_0 \circ \theta_{X_n} &= \inf_{q\in\mathbb{Z}} |X_0 \circ \theta_{X_n} - q| \\ &= \inf_{q\in\mathbb{Z}} |-q| \\ &= 0 \neq d_n. \end{aligned}$$

Exercise 7.4.8. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ be a point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ with intensity $\lambda \in \mathbb{R}_+^*$, and $\{P(t)\}_{t\in\mathbb{R}^d}$ be an $\mathbb{R}_+$-valued stochastic process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Let*

$$I(t) = \sum_{n\in\mathbb{Z}} \frac{P(X_n)}{\ell(|t - X_n|)}, \quad t \in \mathbb{R}^d,$$

for some measurable function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$.

1. *Compute $\mathbf{E}[I(0)]$. Give a sufficient condition for $\mathbf{E}[I(0)]$ to be finite.*
2. *Show that $\{I(t)\}_{t\in\mathbb{R}^d}$ is compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$.*
3. *Let*

$$Z_n = \frac{1}{\left|\tilde{V}_n\right|} \int_{\tilde{V}_n} \frac{P(X_n)/\ell(|t - X_n|)}{N + I(t)} \mathrm{d}t, \quad n \in \mathbb{Z},$$

where N is a positive constant and $\tilde{V}_n = \{y \in \mathbb{R}^d : |y - X_n| \leq |y - X_k|, \forall k \in \mathbb{Z}\}$. Assume the enumeration property (7.1.6). Show that $Z_n = Z_0 \circ \theta_{X_n}$.

Solution 7.4.8. *1. Let $f(x,p) = \frac{p}{\ell(|x|)}$. By Proposition 7.2.4*

$$\begin{aligned}
\mathbf{E}[I(0)] &= \mathbf{E}\left[\sum_{n\in\mathbb{Z}} f(X_n, P(X_n))\right] \\
&= \lambda \int_{\mathbb{R}^d} \mathbf{E}^0[f(x, P(0))]\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^d} \mathbf{E}^0\left[\frac{P(0)}{\ell(|x|)}\right] \mathrm{d}x = \lambda \mathbf{E}^0[P(0)] \int_{\mathbb{R}^d} \frac{1}{\ell(|x|)} \mathrm{d}x.
\end{aligned}$$

If $\mathbf{E}^0[P(0)] < \infty$ and $\int_{\mathbb{R}^d} \frac{1}{\ell(|x|)} \mathrm{d}x < \infty$ then $\mathbf{E}[I(0)] < \infty$.

2. We have to prove that $I(0) \circ \theta_t = I(t)$ for all $t \in \mathbb{R}^d$. Let $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{X_n, P(X_n)}$. By Proposition 7.1.8(i), $\tilde{\Phi} \circ \theta_t = \sum_{m\in\mathbb{Z}} \delta_{X_m - t, P(X_m)}$. Note that

$$I(0) = \sum_{n\in\mathbb{Z}} \frac{P(X_n)}{\ell(|X_n|)}.$$

Then

$$I(0) \circ \theta_t = \sum_{m\in\mathbb{Z}} \frac{P(X_m)}{\ell(|X_m - t|)} = I(t).$$

3. Let

$$V_n = \tilde{V}_n - X_n, \quad n \in \mathbb{Z}.$$

It is shown in Exercise 7.4.6 that $V_n = V_0 \circ \theta_{X_n}$. Note that

$$\begin{aligned}
Z_0 &= \frac{1}{|V_0 + X_0|} \int_{V_0 + X_0} \frac{P(X_0)/\ell(|t - X_0|)}{N + I(t)} \mathrm{d}t \\
&= \frac{1}{|V_0|} \int_{V_0 + X_0} \frac{P(X_0)/\ell(|t - X_0|)}{N + I(t)} \mathrm{d}t.
\end{aligned}$$

Since $X_0 \circ \theta_{X_n} = 0$, then

$$\begin{aligned} Z_0 \circ \theta_{X_n} &= \frac{1}{|V_0 \circ \theta_{X_n}|} \int_{V_0 \circ \theta_{X_n}} \frac{P(0)/\ell(|t|)}{N + I(t) \circ \theta_{X_n}} \mathrm{d}t \\ &= \frac{1}{|V_n|} \int_{V_n} \frac{P(0)/\ell(|t|)}{N + I(t + X_n)} \mathrm{d}t \\ &= \frac{1}{|V_n|} \int_{\tilde{V}_n - X_n} \frac{P(0)/\ell(|t|)}{N + I(t + X_n)} \mathrm{d}t \\ &= \frac{1}{\left|\tilde{V}_n\right|} \int_{\tilde{V}_n} \frac{P(0)/\ell(|s - X_n|)}{N + I(s)} \mathrm{d}t = Z_n, \end{aligned}$$

where for the second equality we use the fact that $I(t) \circ \theta_{X_n} = I(t + X_n)$ which follows from (6.1.6) and for the fourth equality we make the change of variable $s = t + X_n$.

Exercise 7.4.9. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t \in \mathbb{R}^d}, \mathbf{P})$ be a stationary framework, $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ be a l.c.s.h. space, and let $\tilde{\Phi} = \sum_{k \in \mathbb{Z}} \delta_{(X_k, Z_k)}$.be marked point process on $\mathbb{R}^d \times \mathbb{K}$ with simple ground process $\Phi = \sum_{k \in \mathbb{Z}} \delta_{X_k}$. Show that $\tilde{\Phi}$ is compatible with the flow iff Φ is so and*

$$Z_k = Y \circ \theta_{X_k}, \quad k \in \mathbb{Z}, \tag{7.4.1}$$

for some random variable Y.

Solution 7.4.9. Sufficiency. *Consider the stochastic process $Z(t) := Y \circ \theta_t$. Observe that*

$$Z(t, \theta_x \omega) = Y(\theta_t \theta_x \omega) = Y(\theta_{t+x} \omega) = Z(t + x, \omega),$$

which, by Lemma 6.1.9, shows that $\{Z(t)\}_{t \in \mathbb{R}^d}$ is compatible with the flow. Moreover,

$$Z_k = Y \circ \theta_{X_k} = Z(X_k).$$

Then by Proposition 7.1.8(i), $\tilde{\Phi}$ is compatible with the flow.

Necessity. *Assume that $\tilde{\Phi}$ is compatible with the flow. Then, by Proposition 7.1.8(ii), Φ is so and there exists a stochastic process $\{Z(t)\}_{t \in \mathbb{R}^d}$ compatible with the flow such that*

$$Z_k = Z(X_k) = Z(0) \circ \theta_{X_k}, \quad k \in \mathbb{Z}.$$

Chapter 8

Ergodicity

8.1 Motivation

Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, Φ be a random measure on $\mathbb{R}^d$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$, $f : \Omega \to \mathbb{R}_+$ be a measurable function and $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}^d$ increasing to $\mathbb{R}^d$. We are interested in the almost sure limits as $n \to \infty$ of the following two empirical means:

$$\frac{1}{|B_n|}\int_{B_n} f \circ \theta_x \mathrm{d}x \quad \text{and} \quad \frac{1}{\Phi(B_n)}\int_{B_n} f \circ \theta_x \Phi\left(\mathrm{d}x\right).$$

In the first case, taking the expectation we get that for all n,

$$\begin{aligned}\mathbf{E}\left[\frac{1}{|B_n|}\int_{B_n} f \circ \theta_x \mathrm{d}x\right] &= \frac{1}{|B_n|}\int_{B_n} \mathbf{E}\left[f \circ \theta_x\right]\mathrm{d}x \\ &= \frac{1}{|B_n|}\int_{B_n} \mathbf{E}\left[f\right]\mathrm{d}x = \mathbf{E}\left[f\right],\end{aligned}$$

where the first equality is due to the Fubini-Tonelli theorem and the second one follows from stationarity. So a first natural question is whether $\frac{1}{|B_n|}\int_{B_n} f\circ\theta_x \mathrm{d}x$ tends to $\mathbf{E}\left[f\right]$, or the conditions under which this property holds.

In the second case, we have

$$\begin{aligned}\frac{1}{\Phi(B_n)}\int_{B_n} f \circ \theta_x \Phi\left(\mathrm{d}x\right) &= \frac{|B_n|}{\Phi(B_n)}\frac{1}{|B_n|}\int_{B_n} f \circ \theta_x \Phi\left(\mathrm{d}x\right) \\ &= \frac{\frac{1}{|B_n|}\int_{B_n} f \circ \theta_x \Phi\left(\mathrm{d}x\right)}{\frac{1}{|B_n|}\int_{B_n} 1 \circ \theta_x \Phi\left(\mathrm{d}x\right)}.\end{aligned}$$

By the C-L-M-M theorem 6.1.28, the expectation of the numerator is

$$\mathbf{E}\left[\frac{1}{|B_n|}\int_{B_n} f \circ \theta_x \Phi\left(\mathrm{d}x\right)\right] = \frac{\lambda}{|B_n|}\int_{B_n} \mathbf{E}^0\left[f\right]\mathrm{d}x = \lambda\mathbf{E}^0\left[f\right],$$

whereas the expectation of the denominator is λ. Thus, a second natural question is whether, or under what conditions, $\frac{1}{\Phi(B_n)}\int_{B_n} f\circ\theta_x \Phi(\mathrm{d}x) \to \mathbf{E}^0[f]$.

These two questions are the main topic of the ergodic theorems discussed in this section.

8.2 Birkhoff's pointwise ergodic theorem

8.2.1 Ergodic theory

In the present section, we give some basic results of the general ergodic theory without explicit reference to random measures.

Definition 8.2.1. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework. An event $A \in \mathcal{A}$ is said to be $\{\theta_t\}$-*invariant *if for all $t \in \mathbb{R}^d$, $\mathbf{P}(A \bigtriangleup \theta_{-t}A) = 0$, where $\bigtriangleup$ denotes the symmetrical difference. Let*

$$\mathcal{I} := \{A \in \mathcal{A} : A \text{ is } \{\theta_t\}\text{-invariant}\}.$$

Using the facts that $A^c \bigtriangleup B^c = A \bigtriangleup B$ and $\left(\bigcup_n A\right) \bigtriangleup \left(\bigcup_n B\right) \subset \bigcup_n (A_n \bigtriangleup B_n)$, it is easy to check that $\mathcal{I}$ is a σ-algebra. It is called the invariant σ-algebra.

Example 8.2.2. *Any $A \in \mathcal{A}$ such that $\mathbf{P}(A) = 1$ is $\{\theta_t\}$-invariant. Indeed, since $\mathbf{P}$ is invariant with respect to $\{\theta_t\}_{t\in\mathbb{R}^d}$, $\mathbf{P}(\theta_{-t}A) = 1$ and therefore $\mathbf{P}(A \bigtriangleup \theta_{-t}A) = 0$ for all $t \in \mathbb{R}^d$.*

Definition 8.2.3. *A sequence of sets $\{B_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ is said to be a* convex averaging sequence *if each B_n is bounded, Borel and convex; $B_n \subset B_{n+1}$, $\forall n$; and $\sup\{r \geq 0 : B_n$ contains a ball of radius $r\} \longrightarrow \infty$ when $n \to \infty$.*

Theorem 8.2.4. *[31, Proposition 12.2.II]* Birkhoff's pointwise ergodic theorem. *Consider a stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$. Let $\mathcal{I}$ be its invariant σ-algebra. Let $\{B_n\}_{n\in\mathbb{N}}$ be a convex averaging sequence in $\mathbb{R}^d$. Then for all $f \in L^1_{\mathbb{R}}(\mathbf{P}, \Omega)$ (i.e., $f : \Omega \to \mathbb{R}$ measurable and such that $\mathbf{E}[|f|] < \infty$),*

$$\frac{1}{|B_n|}\int_{B_n} f\circ\theta_x \mathrm{d}x \underset{n\to\infty}{\longrightarrow} \mathbf{E}[f|\mathcal{I}], \quad \mathbf{P}\text{-a.s.}$$

Proof. Cf. [95]. □

Definition 8.2.5. *The invariant σ-algebra $\mathcal{I}$ is said* $\mathbf{P}$-trivial *if $\forall A \in \mathcal{I}, \mathbf{P}(A) \in \{0,1\}$. We say in this case that $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ is* metrically transitive.

Lemma 8.2.6. *If $\mathcal{I}$ is $\mathbf{P}$-trivial, then for all $f \in L^1_{\mathbb{R}}(\mathbf{P}, \Omega)$, $\mathbf{E}[f|\mathcal{I}] = \mathbf{E}[f]$.*

Proof. Recall that a version of the conditional expectation of f given $\mathcal{I}$ is any integrable $\mathcal{I}$-measurable random variable Z such that $\mathbf{E}[f1_A] = \mathbf{E}[Z1_A]$, for all $A \in \mathcal{I}$. For any $A \in \mathcal{I}$, either $\mathbf{P}(A) = 0$ in which case

$$\mathbf{E}[f1_A] = 0, \quad \mathbf{E}[\mathbf{E}[f]1_A] = \mathbf{E}[f]\mathbf{P}(A) = 0,$$

or $\mathbf{P}(A) = 1$ which implies

$$\mathbf{E}[f1_A] = \mathbf{E}[f], \quad \mathbf{E}[\mathbf{E}[f]1_A] = \mathbf{E}[f]\mathbf{P}(A) = \mathbf{E}[f].$$

It follows that

$$\mathbf{E}[f1_A] = \mathbf{E}[\mathbf{E}[f]1_A], \quad \forall A \in \mathcal{I}.$$

Therefore $\mathbf{E}[f|\mathcal{I}] = \mathbf{E}[f]$. □

Definition 8.2.7. *A stationary framework* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ *is said to be* ergodic *if*

$$\lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{P}(A_1 \cap \theta_{-x}A_2)\mathrm{d}x = \mathbf{P}(A_1)\mathbf{P}(A_2), \quad \forall A_1, A_2 \in \mathcal{A}. \quad (8.2.1)$$

It is said to be mixing *if*

$$\lim_{|x|\to\infty} \mathbf{P}(A_1 \cap \theta_{-x}A_2) = \mathbf{P}(A_1)\mathbf{P}(A_2), \quad \forall A_1, A_2 \in \mathcal{A}. \quad (8.2.2)$$

Lemma 8.2.8. *If a stationary framework* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ *is mixing, then it is ergodic.*

Proof. Making the change of variable $y = x/a$, we get

$$\frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{P}(A_1 \cap \theta_{-x}A_2)\,\mathrm{d}x = \frac{1}{2^d} \int_{[-1,1]^d} \mathbf{P}(A_1 \cap \theta_{-ay}A_2)\mathrm{d}y.$$

For any $y \in [-1,1]^d \setminus \{0\}$, we deduce from the mixing property that

$$\lim_{a\to\infty} \mathbf{P}(A_1 \cap \theta_{-ay}A_2) = \mathbf{P}(A_1)\mathbf{P}(A_2).$$

Then, by the dominated convergence theorem [11, Theorem 16.4 p.209],

$$\lim_{a\to\infty} \frac{1}{2^d} \int_{[-1,1]^d} \mathbf{P}(A_1 \cap \theta_{-ay}A_2)\mathrm{d}y = \frac{1}{2^d} \int_{[-1,1]^d} \mathbf{P}(A_1)\mathbf{P}(A_2)\mathrm{d}y = \mathbf{P}(A_1)\mathbf{P}(A_2).$$

□

Theorem 8.2.9. *A stationary framework* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ *is metrically transitive if and only if it is ergodic.*

Proof. **Direct part**. Assume that the framework is ergodic. Consider some $A \in \mathcal{I}$. For all $t \in \mathbb{R}^d$, $\mathbf{P}(A \triangle \theta_{-t}A) = 0$; since $A \cap \theta_{-t}A = A \setminus (A \setminus \theta_{-t}A)$ and $A \setminus \theta_{-t}A \subset A \triangle \theta_{-t}A$, then $\mathbf{P}(A \cap \theta_{-t}A) = \mathbf{P}(A) - \mathbf{P}(A \setminus \theta_{-t}A) = \mathbf{P}(A)$. On the other hand, we deduce from ergodicity that

$$\lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{P}(A \cap \theta_{-x}A)\mathrm{d}x = \mathbf{P}(A)^2.$$

Then $\mathbf{P}(A) = \mathbf{P}(A)^2$, thus $\mathbf{P}(A) \in \{0,1\}$. Therefore the invariant σ-algebra $\mathcal{I}$ is $\mathbf{P}$-trivial, and the framework is consequently metrically transitive. **Converse**

part. Assume that the framework is metrically transitive. Let $A_1, A_2 \in \mathcal{A}$. By Birkhoff's theorem 8.2.4 and Lemma 8.2.6, we have

$$\lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} 1\left\{\theta_x\omega \in A_2\right\} \mathrm{d}x = \mathbf{E}[\mathbf{1}\{\omega \in A_2\}] = \mathbf{P}(A_2).$$

Then

$$\begin{aligned}
\mathbf{P}(A_1)\mathbf{P}(A_2) &= \mathbf{E}[\mathbf{1}\{\omega \in A_1\}] \left(\lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{1}\left\{\theta_x\omega \in A_2\right\} \mathrm{d}x\right) \\
&= \mathbf{E}\left[\mathbf{1}\{\omega \in A_1\} \lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{1}\left\{\theta_x\omega \in A_2\right\} \mathrm{d}x\right] \\
&= \lim_{a\to\infty} \mathbf{E}\left[\mathbf{1}\{\omega \in A_1\} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{1}\left\{\theta_x\omega \in A_2\right\} \mathrm{d}x\right] \\
&= \lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{E}\left[\mathbf{1}\{\omega \in A_1\}\mathbf{1}\left\{\theta_x\omega \in A_2\right\}\right] \mathrm{d}x \\
&= \lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbf{P}\left(A_1 \cap \theta_{-x}A_2\right) \mathrm{d}x,
\end{aligned}$$

where the third equality follows from the dominated convergence theorem [11, Theorem 16.4 p.209], the fourth one is due to the Fubini-Tonelli theorem. □

The following lemma will be useful to prove the ergodicity of some random measures in the following section.

Lemma 8.2.10. *For a stationary framework to be ergodic (respectively mixing) it is enough that the limit in (8.2.1) (respectively (8.2.2)) holds for all A_1, A_2 in a semiring generating $\mathcal{A}$. A* semiring *is a family of sets $\mathcal{S}$ closed under intersections such that every symmetric difference of sets in $\mathcal{S}$ can be represented as a finite union of disjoint sets in $\mathcal{S}$.*

Proof. Cf. [31, Lemma 12.3.II]. □

8.3 Ergodic theorems for random measures

8.3.1 Ergodicity of random measures

Definition 8.3.1. *A stationary random measure Φ on $\mathbb{R}^d$ is said to be* ergodic *(respectively* mixing*) if and only if the stationary framework $\left(\mathbb{M}\left(\mathbb{R}^d\right), \mathcal{M}\left(\mathbb{R}^d\right), \{S_t\}_{t\in\mathbb{R}^d}, \mathbf{P}_\Phi\right)$ is ergodic (respectively mixing).*

Proposition 8.3.2. *Let Φ be a stationary random measure with Laplace transform $\mathcal{L}_\Phi$. Then* *(i) Φ is ergodic if and only if*

$$\lim_{a\to\infty}\frac{1}{(2a)^d}\int_{[-a,a]^d}\mathcal{L}_\Phi(f_1+S_xf_2)\mathrm{d}x=\mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(f_2),$$

for all measurable $f_1,f_2:\mathbb{R}^d\to\mathbb{R}_+$ bounded with bounded support.

(ii) Φ is mixing if and only if

$$\lim_{|x|\to\infty}\mathcal{L}_\Phi(f_1+S_xf_2)=\mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(f_2),$$

for all measurable $f_1,f_2:\mathbb{R}^d\to\mathbb{R}_+$ bounded with bounded support.

Proof. Cf. [31, Proposition 12.3.VI]. □

Corollary 8.3.3. *A homogeneous Poisson point process on $\mathbb{R}^d$ is mixing and ergodic.*

Proof. Let Φ be a homogeneous Poisson point process on $\mathbb{R}^d$ and let $f_1,f_2:\mathbb{R}^d\to\mathbb{R}_+$ be measurable bounded with bounded support. Let $B_1,B_2\in\mathcal{B}_\mathrm{c}\left(\mathbb{R}^d\right)$ be their respective supports. For $|x|$ sufficiently large, the sets B_1 and B_2+x are disjoint, and then the restrictions of Φ to these sets are independent. For such x,

$$\begin{aligned}\mathcal{L}_\Phi(f_1+S_xf_2)&=\mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d}(f_1+S_xf_2)\,\mathrm{d}\Phi\right)\right]\\&=\mathbf{E}\left[\exp\left(-\int_{B_1}f_1\mathrm{d}\Phi\right)\exp\left(-\int_{B_2}S_xf_2\mathrm{d}\Phi\right)\right]\\&=\mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(S_xf_2)=\mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(f_2),\end{aligned}$$

where the last equality follows from stationarity. Thus Φ is mixing by Proposition 8.3.2(ii). It follows from Lemma 8.2.8 that Φ is also ergodic. □

8.3.2 Ergodic theorem for random measures

Theorem 8.3.4. Ergodic theorem for random measures. *Let $(\Omega,\mathcal{A},\{\theta_t\}_{t\in\mathbb{R}^d},\mathbf{P})$ be a stationary and ergodic framework and Φ be a random measure on $\mathbb{R}^d$, compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ and with intensity $\lambda\in\mathbb{R}_+^*$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a convex averaging sequence in $\mathbb{R}^d$. Then for all measurable $f:\Omega\to\bar{\mathbb{R}}_+$ in $L^1_\mathbb{R}\left(\mathbf{P}^0,\Omega\right)$ (i.e., such that $\mathbf{E}^0\left[|f|\right]<\infty$), we have*

$$\frac{1}{|B_n|}\int_{B_n}f\circ\theta_x\Phi\left(\mathrm{d}x\right)\underset{n\to\infty}{\longrightarrow}\lambda\mathbf{E}^0\left[f\right],\quad\mathbf{P}\text{-}a.s.$$

Proof. Cf. [31, Theorem 12.2.IV]. **Consider first the case** $f\equiv1$. For $\epsilon>0$, let $g_\epsilon:\mathbb{R}^d\to\mathbb{R}_+$ be a measurable mapping with the following properties: (i)

$g_\epsilon(x)$ is nonnegative and continuous; (ii) $g_\epsilon(x) \equiv 0$ for $x \notin B(0,\epsilon)$; and (iii) $\int_{\mathbb{R}^d} g_\epsilon(x)\mathrm{d}u = 1$. Let

$$h(\omega) = \int_{\mathbb{R}^d} g_\epsilon(x)\Phi(\mathrm{d}x).$$

By Campbell's averaging formula (1.2.2)

$$\mathbf{E}\left[h(\omega)\right] = \int_{\mathbb{R}^d} g_\epsilon(x)\lambda \mathrm{d}x = \lambda.$$

In addition,

$$h(\theta_t\omega) = \int_{\mathbb{R}^d} g_\epsilon(x)\Phi\circ\theta_t\left(\mathrm{d}x\right) = \int_{\mathbb{R}^d} g_\epsilon(x-t)\Phi(\mathrm{d}x).$$

Let

$$\begin{aligned} B_n^\epsilon &= \bigcup_{x\in B_n} B(x,\epsilon) \\ B_n^{-\epsilon} &= \{x\in B_n : B(x,\epsilon)\subset B_n\}. \end{aligned}$$

From our assumptions on g_ϵ

$$y\in B_n \Rightarrow \int_{B_n^\epsilon} g_\epsilon(y-t)\mathrm{d}t = 1.$$

Indeed, by the change of variable $v := y-t$, the last integral is $\int_{y-B_n^\epsilon} g_\epsilon(v)\,\mathrm{d}v$, and

$$\begin{aligned} B(0,\epsilon)\subset y-B_n^\epsilon &\iff -B(0,\epsilon)\subset y-B_n^\epsilon \\ &\iff -B(y,\epsilon)\subset -B_n^\epsilon \\ &\iff B(y,\epsilon)\subset B_n^\epsilon, \end{aligned}$$

which holds true when $y\in B_n$. Moreover,

$$y\notin B_n \Rightarrow \int_{B_n^{-\epsilon}} g_\epsilon(y-t)\mathrm{d}t = 0.$$

Indeed,

$$\begin{aligned} y\notin B_n, t\in B_n^{-\epsilon} &\Rightarrow y\notin B_n, t\in B_n : B(t,\epsilon)\subset B_n \\ &\Rightarrow |y-t|\geq\epsilon \\ &\Rightarrow g_\epsilon(y-t) = 0. \end{aligned}$$

Hence

$$\int_{B_n^{-\epsilon}} g_\epsilon(y-t)\mathrm{d}t \leq 1_{B_n}(y) \leq \int_{B_n^\epsilon} g_\epsilon(y-t)\mathrm{d}t. \tag{8.3.1}$$

Integrating over $\mathbb{R}^d$ w.r.t. $\Phi(\mathrm{d}y)$ we get

$$\int_{\mathbb{R}^d} \int_{B_n^{-\epsilon}} g_\epsilon(y-t)\mathrm{d}t\Phi(\mathrm{d}y) \le \Phi(B_n) \le \int_{\mathbb{R}^d} \int_{B_n^{\epsilon}} g_\epsilon(y-t)\mathrm{d}t\Phi(\mathrm{d}y).$$

But, for all $A \in \mathcal{B}\left(\mathbb{R}^d\right)$

$$\int_{\mathbb{R}^d} \left(\int_A g_\epsilon(y-t)\mathrm{d}t \right) \Phi(\mathrm{d}y) = \int_A \left(\int_{\mathbb{R}^d} g_\epsilon(y-t)\Phi(\mathrm{d}y) \right) \mathrm{d}t = \int_A h \circ \theta_t \mathrm{d}t.$$

Then,

$$\int_{B_n^{-\epsilon}} h \circ \theta_t \mathrm{d}t \le \Phi(B_n) \le \int_{B_n^{\epsilon}} h \circ \theta_t \mathrm{d}t.$$

Hence

$$\limsup_n \frac{\Phi(B_n)}{|B_n|} \le \limsup_n \frac{|B_n^\epsilon|}{|B_n|} \times \left(\frac{1}{|B_n^\epsilon|} \int_{B_n^\epsilon} h \circ \theta_t \mathrm{d}t \right).$$

The second term in the right-hand side of the above inequality tends **P**-a.s. to λ from Birkhoff's theorem 8.2.4. The first term tends to 1 because B_n are convex with $\sup\{r \ge 0 : B_n$ contains a ball of radius $r\} \longrightarrow \infty$ when $n \to \infty$. By the same arguments

$$\liminf_n \frac{\Phi(B_n)}{|B_n|} \ge \liminf_n \frac{|B_n^{-\epsilon}|}{|B_n|} \times \frac{1}{\left|B_n^{-\epsilon}\right|} \int_{B_n^{-\epsilon}} h \circ \theta_t \mathrm{d}t = \lambda, \quad \mathbf{P}\text{-a.s.}$$

Hence

$$\lim_{n\to\infty} \frac{\Phi(B_n)}{|B_n|} = \lambda \quad \mathbf{P} \text{ a.s.}$$

Consider now a general f. Let

$$h(\omega) = \int_{\mathbb{R}^d} g_\epsilon(x) f \circ \theta_x \Phi(\mathrm{d}x).$$

By the C-L-M-M theorem 6.1.28

$$\mathbf{E}\left[h(\omega)\right] = \lambda \int_{\mathbb{R}^d} \mathbf{E}^0\left[g_\epsilon(x) f\right] \mathrm{d}x = \lambda \mathbf{E}^0\left[f\right].$$

Observe that,

$$h(\theta_t\omega) = \int_{\mathbb{R}^d} g_\epsilon(x) f \circ \theta_x \circ \theta_t \;\, \Phi \circ \theta_t\left(\mathrm{d}x\right) = \int_{\mathbb{R}^d} g_\epsilon(x-t) f \circ \theta_x \Phi(\mathrm{d}x).$$

Multiplying (8.3.1) by $f \circ \theta_y$ and then integrating over $\mathbb{R}^d$ w.r.t. $\Phi(\mathrm{d}y)$ we get

$$\begin{aligned}\int_{\mathbb{R}^d} \left(\int_{B_n^{-\epsilon}} g_\epsilon(y-t)\mathrm{d}t \right) f \circ \theta_y \Phi(\mathrm{d}y) &\le \int_{B_n} f \circ \theta_y \Phi(\mathrm{d}y) \\ &\le \int_{\mathbb{R}^d} \left(\int_{B_n^\epsilon} g_\epsilon(y-t)\mathrm{d}t \right) f \circ \theta_y \Phi(\mathrm{d}y).\end{aligned}$$

But, for all $A \in \mathcal{B}\left(\mathbb{R}^d\right)$

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_A g_\epsilon(y-t)\mathrm{d}t\right) f\circ\theta_y\Phi(\mathrm{d}y) &= \int_A\left(\int_{\mathbb{R}^d} g_\epsilon(y-t) f\circ\theta_y\Phi(\mathrm{d}y)\right)\mathrm{d}t\\ &= \int_A h\circ\theta_t\mathrm{d}t.\end{aligned}$$

Then,

$$\int_{B_n^{-\epsilon}} h\circ\theta_t\mathrm{d}t \le \int_{B_n} f\circ\theta_y\Phi(\mathrm{d}y) \le \int_{B_n^{\epsilon}} h\circ\theta_t\mathrm{d}t.$$

Hence

$$\limsup_n \frac{\int_{B_n} f\circ\theta_y\Phi(\mathrm{d}y)}{|B_n|} \le \limsup_n \frac{|B_n^\epsilon|}{|B_n|}\times\frac{1}{|B_n^\epsilon|}\int_{B_n^\epsilon} h\circ\theta_t\mathrm{d}t.$$

The second term in the right-hand side of the above inequality tends $\mathbf{P}$-a.s. to $\mathbf{E}[h] = \lambda\mathbf{E}^0[f]$ from Birkhoff's theorem 8.2.4. The first term tends to 1 as seen in the case $f\equiv 1$. By the same arguments,

$$\liminf_n \frac{\int_{B_n} f\circ\theta_y\Phi(\mathrm{d}y)}{|B_n|} \ge \liminf_n \frac{|B_n^{-\epsilon}|}{|B_n|}\times\frac{1}{\left|B_n^{-\epsilon}\right|}\int_{B_n^{-\epsilon}} h\circ\theta_t\mathrm{d}t = \lambda\mathbf{E}^0[f], \quad \mathbf{P}\text{-a.s.}$$

Hence

$$\lim_{n\to\infty}\frac{\int_{B_n} f\circ\theta_y\Phi(\mathrm{d}y)}{|B_n|} = \lambda\mathbf{E}^0[f], \quad \mathbf{P}\text{-a.s.}$$

□

Taking $f\equiv 1$ in Theorem 8.3.4, we get

$$\frac{\Phi(B_n)}{|B_n|} \underset{n\to\infty}{\longrightarrow} \lambda, \quad \mathbf{P}\text{-a.s.} \tag{8.3.2}$$

Then

$$\frac{1}{\Phi(B_n)}\int_{B_n} f\circ\theta_x\Phi(\mathrm{d}x) = \frac{|B_n|}{\Phi(B_n)}\frac{1}{|B_n|}\int_{B_n} f\circ\theta_x\Phi(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \mathbf{E}^0[f].$$

Notice that (8.3.2) also implies that $\Phi(B_n)$ almost surely tends to infinity as n tends to infinity. That is,

Corollary 8.3.5. *For all stationary and ergodic point processes Φ with a positive intensity, $\mathbf{P}(\Phi(\mathbb{R}^d)=\infty)=1$.*

Example 8.3.6. Ergodic interpretation of Palm probability. *Let $\Phi = \sum_{k\in\mathbb{Z}}\delta_{X_k}$ be a stationary and ergodic point process on $\mathbb{R}^d$ with finite and non-null intensity* *and with Palm probability $\mathbf{P}^0$, and let $\{B_n\}_{n\in\mathbb{N}}$ be a convex averaging sequence in $\mathbb{R}^d$. Then, for all $A \in \mathcal{A}$,*

$$\begin{aligned}\mathbf{P}^0(A) &= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \int_{B_n} \mathbf{1}_A(\theta_x)\,\Phi(\mathrm{d}x)\\ &= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_k)\,\mathbf{1}_A(\theta_{X_k}), \quad \mathbf{P}\text{-}a.s.\end{aligned}$$

and for all measurable functions $f : \Omega \to \bar{\mathbb{R}}_+$,

$$\begin{aligned}\mathbf{E}^0[f] &= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \int_{B_n} f\circ\theta_x \Phi(\mathrm{d}x)\\ &= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_k)\, f\circ\theta_{X_k}, \quad \mathbf{P}\text{-}a.s. \end{aligned}\tag{8.3.3}$$

Example 8.3.7. Ergodic interpretation of the exchange formula. *Consider the context of Theorem 6.3.7 and assume that the framework $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ is ergodic. Neveu's exchange formula (6.3.4) reads*

$$\lambda'\mathbf{E}^{0\prime}[f] = \lambda\mathbf{E}^0\left[\int_V f\circ\theta_y \Phi'(\mathrm{d}y)\right].$$

Denote the points of the two considered point processes by $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ and $\Phi' = \sum_{k\in\mathbb{Z}} \delta_{X'_k}$, respectively. By the ergodic theorem 8.3.4, the left-hand side of the above relation is

$$\lambda'\mathbf{E}^{0\prime}[f] = \lim_{n\to\infty} \frac{1}{|B_n|} \sum_{k\in\mathbb{Z}} \mathbf{1}_{B_n}(X'_k)\, f\circ\theta_{X'_k},$$

whereas the right-hand side is

$$\begin{aligned}\lambda\mathbf{E}^0\left[\int_V f\circ\theta_y\Phi'(\mathrm{d}y)\right] &= \lim_{n\to\infty}\frac{1}{|B_n|}\sum_{l\in\mathbb{Z}}\mathbf{1}_{B_n}(X_l)\left(\int_V f\circ\theta_y\Phi'(\mathrm{d}y)\right)\circ\theta_{X_l}\\ &= \lim_{n\to\infty}\frac{1}{|B_n|}\sum_{l\in\mathbb{Z}}\mathbf{1}_{B_n}(X_l)\left(\int_{\tilde V(X_l)} f\circ\theta_y\Phi'(\mathrm{d}y)\right)\\ &= \lim_{n\to\infty}\frac{1}{|B_n|}\sum_{l\in\mathbb{Z}}\mathbf{1}_{B_n}(X_l)\sum_{k\in\mathbb{Z}}\mathbf{1}_{\tilde V(X_l)}(X'_k)\, f\circ\theta_{X'_k},\end{aligned}$$

where $\tilde V(X_l)$ is the Voronoi cell associated to X_l defined by (6.2.2) and the second equality may be justified as follows

$$\begin{aligned}\left(\int_V f\circ\theta_y\Phi'(\mathrm{d}y)\right)\circ\theta_{X_l} &= \int_{V\circ\theta_{X_l}} f\circ\theta_y\circ\theta_{X_l}\Phi'\circ\theta_{X_l}(\mathrm{d}y)\\ &= \int_{V(X_l)} f\circ\theta_{y+X_l}\Phi'(\mathrm{d}(y+X_l))\\ &= \int_{V(X_l)+X_l} f\circ\theta_z\Phi'(\mathrm{d}z) = \int_{\tilde V(X_l)} f\circ\theta_z\Phi'(\mathrm{d}z),\end{aligned}$$

where the last equality is due to (6.2.3). Consequently,

$$\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{k\in\mathbb{Z}}\mathbf{1}_{B_n}\left(X'_k\right)f\circ\theta_{X'_k}=\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{l\in\mathbb{Z}}\mathbf{1}_{B_n}\left(X_l\right)\sum_{k\in\mathbb{Z}}\mathbf{1}_{\tilde{V}(X_l)}\left(X'_k\right)f\circ\theta_{X'_k}.$$

In the left-hand side we consider points $X'_k\in B_n$ whereas in the right-hand side we consider points $X'_k\in\bigcup_{X_l\in B_n}\tilde{V}\left(X_l\right)$.

Corollary 8.3.8. *Let Φ be a simple stationary and ergodic point process on $\mathbb{R}^d$ with intensity $\lambda\in\mathbb{R}_+^*$. Then, $\mathbf{P}$-almost surely,*

$$\tilde{V}\left(X\right)\ \text{is bounded},\quad\forall X\in\Phi.$$

Proof. It follows from (6.2.7) that

$$\mathbf{E}^0[|\tilde{V}\left(0\right)|]=\frac{1}{\lambda}<\infty.$$

Then, by the ergodic theorem 8.3.4, taking $B_n=B\left(0,n\right)$, we get

$$\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{X\in B_n\cap\Phi}\left|\tilde{V}\left(X\right)\right|=\lambda\mathbf{E}^0\left[\tilde{V}\left(0\right)\right]=1,\quad\mathbf{P}\text{-a.s.}$$

Observe that, for all $\omega\in\Omega$, if $\left|\tilde{V}\left(X\left(\omega\right)\right)\right|=\infty$ for some $X\left(\omega\right)\in\Phi\left(\omega\right)$ then $\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{X\in B_n\cap\Phi(\omega)}\left|\tilde{V}\left(X\left(\omega\right)\right)\right|=\infty$. Thus $\mathbf{P}$-almost surely,

$$\left|\tilde{V}\left(X\right)\right|<\infty,\quad\forall X\in\Phi,$$

which together with the fact that $\tilde{V}\left(X\right)$ is a polygon implies that $\tilde{V}\left(X\right)$ is bounded. □

Example 8.3.9. Non-ergodic point process. *Let X be a $\{0,1\}$ random variable with $0<\mathbf{P}(X=0)=p<1$. Let Φ_1 and Φ_2 be two stationary and ergodic point processes with intensities $\lambda_1\neq\lambda_2$, respectively. Let $\Phi=X\Phi_1+(1-X)\Phi_2$. Then Φ is stationary but not ergodic. On $X=1$, $\Phi(B_n)/B_n$ tends to λ_1 whereas on the complementary event, it tends to λ_2.*

Example 8.3.10. *Let Φ be a stationary and ergodic point process on $\mathbb{R}^d$ with finite and non-null intensity. Let*

$$f\left(\omega\right)=\left|V\right|,$$

where V be the virtual cell given by (6.2.1). Then if follows from (6.2.3) that for all $X\in\Phi$,

$$f\circ\theta_X=\left|V\circ\theta_X\right|=\left|\tilde{V}(X)-X\right|=\left|\tilde{V}(X)\right|.$$

Thus

$$\lim_{n\to\infty}\frac{1}{\Phi\left(B_n\right)}\sum_{X\in\Phi\cap B_n}\left|\tilde{V}(X)\right|=\mathbf{E}^0\left[\left|V\right|\right]=\frac{1}{\lambda}.$$

Example 8.3.11. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary and ergodic framework, and let $\tilde{\Phi} = \sum_{k\in\mathbb{Z}} \delta_{(X_k, Z(X_k))}$ be a stationary marked point process on $\mathbb{R}^d$ with marks process $\{Z(t)\}_{t\in\mathbb{R}^d}$ taking values in some measurable space $(\mathbb{K}, \mathcal{K})$. Then for all measurable functions $g : \mathbb{K} \to \mathbb{R}$ such that $\mathbf{E}^0[|g(Z(0))|] < \infty$*

$$\frac{1}{|B_n|}\int_{B_n\times\mathbb{K}} g(z)\,\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}z) \underset{n\to\infty}{\longrightarrow} \lambda\mathbf{E}^0[g(Z(0))], \quad \mathbf{P}\text{-}a.s.$$

where λ and $\mathbf{E}^0$ respectively denote the intensity and the Palm expectation associated to the ground process $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$. Indeed,

$$\begin{aligned}\frac{1}{|B_n|}\int_{B_n\times\mathbb{K}} g(z)\,\tilde{\Phi}(\mathrm{d}x\times\mathrm{d}z) &= \frac{1}{|B_n|}\sum_{X\in\Phi\cap B_n} g(Z(X))\\ &= \frac{1}{|B_n|}\int_{B_n} g(Z(x))\,\Phi(\mathrm{d}x)\\ &= \frac{1}{|B_n|}\int_{B_n} g\circ Z(0)\circ\theta_x\Phi(\mathrm{d}x)\,.\end{aligned}$$

Example 8.3.12. *Let Φ be a stationary and ergodic point process on $\mathbb{R}^d$ with finite and non-null intensity and let $\{Z(t)\}_{t\in\mathbb{R}^d}$ be a stochastic process on some measurable space $(\mathbb{K}, \mathcal{K})$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$. Then for all measurable functions $g : \mathbb{K} \to \mathbb{R}$ such that $\mathbf{E}^0[|g(Z(0))|] < \infty$*

$$\lim_{n\to\infty}\frac{1}{\Phi(B_n)}\sum_{X\in\Phi\cap B_n} g(Z(X)) = \mathbf{E}^0[g(Z(0))], \quad \mathbf{P}\text{-}a.s. \tag{8.3.4}$$

Example 8.3.13. *Let $\Phi = \sum_{k\in\mathbb{Z}} \delta_{X_k}$ be a Poisson point process on $\mathbb{R}^2$ with intensity 1. For all $\lambda \in \mathbb{R}_+^*$, let $\tilde{\Phi}_\lambda := \sum_{k\in\mathbb{Z}} \delta_{X_k/\sqrt{\lambda}}$ which is clearly a Poisson point process on $\mathbb{R}^2$ with intensity λ. Moreover*

$$\tilde{\Phi}_\lambda(B) = \Phi\left(B\sqrt{\lambda}\right), \quad B \in \mathcal{B}_{\mathrm{c}}\left(\mathbb{R}^d\right).$$

Then for any fixed $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^d)$ convex with non empty interior,

$$\lim_{\lambda\to\infty}\frac{\tilde{\Phi}_\lambda(B)}{\lambda|B|} = \lim_{\lambda\to\infty}\frac{\Phi\left(B\sqrt{\lambda}\right)}{\left|B\sqrt{\lambda}\right|} = 1, \quad \mathbf{P}\text{-}a.s.$$

where the last equality is due to (8.3.2).

Example 8.3.14. Ripley's K-function. *Let Φ be a stationary point process on $\mathbb{R}^2$ with intensity $\lambda \in \mathbb{R}_+^*$. The Ripley's K-function is defined by*

$$K(r) = \frac{1}{\lambda}\mathbf{E}^0[\Phi(B(0,r)) - 1]\,.$$

It is related to the reduced second moment measure $\mathcal{K}$ defined by (6.3.18) through the relation $K(r) = \frac{1}{\lambda}\mathcal{K}(B(0,r))$. We deduce from Corollary 6.1.30 that $\forall A \in \mathcal{B}(\mathbb{R}^d), 0 < |A| < \infty$,

$$\begin{aligned}\mathbf{E}^0[\Phi(B(0,r))] &= \frac{1}{\lambda|A|}\mathbf{E}\left[\sum_{X\in\Phi\cap A}\Phi(\theta_X\omega)(B(0,r))\right]\\ &= \frac{1}{\lambda|A|}\mathbf{E}\left[\sum_{X\in\Phi\cap A}\Phi(B(X,r))\right].\end{aligned}$$

Hence

$$K(r) = \frac{1}{\lambda^2|A|}\mathbf{E}\left[\sum_{X\in\Phi\cap A}(\Phi(B(X,r))-1)\right].$$

If Φ is ergodic, $\{A_n\}_{n\in\mathbb{N}}$ a convex averaging sequence in $\mathbb{R}^d$ and $\mathbf{E}^0[\Phi(B(0,r))] < \infty$, then by Theorem 8.3.4

$$K(r) = \lim_{n\to\infty}\frac{1}{\lambda^2|A_n|}\sum_{X\in\Phi\cap A_n}(\Phi(B(X,r))-1), \quad \mathbf{P}\text{-}a.s.,$$

which gives a practical way to estimate $K(r)$ for a given configuration of points (sufficiently large).

In the particular case of Poisson, by Slivnyak's theorem, $\mathbf{E}^0[\Phi(B(0,r))-1] = \mathbf{E}[\Phi(B(0,r))] = \lambda\pi r^2$, then $K(r) = \pi r^2$.

Going back to the general case, we study

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{\pi r^2}K(r) &= \lim_{r\to\infty}\frac{1}{\lambda\pi r^2}\mathbf{E}^0[\Phi(B(0,r))]\\ &= \lim_{r\to\infty}\mathbf{E}^0\left[\frac{1}{\lambda\pi r^2}\Phi(B(0,r))\right].\end{aligned}$$

If Φ is ergodic, then, by Theorem 8.3.4,

$$\lim_{r\to\infty}\frac{1}{\lambda\pi r^2}\Phi(B(0,r)) = \lim_{r\to\infty}\frac{1}{|B(0,r)|}\int_{B(0,r)}\Phi(\mathrm{d}x) = 1,$$

$\mathbf{P}$*-a.s. If we can swap the expectation and the limit in the above equation, then we would get $\lim_{r\to\infty}\frac{1}{\pi r^2}K(r) = 1$.*

Example 8.3.15. Wireless networks. *Consider the context of Corollary 6.3.15 and assume that Φ is ergodic. Moreover let $\{B_n\}_{n\in\mathbb{N}}$ be a convex averaging sequence in $\mathbb{R}^d$.*

(i) Assume that

$$h(\omega) = \int_{V(\omega)} g(y,\Phi(\omega))\,\mathrm{d}y$$

is $\mathbf{P}^0$-integrable. Then, by Theorem 8.3.4

$$\begin{aligned}\lambda\mathbf{E}^0\left[\int_V g(y,\Phi)\,\mathrm{d}y\right] &= \lambda\mathbf{E}^0[h(\omega)]\\ &= \lim_{n\to\infty}\frac{1}{|B_n|}\int_{B_n} h\circ\theta_x\Phi(\mathrm{d}x)\\ &= \lim_{n\to\infty}\frac{1}{|B_n|}\sum_{X\in\Phi\cap B_n}\int_{\tilde V(X)} g(y,\Phi)\,\mathrm{d}y,\end{aligned}$$

where for the last equality we use (6.3.12).

(ii) Similarly, assuming that

$$l(\omega)=\int_{V(\omega)} g(y,\Phi(\omega))\,\Phi'(\mathrm{d}y)$$

is $\mathbf{P}^0$-integrable, then by Theorem 8.3.4

$$\begin{aligned}\frac{\lambda}{\lambda'}\mathbf{E}^0\left[\int_V g(y,\Phi)\,\Phi'(\mathrm{d}y)\right] &= \frac{\lambda}{\lambda'}\mathbf{E}^0[l(\omega)]\\ &= \frac{1}{\lambda'}\lim_{n\to\infty}\frac{1}{|B_n|}\int_{B_n} l\circ\theta_x\Phi(\mathrm{d}x)\\ &= \frac{1}{\lambda'}\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{X\in\Phi\cap B_n}\int_{\tilde V(X)} g(y,\Phi)\,\Phi'(\mathrm{d}y).\end{aligned}$$

(iii) On the other hand, assuming that $\mathbf{E}[|g(0,\Phi)|]<\infty$, then by Theorem 8.2.4

$$\mathbf{E}[g(0,\Phi)] = \lim_{n\to\infty}\frac{1}{|B_n|}\int_{B_n} g(0,\Phi\circ\theta_y)\,\mathrm{d}y = \lim_{n\to\infty}\frac{1}{|B_n|}\int_{B_n} g(y,\Phi)\,\mathrm{d}y.$$

Thus, under the above assumptions, we may complete the equalities in Corollary 6.3.15 by the following three new ones

$$\begin{aligned}\mathbf{E}[g(0,\Phi)] &= \frac{\lambda}{\lambda'}\mathbf{E}^0\left[\int_V g(y,\Phi)\,\Phi'(\mathrm{d}y)\right]\\ &= \lambda\mathbf{E}^0\left[\int_V g(y,\Phi)\,\mathrm{d}y\right]\\ &= \frac{1}{|B|}\mathbf{E}\left[\sum_{X\in\Phi\cap B}\int_{\tilde V(X)} g(y,\Phi)\,\mathrm{d}y\right]\\ &= \lim_{n\to\infty}\frac{1}{|B_n|}\sum_{X\in\Phi\cap B_n}\int_{\tilde V(X)} g(y,\Phi)\,\mathrm{d}y\\ &= \lim_{n\to\infty}\frac{1}{\lambda'|B_n|}\sum_{X\in\Phi\cap B_n}\int_{\tilde V(X)} g(y,\Phi)\,\Phi'(\mathrm{d}y)\\ &= \lim_{n\to\infty}\frac{1}{|B_n|}\int_{B_n} g(y,\Phi)\,\mathrm{d}y.\end{aligned}$$

Note that Φ' is only assumed to be stationary (not necessarily ergodic).

8.3.3 Cross-ergodicity

Lemma 8.3.16. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary framework, Φ a point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ with finite and non-null intensity, such that $\mathbf{P}\left(\Phi\left(\mathbb{R}^d\right)=0\right)=0$. Let $A \in \mathcal{A}$ be* strictly $\{\theta_t\}$-invariant*; i.e., for all $t \in \mathbb{R}^d$, $A = \theta_{-t}A$. Then $\mathbf{P}(A) = 1$ if and only if $\mathbf{P}^0(A) = 1$.*

Proof. Note first that

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{E}[1_A] \\ &= \lambda \mathbf{E}^0\left[\int_V 1_A \circ \theta_x \mathrm{d}x\right] \\ &= \lambda \mathbf{E}^0\left[\int_V 1_A \mathrm{d}x\right] \\ &= \lambda \mathbf{E}^0\left[1_A |V|\right] \\ &= 1 - \mathbf{E}^0\left[1_{A^c} |V|\right],\end{aligned}$$

where we use Theorem 6.2.8 for the second equality. The third equality is due to the $\{\theta_t\}$-invariance of A and the fifth one follows from (6.2.7). Assume that $\mathbf{P}(A) = 1$, then by the above equation $\mathbf{E}^0[1_{A^c}|V|] = 0$. Thus, $\mathbf{P}^0$-almost surely, $1_{A^c}|V| = 0$ which, together with the fact that $|V| \neq 0$, implies $1_{A^c} = 0$. Therefore $\mathbf{P}^0(A) = 1$ Conversely, if $\mathbf{P}^0(A) = 1$, then $\mathbf{E}^0[1_{A^c}|V|] = 0$ thus by the above equation $\mathbf{P}(A) = 1$. $\square$

Proposition 8.3.17. Cross-ergodicity. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary and ergodic framework, Φ a point process compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ with finite and non-null intensity, such that $\mathbf{P}\left(\Phi\left(\mathbb{R}^d\right)=0\right)=0$. Let $B_n = B(0,n)$ the open ball in $\mathbb{R}^d$ of center 0 and radius n. Then for all $f \in L^1_{\mathbb{R}}(\mathbf{P}, \Omega)$,*

$$\frac{1}{|B_n|}\int_{B_n} f \circ \theta_x \mathrm{d}x \underset{n\to\infty}{\longrightarrow} \mathbf{E}[f], \quad \mathbf{P}^0\text{-}a.s. \tag{8.3.5}$$

and for all $f \in L^1_{\mathbb{R}}\left(\mathbf{P}^0, \Omega\right)$,

$$\frac{1}{|B_n|}\int_{B_n} f \circ \theta_x \Phi(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \lambda \mathbf{E}^0[f], \quad \mathbf{P}^0\text{-}a.s.$$

Proof. Let $f \in L^1_{\mathbb{R}}(\mathbf{P}, \Omega)$ such that $f \geq 0$ and

$$A = \left\{\omega \in \Omega : \frac{1}{|B_n|}\int_{B_n} f \circ \theta_x(\omega)\,\mathrm{d}x \underset{n\to\infty}{\longrightarrow} \mathbf{E}[f]\right\}.$$

We will show that A is strictly $\{\theta_t\}$-invariant. Indeed, let $\omega \in A$ and $t \in \mathbb{R}^d$. We have

$$\begin{aligned}\int_{B_n} f \circ \theta_x(\theta_t\omega)\,\mathrm{d}x &= \int_{B_n} f \circ \theta_{x+t}(\omega)\,\mathrm{d}x \\ &= \int_{B_n+t} f \circ \theta_y(\omega)\,\mathrm{d}y.\end{aligned}$$

Note that $B_{n-\lceil t \rceil} \subset B_n + t \subset B_{n+\lceil t \rceil}$, then for all,

$$\int_{B_{n-\lceil |t| \rceil}} f \circ \theta_y(\omega)\,\mathrm{d}y \leq \int_{B_n+t} f \circ \theta_y(\omega)\,\mathrm{d}y \leq \int_{B_{n+\lceil |t| \rceil}} f \circ \theta_y(\omega)\,\mathrm{d}y,$$

thus

$$\begin{aligned}\frac{B_{n-\lceil |t| \rceil}}{|B_n|}\frac{1}{B_{n-\lceil |t| \rceil}}\int_{B_{n-\lceil |t| \rceil}} f \circ \theta_y(\omega)\,\mathrm{d}y &\leq \frac{1}{|B_n|}\int_{B_n+t} f \circ \theta_y(\omega)\,\mathrm{d}y \\ &\leq \frac{B_{n+\lceil |t| \rceil}}{|B_n|}\frac{1}{B_{n+\lceil |t| \rceil}}\int_{B_{n+\lceil |t| \rceil}} f \circ \theta_y(\omega)\,\mathrm{d}y.\end{aligned}$$

Letting $n \to \infty$, it follows that

$$\frac{1}{|B_n|}\int_{B_n+t} f \circ \theta_y(\omega)\,\mathrm{d}y \to \mathbf{E}[f].$$

Thus $\theta_t \omega \in A$. It follows that $A \subset \theta_{-t}A$ which implies that $\theta_t A \subset A$. This being true for all $t \in \mathbb{R}^d$, it follows that $A = \theta_{-t}A$. Then A is strictly $\{\theta_t\}$-invariant. On the other hand, by ergodicity and Theorem 8.2.4 $\mathbf{P}(A) = 1$. Thus by Lemma 8.3.16, $\mathbf{P}^0(A) = 1$, which proves (8.3.5) for $f \geq 0$. For general $f \in L^1_{\mathbb{R}}(\mathbf{P}, \Omega)$, its is enough to write $f = f^+ - f^-$ where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, apply (8.3.5) for f^+ and f^- separately and subtract the obtained limits. The proof of the second statement is similar; we omit the details for brevity. □

8.4 Ergodicity of marked random measures

Definition 8.4.1. *A stationary marked random measure $\tilde{\Phi}$ on $\mathbb{R}^d \times \mathbb{K}$ is said to be* ergodic *(respectively* mixing*) if and only if the stationary framework $\left(\tilde{\mathbb{M}}\left(\mathbb{R}^d \times \mathbb{K}\right), \tilde{\mathcal{M}}\left(\mathbb{R}^d \times \mathbb{K}\right), \{S_t\}_{t \in \mathbb{R}^d}, \mathbf{P}_{\tilde{\Phi}}\right)$ is ergodic (respectively mixing).*

Proposition 8.4.2. *Let $\tilde{\Phi}$ be a stationary marked random measure on $\mathbb{R}^d \times \mathbb{K}$ with Laplace transform $\mathcal{L}_{\tilde{\Phi}}$. Then*

(i) $\tilde{\Phi}$ is ergodic if and only if

$$\lim_{a \to \infty} \frac{1}{(2a)^d}\int_{[-a,a]^d} \mathcal{L}_{\tilde{\Phi}}(f_1 + S_x f_2)\mathrm{d}x = \mathcal{L}_{\tilde{\Phi}}(f_1)\mathcal{L}_{\tilde{\Phi}}(f_2),$$

for all measurable $f_1, f_2 : \mathbb{R}^d \times \mathbb{K} \to \mathbb{R}_+$ bounded with bounded support.

(ii) $\tilde{\Phi}$ is mixing if and only if

$$\lim_{|x| \to \infty} \mathcal{L}_{\tilde{\Phi}}(f_1 + S_x f_2) = \mathcal{L}_{\tilde{\Phi}}(f_1)\mathcal{L}_{\tilde{\Phi}}(f_2),$$

for all measurable $f_1, f_2 : \mathbb{R}^d \times \mathbb{K} \to \mathbb{R}_+$ bounded with bounded support.

Proof. Cf. [31, Proposition 12.3.VI p.210]. □

Corollary 8.4.3. *Let $\tilde{\Phi}$ be an i.i.d. marked point process on $\mathbb{R}^d \times \mathbb{K}$ associated to a homogeneous Poisson point process Φ on $\mathbb{R}^d$. Then $\tilde{\Phi}$ is mixing and ergodic.*

Proof. Let $f_1, f_2 : \mathbb{R}^d \times \mathbb{K} \to \mathbb{R}_+$ be measurable bounded with bounded support. For $|x|$ sufficiently large, the supports of f_1 and $S_x f_2$ are disjoint, and then the restrictions of $\tilde{\Phi}$ to these sets are independent. For such x,

$$\begin{aligned}\mathcal{L}_{\tilde{\Phi}}(f_1 + S_x f_2) &= \mathbf{E}\left[\exp\left(-\int_{\mathbb{R}^d} (f_1 + S_x f_2)\, \mathrm{d}\tilde{\Phi}\right)\right] \\ &= \mathbf{E}\left[\exp\left(-\int_{B_1} f_1 \mathrm{d}\tilde{\Phi}\right) \exp\left(-\int_{B_2} S_x f_2 \mathrm{d}\tilde{\Phi}\right)\right] \\ &= \mathcal{L}_{\tilde{\Phi}}(f_1)\mathcal{L}_{\tilde{\Phi}}(S_x f_2) = \mathcal{L}_{\tilde{\Phi}}(f_1)\mathcal{L}_{\tilde{\Phi}}(f_2),\end{aligned}$$

where the last equality follows from stationarity. Thus $\tilde{\Phi}$ is mixing by Proposition 8.3.2(ii). It follows from Lemma 8.2.8 that $\tilde{\Phi}$ is also ergodic. □

We now give an extension of Theorem 8.3.4 to the marked case.

Theorem 8.4.4. Ergodic theorem for marked random measures. *Let $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbf{P})$ be a stationary and ergodic framework, $\tilde{\Phi}$ be a marked random measure on $\mathbb{R}^d \times \mathbb{K}$ compatible with the flow $\{\theta_t\}_{t\in\mathbb{R}^d}$ with finite and non-null ground intensity, $\{B_n\}_{n\in\mathbb{N}}$ a convex averaging sequence in $\mathbb{R}^d$ and $f : \mathbb{K} \times \Omega \to \mathbb{R}_+$ be measurable. Then*

$$\frac{1}{|B_n|}\int_{B_n} f(z, \theta_x \omega)\, \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z) \underset{n\to\infty}{\longrightarrow} \lambda \int \mathbf{E}^{(0,z)}[f(z,\omega)]\, \pi(\mathrm{d}z), \quad \mathbf{P}\text{-}a.s.,$$

where $\mathbf{E}^{(0,z)}$ is the expectation with respect to $\mathbf{P}^{(0,z)}$ the Palm probability conditional on the mark, and π is the Palm distribution of the mark.

Proof. Cf. [31, Proposition 13.4.I]. □

Corollary 8.4.5. *In the conditions of Theorem 8.4.4, for all measurable $g : \mathbb{K} \to \mathbb{R}_+$,*

$$\frac{1}{|B_n|}\int_{B_n} g(z)\, \tilde{\Phi}(\mathrm{d}x \times \mathrm{d}z) \underset{n\to\infty}{\longrightarrow} \lambda \int g(z)\, \pi(\mathrm{d}z), \quad \mathbf{P}\text{-}a.s.$$

In the particular case when $\tilde{\Phi}$ is a stationary marked point process, then the above convergence may be written as follows

$$\frac{1}{|B_n|}\sum_{k\in\mathbb{Z}:X_k\in B_n} g(Z_k) \underset{n\to\infty}{\longrightarrow} \lambda \mathbf{E}^0[g(Z(0))], \quad \mathbf{P}\text{-}a.s.$$

(We retrieve (8.3.4).)

8.5 Exercises

Exercise 8.5.1. *Give an ergodic interpretation of the mass transport formula for graph (6.1.20) assuming that* $\mathbf{E}^0[\mathrm{card}\,(H^+(0))] < \infty$.

Solution 8.5.1. *Let* $\Phi = \sum_{m\in\mathbb{Z}} \delta_{X_m}$ *and* B_n *be a ball of center* 0 *and radius* n ($n \in \mathbb{N}$). *The ergodic interpretation (8.3.3) says that, for all measurable functions* $f : \Omega \to \bar{\mathbb{R}}_+$,

$$\mathbf{E}^0[f] = \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m)\, f \circ \theta_{X_m}.$$

Recall that $\mathrm{card}\,(H^+(0))$ *is the number of out-neighbors of the origin, then*

$$\mathrm{card}\,\big(H^+(0)\big) = \sum_{k\in\mathbb{Z}} \mathbf{1}_{(0\to X_k)},$$

where $x \to y$ *means there is a directed edge from* x *to* y. *Thus*

$$\mathrm{card}\,\big(H^+(0)\circ\theta_{X_m}\big) = \sum_{k\in\mathbb{Z}} \mathbf{1}_{(X_m\to X_k)}.$$

Therefore,

$$\begin{aligned}
\mathbf{E}^0[\mathrm{card}\,\big(H^+(0)\big)] &= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m) \sum_{k\in\mathbb{Z}} \mathbf{1}_{(X_m\to X_k)} \\
&= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m,k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m)\,\mathbf{1}_{B_n}(X_k)\,\mathbf{1}_{(X_m\to X_k)} \\
&+ \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m,k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m)\,\mathbf{1}_{B_n^c}(X_k)\,\mathbf{1}_{(X_m\to X_k)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{E}^0[\mathrm{card}\,\big(H^-(0)\big)] &= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m) \sum_{k\in\mathbb{Z}} \mathbf{1}_{(X_k\to X_m)} \\
&= \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m,k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m)\,\mathbf{1}_{B_n}(X_k)\,\mathbf{1}_{(X_k\to X_m)} \\
&+ \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m,k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m)\,\mathbf{1}_{B_n^c}(X_k)\,\mathbf{1}_{(X_k\to X_m)}.
\end{aligned}$$

Hence, under the assumption $\mathbf{E}^0[\mathrm{card}\,(H^+(0))] < \infty$, *it follows from (6.1.20) and the above two expressions that*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m,k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m)\,\mathbf{1}_{B_n^c}(X_k)\,\mathbf{1}_{(X_m\to X_k)} \\
&\quad = \lim_{n\to\infty} \frac{1}{\Phi(B_n)} \sum_{m,k\in\mathbb{Z}} \mathbf{1}_{B_n}(X_m)\,\mathbf{1}_{B_n^c}(X_k)\,\mathbf{1}_{(X_k\to X_m)}.
\end{aligned}$$

The left-hand side counts the number of out edges from B_n *whereas the right-hand side counts the number of in edges to* B_n*.*

Part III

Stochastic geometry

Chapter 9

Framework for stochastic geometry

We have already seen examples of random sets, mainly generated by point processes, such as the cells of a Voronoi tessellation in Section 6.2.1. There is however a need, motivated by many applied sciences (e.g. material sciences and biology) for a framework allowing one to consider even more general random sets and collections of such sets; in particular point processes whose atoms are closed sets.

The theory of *random closed sets* offers such possibility through a natural extension of the theory of point processes.

In this regard, we will induce the space $\mathcal{F}(\mathbb{G})$ of closed subsets of a l.c.s.h. space $\mathbb{G}$ with a particular topology called the *Fell topology*. Then we will show that $\mathcal{F}(\mathbb{G})$ is compact, second countable and Hausdorff for this topology. This will allow us, in particular, to define point processes whose atoms are closed subsets of $\mathbb{G}$. Our main references in this chapter are [67, 87, 72].

9.1 Space of closed sets

Let $\mathbb{G}$ be a l.c.s.h. space and let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of $\mathbb{G}$ which will be the space of realizations of our random closed sets, with some topology and σ-field considered on it. Note that the elements (points) of $\mathcal{F}(\mathbb{G})$ are closed subsets of $\mathbb{G}$, and subsets of $\mathcal{F}(\mathbb{G})$ are collections of subsets of $\mathbb{G}$.

Let $\mathcal{K}(\mathbb{G})$ (respectively $\mathcal{O}(\mathbb{G})$) be the space of compact (respectively open) subsets of $\mathbb{G}$. We need the following notation to define the topology on $\mathcal{F}(\mathbb{G})$. For any $A \subset \mathbb{G}$, we denote by $\mathcal{F}_A$ the class of closed sets which intersect (*hit*) A; that is

$$\mathcal{F}_A = \{F \in \mathcal{F}(\mathbb{G}) : F \cap A \neq \emptyset\}.$$

Similarly, we denote by

$$\mathcal{F}^A = \{F \in \mathcal{F}(\mathbb{G}) : F \cap A = \emptyset\}$$

the class of closed sets which do not intersect (*miss*) A. For any $n \in \mathbb{N}$, $A, A_1, \ldots, A_n \subset \mathbb{G}$, let

$$\mathcal{F}^{A}_{A_1,\ldots,A_n} = \mathcal{F}^A \cap \mathcal{F}_{A_1} \cap \ldots \cap \mathcal{F}_{A_n},$$

the class of closed sets which hit all sets $A_1, \ldots, A_n$ but miss A. For $n = 0$, the above set is by convention $\mathcal{F}^A$.

Remark 9.1.1. *Let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of a l.c.s.h. space $\mathbb{G}$. The next relations, which follow from first principles, will be used below:*

(i) $\mathcal{F}^A = \mathcal{F}^{\mathrm{c}}_A, \quad \forall A \subset \mathbb{G}$;

(ii) $\mathcal{F}^{\emptyset} = \mathcal{F}$;

(iii) $\mathcal{F}_{\emptyset} = \emptyset$; [1]

(iv) $\mathcal{F}^{\mathbb{G}} = \{\emptyset\}$; [2]

(v) $\mathcal{F}_{\mathbb{G}} = \mathcal{F}(\mathbb{G}) \setminus \{\emptyset\}$;

(vi) $\mathcal{F}^{\emptyset}_A = \mathcal{F}_A, \quad \forall A \subset \mathbb{G}$;

(vii) $\emptyset \in \mathcal{F}^A, \quad \forall A \subset \mathbb{G}$;

(viii) $\emptyset \notin \mathcal{F}^A_{A_1,\ldots,A_n}, \quad \forall n \in \mathbb{N}^*, A, A_1, \ldots, A_n \subset \mathbb{G}$;

(ix) $\mathcal{F}^A_{A_1,\ldots,A_n} = \emptyset, \quad$ *if* $A_i \subset A$ *for some* $i \in \{1, \ldots, n\}$;

(x) $\mathcal{F}^A \setminus \{\emptyset\} \subset \mathcal{F}_{A^{\mathrm{c}}}$.

Recall that defining a topology on $\mathcal{F}(\mathbb{G})$ consists in specifying a family $\mathcal{T}(\mathbb{G})$ of open subsets of $\mathcal{F}(\mathbb{G})$. Recall also that a topology may be defined as the family of all unions of elements of a given base $\mathcal{U}$, which needs to satisfy in this regard some conditions [57, Theorem 11 p.47].

Definition 9.1.2. *Let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of a l.c.s.h. space $\mathbb{G}$. The* Fell topology *is the topology $\mathcal{T}(\mathbb{G})$ on $\mathcal{F}(\mathbb{G})$ generated by the base*

$$\mathcal{U} = \left\{\mathcal{F}^K_{G_1,\ldots,G_n} : K \in \mathcal{K}(\mathbb{G}), n \in \mathbb{N}, G_1, \ldots, G_n \in \mathcal{O}(\mathbb{G})\right\}, \tag{9.1.1}$$

where $\mathcal{K}(\mathbb{G})$ (respectively $\mathcal{O}(\mathbb{G})$) is the space of compact (respectively open) subsets of $\mathbb{G}$.

We have to check that the collection $\mathcal{U}$ given by (9.1.1) is a base. This follows from the fact that $\mathcal{F}(\mathbb{G}) = \mathcal{F}^{\emptyset} \in \mathcal{U}$ and from the fact that $\mathcal{U}$ is closed under finite intersections, since

$$\mathcal{F}^A_{A_1,\ldots,A_n} \cap \mathcal{F}^B_{B_1,\ldots,B_m} = \mathcal{F}^{A\cup B}_{A_1,\ldots,A_n,B_1,\ldots,B_m}.$$

[1] Note $\emptyset$ on the left-hand-side is the empty subset of $\mathbb{G}$, while $\emptyset$ on the right-hand-side is the empty subset of $\mathcal{F}$.

[2] This is a singleton (one element subset) of $\mathcal{F}$, consisting of the empty subset of $\mathbb{G}$.

Then $\mathcal{U}$ is a base for a topology on $\mathcal{F}(\mathbb{G})$ by [57, Theorem 11 p.47].

The Fell topology is an example of hit-and-miss topology (*hit open* sets and *miss compact* sets).

Remark 9.1.3. *A* neighborhood *of $F \in \mathcal{F}(\mathbb{G})$ is a subset of $\mathcal{F}(\mathbb{G})$ that includes an open set (of the Fell topology) containing F. For example, if $F \cap K = \emptyset$ and $F \cap G_i \neq \emptyset$ for $i = 1, \ldots, n$ for some $K \in \mathcal{K}(\mathbb{G}), n \in \mathbb{N}, G_1, \ldots, G_n \in \mathcal{O}(\mathbb{G})$, then $\mathcal{F}^K_{G_1,\ldots,G_n}$ is a neighborhood of F.*

We will also consider the space of all non-empty closed sets $\mathcal{F}'(\mathbb{G}) = \mathcal{F}(\mathbb{G}) \setminus \{\emptyset\}$, with the corresponding subspace-topology induced by the Fell topology on $\mathcal{F}(\mathbb{G})$ (the open sets on $\mathcal{F}'(\mathbb{G})$ are open sets of $\mathcal{T}(\mathbb{G})$ intersected with $\mathcal{F}'(\mathbb{G})$). This restriction is important both from the theoretical and practical point of view, with the empty set not being observable.

Lemma 9.1.4. Base for the Fell subspace-topology on $\mathcal{F}'(\mathbb{G})$. *Let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of a l.c.s.h. space $\mathbb{G}$. The collection*

$$\mathcal{U}' = \left\{\mathcal{F}^K \setminus \{\emptyset\} : K \in \mathcal{K}(\mathbb{G})\right\} \cup \left\{\mathcal{F}^K_{G_1,\ldots,G_n} : K \in \mathcal{K}(\mathbb{G}), n \in \mathbb{N}^*, G_1, \ldots, G_n \in \mathcal{O}(\mathbb{G})\right\}$$

is a base of the Fell subspace-topology on $\mathcal{F}'(\mathbb{G}) = \mathcal{F}(\mathbb{G}) \setminus \{\emptyset\}$, where $\mathcal{K}(\mathbb{G})$ (respectively $\mathcal{O}(\mathbb{G})$) is the space of compact (respectively open) subsets of $\mathbb{G}$.

Proof. Let $\mathcal{U}$ be the base of the Fell topology given by (9.1.1). Then

$$\mathcal{U}' = \{C \setminus \{\emptyset\} : C \in \mathcal{U}\}$$

is a base of the subspace-topology on $\mathcal{F}'(\mathbb{G})$. The announced result follows from the observation that

$$\emptyset \in \mathcal{F}^A, \quad \forall A \subset \mathbb{G}$$

and

$$\emptyset \notin \mathcal{F}^A_{A_1,\ldots,A_n}, \quad \forall n \in \mathbb{N}^*, A, A_1, \ldots, A_n \subset \mathbb{G}.$$

□

We will show that $\mathcal{F}(\mathbb{G})$ and $\mathcal{F}'(\mathbb{G})$ with the Fell topology inherit the fundamental topological properties assumed for $\mathbb{G}$. This allows us to define and study point processes on these two spaces by a straightforward extension of the theory previously considered on $\mathbb{G}$.

Theorem 9.1.5. *The space $\mathcal{F}(\mathbb{G})$ of closed subsets of a l.c.s.h. space $\mathbb{G}$, equipped with the Fell topology $\mathcal{T}(\mathbb{G})$, is compact, second countable, and Hausdorff (c.s.h).*

Proof. For a complete proof, see [67, Theorem 1-2-1 and Proposition 1-2-1]. We give below a proof of compactness of $\mathcal{F}(\mathbb{G})$ based on Alexander's compactness theorem. The family of open sets

$$\mathcal{S} = \{\mathcal{F}^K : K \in \mathcal{K}(\mathbb{G})\} \cup \{\mathcal{F}_G : G \in \mathcal{O}(\mathbb{G})\}$$

is a sub-base of the Fell topology (since the family of all finite intersections $\cap_{i=1}^{k} U_i$, where $U_i \in \mathcal{S}$ for $i = 1, 2, \ldots, k$ is a base for $\mathcal{T}(\mathbb{G})$). Therefore, from Alexander's theorem, in order to show compactness, it is enough to show that from all covers of $\mathcal{F}(\mathbb{G})$ by open sets of the sub-base, one can extract a finite subcover. Such a cover features a collection K_i, $i \in I$ (resp. G_j, $j \in J$) of compact (resp. open) sets such that

$$\mathcal{F}(\mathbb{G}) = \left(\cup_{i \in I} \mathcal{F}^{K_i}\right) \cup \left(\cup_{j \in J} \mathcal{F}_{G_j}\right).$$

For all such covers,

$$\emptyset = \left(\cap_{i \in I} \mathcal{F}_{K_i}\right) \cap \left(\cap_{j \in J} \mathcal{F}^{G_j}\right) = \left(\cap_{i \in I} \mathcal{F}_{K_i}\right) \cap \mathcal{F}^{G},$$

with $G = \cup_{j \in J} G_j$. Let us show that this implies that there exists an i^* such that $K_{i^*} \subset G$. The proof is by contradiction. If for all i, $K_i \cap G^{\mathrm{c}} \neq \emptyset$, then

$$G^{\mathrm{c}} \in \left(\cap_{i \in I} \mathcal{F}_{K_i}\right) \cap \mathcal{F}^{G},$$

which is not possible in view of the last relation. Hence, K_{i^*} is a compact of $\mathbb{G}$ covered by the open sets $G_j, j \in J$, which in turn implies that K_{i^*} has a finite subcover $G_j, j \in J^*$. This implies that

$$\emptyset = \mathcal{F}_{K_{i^*}} \cap \left(\cap_{j \in J^*} \mathcal{F}^{G_j}\right)$$

or, equivalently, that

$$\mathcal{F}(\mathbb{G}) = \mathcal{F}^{K_{i^*}} \cup \left(\cup_{j \in J^*} \mathcal{F}_{G_j}\right),$$

which is the announced finite subcover of $\mathcal{F}(\mathbb{G})$. □

Proposition 9.1.6. *Consider the space $\mathcal{F}(\mathbb{G})$ of closed subsets of a l.c.s.h. space $\mathbb{G}$ with the Fell topology.*

(i) $\{\emptyset\}$ *is closed and compact. If $\mathbb{G}$ is compact, then $\{\emptyset\}$ is open.*

(ii) $\left\{\mathcal{F}^{K} : K \in \mathcal{K}(\mathbb{G})\right\}$ *is a base of neighborhoods of $\emptyset$.*

(iii) Each closed subset of $\mathcal{F}(\mathbb{G})$ is compact. In particular, $\mathcal{F}_K$, where $K \in \mathcal{K}(\mathbb{G})$, or $\mathcal{F}^G$, where $G \in \mathcal{O}(\mathbb{G})$, are compact.

(iv) For any closed $\mathcal{H} \subset \mathcal{F}(\mathbb{G}) \setminus \{\emptyset\}$, there is some $K \in \mathcal{K}(\mathbb{G})$ such that $\mathcal{H} \subset \mathcal{F}_K$.

Proof.

(i) The fact that $\{\emptyset\}$ is closed follows from item (v) of Remark 9.1.1. The fact that it is compact follows since $\mathcal{F}(\mathbb{G})$ is compact. If $\mathbb{G}$ is compact, then $\{\emptyset\}$ is open since $\{\emptyset\} = \mathcal{F}^{\mathbb{G}}$.

(ii) Since $\mathcal{U}$ is a base, for all $F \in \mathcal{F}(\mathbb{G})$, for all open sets U containing F, there exists a compact K, an integer n, and a collection of open sets $G_1, \ldots, G_n$ such that $F \in \mathcal{F}^K_{G_1,\ldots,G_n} \subset U$. In the particular case when $F = \emptyset$, $n = 0$ necessarily (item (viii) of Remark 9.1.1), which shows that there exists a $K \in \mathcal{K}(\mathbb{G})$ such that $\emptyset \in \mathcal{F}^K \subset U$.

(iii) Since $\mathcal{F}(\mathbb{G})$ is compact, each closed subset of $\mathcal{F}(\mathbb{G})$ is compact.

(iv) The complement $\mathcal{H}^c$ of $\mathcal{H}$ is open. Since $\mathcal{H}^c$ contains $\emptyset$, it then follows from (ii) that there is some $K \in \mathcal{K}(\mathbb{G})$ such that $\mathcal{F}^K \subset \mathcal{H}^c$ and therefore $\mathcal{H} \subset \mathcal{F}_K$.

□

Proposition 9.1.7. *Consider the space $\mathcal{F}'(\mathbb{G})$ (of all nonempty closed subsets of a l.c.s.h. space $\mathbb{G}$) with the Fell subspace-topology.*

(i) For $K \in \mathcal{K}(\mathbb{G})$, $\mathcal{F}_K$ is closed and compact.

(ii) For all compact sets $\mathcal{H}$, there exists $K \in \mathcal{K}(\mathbb{G})$ such that $\mathcal{H} \subset \mathcal{F}_K$.

(iii) For all $G \in \mathcal{O}(\mathbb{G})$, $\mathcal{F}^G \setminus \{\emptyset\}$ is closed.

Proof.

(i) The complement of $\mathcal{F}_K$ in $\mathcal{F}'(\mathbb{G})$ is $\mathcal{F}^K \setminus \{\emptyset\}$, which is open in $\mathcal{F}'(\mathbb{G})$. Hence $\mathcal{F}_K$ is closed in $\mathcal{F}'(\mathbb{G})$. It remains to prove that it is compact in $\mathcal{F}'(\mathbb{G})$. Let $G_i,\ i \in I$ be any collection of open sets of $\mathcal{F}'(\mathbb{G})$ covering $\mathcal{F}_K$. Every G_i admits the representation $G_i = \tilde{G}_i \cap \mathcal{F}'(\mathbb{G})$, with $\tilde{G}_i$ an open set of $\mathcal{F}$. Since $\mathcal{F}'(\mathbb{G}) = \mathcal{F}_{\mathbb{G}}$ is an open set of $\mathcal{F}(\mathbb{G})$, each G_i is an open set of $\mathcal{F}(\mathbb{G})$. On the other hand, the complement of $\mathcal{F}_K$ in $\mathcal{F}(\mathbb{G})$ is $\mathcal{F}^K$ which is open in $\mathcal{F}(\mathbb{G})$. Then $\mathcal{F}_K$ is closed in $\mathcal{F}(\mathbb{G})$ and hence compact in $\mathcal{F}(\mathbb{G})$. Since $\mathcal{F}_K = \cup_{i\in I} G_i$, there exists a finite subcover $\mathcal{F}_K = \cup_{i\in I_0} G_i$, where I_0 is finite. This proves the compactness of $\mathcal{F}_K$ in $\mathcal{F}'(\mathbb{G})$.

(ii) We first show that $\mathcal{H}$ is compact for the $\mathcal{T}(\mathbb{G})$ topology as well. Let $U_i, i \in I$ be a cover of $\mathcal{H}$ by open sets of $\mathcal{T}(\mathbb{G})$. Let $U'_i = U_i \setminus \{\emptyset\}$. Then $U'_i, i \in I$ is a cover of $\mathcal{H}$ by open sets of $\mathcal{T}'(\mathbb{G})$ (the topology induced by $\mathcal{T}(\mathbb{G})$ on $\mathcal{F}'(\mathbb{G})$). Since $\mathcal{H}$ is assumed compact in $\mathcal{T}'(\mathbb{G})$, there exists a finite subcover $U'_i, i \in I_0$. This implies that $U_i, i \in I_0$ is a finite subcover of $\mathcal{H}$. Hence $\mathcal{H}$ is compact for the $\mathcal{T}(\mathbb{G})$ topology. Thus $\mathcal{H}$ is closed ($\mathcal{T}(\mathbb{G})$ is Hausdorff) and since $\mathcal{H}$ is included in $\mathcal{F}(\mathbb{G}) \setminus \{\emptyset\}$, it then follows from (iv) of Proposition 9.1.6 that $H \subset \mathcal{F}^K$ for some $K \in \mathcal{K}(\mathbb{G})$.

(iii) The complement of $\mathcal{F}^G \setminus \{\emptyset\}$ in $\mathcal{F}'(\mathbb{G})$ is $\mathcal{F}_G$, which is open in $\mathcal{F}'(\mathbb{G})$ by Lemma 9.1.4.

□

Theorem 9.1.8. *Let $\mathcal{F}'(\mathbb{G})$ be the space of all nonempty closed subsets of a l.c.s.h. space $\mathbb{G}$. The space $\mathcal{F}'(\mathbb{G})$ with the Fell subspace-topology is l.c.s.h. Moreover, $\mathcal{F}'(\mathbb{G})$ is compact iff $\mathbb{G}$ itself is compact.*

Proof. For a complete proof, see [67, Theorem 1-2-1 and Proposition 1-2-1]. We give a proof of local compactness. Let $F \in \mathcal{F}'(\mathbb{G})$. There exists an open and relatively compact set $G \subset \mathbb{G}$ such that $F \cap G \neq \emptyset$. Hence $F \in \mathcal{F}_G \subset \mathcal{F}_K$ where $K \in \mathcal{K}(\mathbb{G})$ denotes the closure of G. Hence, it follows from Proposition 9.1.7 that F has a relatively compact neighborhood. We now prove that $\mathcal{F}'(\mathbb{G})$ is compact if and only if $\mathbb{G}$ is compact. Assume that $\mathcal{F}'(\mathbb{G})$ is compact. Consider a cover of $\mathbb{G}$ with open sets; that is $\mathbb{G} = \cup_{i\in I} G_i$; where G_i is open for all $i \in I$. Then

$$\cup_{i\in I}\mathcal{F}_{G_i} = \mathcal{F}_{\cup_{i\in I} G_i} = \mathcal{F}_{\mathbb{G}} = \mathcal{F}'(\mathbb{G}).$$

By compactness of $\mathcal{F}'(\mathbb{G})$, there exists a finite $I_0 \subset I$ such that

$$\mathcal{F}'(\mathbb{G}) = \cup_{i\in I_0}\mathcal{F}_{G_i}.$$

For all $x \in \mathbb{G}$, $\{x\} \in \mathcal{F}'(\mathbb{G})$. Thus there exists some $i \in I_0$ such that $\{x\} \in \mathcal{F}_{G_i}$ which implies that $x \in G_i$. Then $\mathbb{G} = \cup_{i\in I_0} G_i$. Thus every open cover of $\mathbb{G}$ has a finite subcover. Therefore, $\mathbb{G}$ is compact. Conversely, if $\mathbb{G}$ is compact, then the relation $\mathcal{F}'(\mathbb{G}) = \mathcal{F}_{\mathbb{G}}$ with (iii) in Proposition 9.1.6 show that $\mathcal{F}'(\mathbb{G})$ is compact. □

Let $\mathcal{B}(\mathcal{F}(\mathbb{G}))$ be the Borel σ-algebra generated by the Fell topology $\mathcal{T}(\mathbb{G})$; i.e., the σ-algebra generated by all open sets in $\mathcal{T}(\mathbb{G})$. Note that $\mathcal{F}'(\mathbb{G}) \in \mathcal{B}(\mathcal{F}(\mathbb{G}))$ since $\mathcal{F}'(\mathbb{G}) = \mathcal{F}_{\mathbb{G}}$ is an open set of $\mathcal{F}$. The Borel σ-algebra $\mathcal{B}(\mathcal{F}'(\mathbb{G}))$ generated by the Fell subspace-topology is equal to

$$\mathcal{B}(\mathcal{F}'(\mathbb{G})) = \{A \cap \mathcal{F}'(\mathbb{G}) : A \in \mathcal{B}(\mathcal{F}(\mathbb{G}))\}.$$

Proposition 9.1.9. *Let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of a l.c.s.h. space $\mathbb{G}$. The Borel σ-algebra $\mathcal{B}(\mathcal{F}(\mathbb{G}))$ is generated by either of the following families*

$$\left\{\mathcal{F}^K : K \in \mathcal{K}(\mathbb{G})\right\},\ \{\mathcal{F}_K : K \in \mathcal{K}(\mathbb{G})\},\{\mathcal{F}_G : G \in \mathcal{O}(\mathbb{G})\},\ \text{or}\ \left\{\mathcal{F}^G : G \in \mathcal{O}(\mathbb{G})\right\},$$

where $\mathcal{K}(\mathbb{G})$ (respectively $\mathcal{O}(\mathbb{G})$) is the space of compact (respectively open) subsets of $\mathbb{G}$.

Proof. For a complete proof, cf. [67, p.27]. We establish the first statement, which will be used below. Because the space $\mathbb{G}$ is locally compact and has a countable basis, for all open sets G, there exists a collection of compact sets $\{K_n\}_{n\in\mathbb{N}}$ such that $G = \cup_{n\in\mathbb{N}} K_n$. Hence

$$\mathcal{F}_G = \cup_{n\in\mathbb{N}}\mathcal{F}_{K_n} = \cup_{n\in\mathbb{N}}\left(\mathcal{F}^{K_n}\right)^{\mathrm{c}}.$$

This shows that every $\mathcal{F}_G$ can be expressed as a countable union of sets which are the complements of sets in the family $\left\{\mathcal{F}^K : K \in \mathcal{K}(\mathbb{G})\right\}$, which concludes the proof. □

Lemma 9.1.10. *Let $\mathcal{F}'(\mathbb{G})$ be the space of all nonempty closed subsets of a l.c.s.h. space $\mathbb{G}$. The system $\mathcal{B}_0 = \{\mathcal{F}_K : K \in \mathcal{K}(\mathbb{G})\}$ satisfies the properties in Lemma 1.3.1 for the l.c.s.h. space $\mathcal{F}'(\mathbb{G})$ (in the role of $\mathbb{G}$ there).*

Proof. Cf. [87, Proof of Lemma 2.3.1 and below]. $\square$

9.2 Random closed sets

Recall that the *basic probability space* is denoted by $(\Omega, \mathcal{A}, \mathbf{P})$.

Definition 9.2.1. *Let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of a l.c.s.h. space $\mathbb{G}$. A* random closed set *(RCS) on $\mathbb{G}$ is a random variable Z with values in $\mathcal{F}(\mathbb{G})$; i.e., a measurable mapping $Z : \Omega \to \mathcal{F}(\mathbb{G})$ (Ω and $\mathcal{F}(\mathbb{G})$ being equipped with the σ-algebras $\mathcal{A}$ and $\mathcal{B}(\mathcal{F}(\mathbb{G}))$ respectively). The probability distribution of Z is as usual $\mathbf{P}_Z = \mathbf{P} \circ Z^{-1}$; i.e.*

$$\mathbf{P}_Z(B) = \mathbf{P}(Z \in B), \quad B \in \mathcal{B}(\mathcal{F}(\mathbb{G})).$$

In the similar way, we define a random non-empty closed set *as a random variable with values in $\mathcal{F}'(\mathbb{G})$.*

We give first the effect of elementary set operations on random closed sets.

Proposition 9.2.2. *Let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of a l.c.s.h. space $\mathbb{G}$.*

(i) Let Z be a random closed set on $\mathbb{G}$. Then the closure of its complement $\overline{Z^{\mathrm{c}}}$ is a random closed sets.

(ii) Let $\{Z_n\}_{n\in\mathbb{N}}$ be random closed sets on $\mathbb{G}$. Then their intersection $\bigcap_{n\in\mathbb{N}} Z_n$ and the closure of their union $\overline{\bigcup_{n\in\mathbb{N}} Z_n}$ are random closed sets.

(iii) Assume that $\mathbb{G}$ is a Banach space. If Z_1 and Z_2 are random closed sets on $\mathbb{G}$, then $\overline{Z_1 \oplus Z_2}$ is a random closed set.

Proof. (i) Let $G \in \mathcal{O}(\mathbb{G})$ and let $\{x_i\}_{i\in\mathbb{Z}}$ be a countable dense subset of G. Observe that

$$\left\{\overline{Z^{\mathrm{c}}} \cap G = \emptyset\right\} = \{G \subset Z\} = \bigcap_{x_i \in G} \{x_i \in Z\}$$

which is measurable. (For the last equality, it is obvious that $\{G \subset Z\} \subset \bigcap_{i\in\mathbb{Z}} \{x_i \in Z\}$. Inversely, assume that $x_i \in Z$ for all $i \in \mathbb{Z}$. Then for any $x \in G$, there exist a subsequence $\left\{x_{\sigma(i)}\right\}_{i\in\mathbb{Z}}$ of $\{x_i\}_{i\in\mathbb{Z}}$ which converges to x, which is necessarily in Z since the latter is closed.) (ii) For the closure of their union, observe that for any $G \in \mathcal{O}(\mathbb{G})$,

$$\left\{\overline{\bigcup_{n\in\mathbb{N}} Z_n} \cap G = \emptyset\right\} = \bigcap_{n\in\mathbb{N}} \{Z_n \cap G = \emptyset\}$$

which is a measurable event. For the intersection, cf. [72, Theorem 2.25(vii) p.37]. (iii) Cf. [72, Theorem 2.25(v) p.37]. $\square$

The following result is useful to show the measurability of certain random closed sets.

Lemma 9.2.3. *Let $\mathcal{F}(\mathbb{G})$ be the space of closed subsets of a l.c.s.h. space $\mathbb{G}$, $Z : \Omega \to \mathcal{F}(\mathbb{G})$, and let ρ be a distance on $\mathbb{G}$. Assume that for any $x \in \mathbb{G}$, the distance between Z and x, denoted by $\rho(Z, x)$, is a measurable function $\Omega \to \mathbb{R}$ (i.e., a random variable). Then Z is a random closed set.*

Proof. Let K be a compact in $\mathbb{G}$. Observe that

$$\{Z \in \mathcal{F}_K\} = \{Z \cap K \neq \emptyset\} = \{\rho(Z, K) = 0\},$$

where

$$\rho(Z, K) = \inf_{x \in K} \rho(Z, x)$$

is the distance between Z and K. It is enough to shows that $\rho(Z, K)$ is a measurable function $\Omega \to \mathbb{R}$. Let $\{x_i\}_{i \in \mathbb{Z}}$ be a countable dense subset of $\mathbb{G}$. Given $n \in \mathbb{N}^*$, since K is compact, it may be covered by a finite number of open balls $B(x_i, 1/n)$, let I_n be the set of such x_i. Obviously,

$$\inf_{x_i \in I_n} \rho(Z, x_i) - \frac{1}{n} \leq \inf_{x \in K} \rho(Z, x) \leq \inf_{x_i \in I_n} \rho(Z, x_i),$$

where the left hand side inequality is due to the triangle inequality. Taking the limit as $n \to \infty$, we get

$$\rho(Z, K) = \lim_{n \to \infty} \inf_{x_i \in I_n} \rho(Z, x_i).$$

Since each $\rho(Z, x_i)$ is measurable, then so is the above limit by [11, Theorem 13.4]. □

Natural examples of random closed sets such as random balls, triangles, orthants, level-sets are generated by random variables, vectors or stochastic processes in the space $\mathbb{G}$, see Exercises 9.3.1, 9.3.2 and 9.3.3. We have already also seen more complicated random sets in previous chapters, which can be now formally recognized as random closed sets.

Example 9.2.4. Support of a point process. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$. The support of Φ, $\mathrm{supp}(\Phi) = \{x \in \mathbb{G} : \Phi(\{x\}) > 0\}$ is a random closed set. Indeed, $\mathrm{supp}(\Phi)$ is a closed set (for all $\omega \in \Omega$, since $\Phi(\omega)$ is locally finite, the set of atoms of $\Phi(\omega)$ do not have accumulation points and hence their union, albeit countable is a closed set). Moreover, for $K \in \mathcal{K}(\mathbb{G})$,*

$$\left\{\omega \in \Omega : Z(\omega) \in \mathcal{F}^K\right\} = \{\omega \in \Omega : \Phi(\omega)(K) = 0\}$$

is a measurable event since Φ is a point process. Since $\left\{\mathcal{F}^K : K \in \mathcal{K}(\mathbb{G})\right\}$ generates $\mathcal{B}(\mathcal{F}(\mathbb{G}))$ by Proposition 9.1.9, then Z is measurable by [11, Theorem 13.1 p.182].

Note that $\mathrm{supp}(\Phi)$ caries the information about the locations of atoms of Φ, but not their multiplicities.

In the last example, if $\mathbf{P}(\Phi(\mathbb{G}) = 0) = 0$, then Z has its support on $\mathcal{F}'(\mathbb{G})$. Although random closed sets are initially defined as $\mathcal{F}(\mathbb{G})$-valued random variables, many of the examples studied below will have the property that their support is actually in $\mathcal{F}'(\mathbb{G})$.

Example 9.2.5. Voronoi cells. *Let $\Phi = \sum_{i\in\mathbb{N}} \delta_{X_i}$ be a simple, stationary point process on $\mathbb{R}^d$. The following random cells considered in Section 6.2.1 are random closed sets.*

- *Centered cell of a given location $x \in \mathbb{R}^d$, $V(x) := \{y \in \mathbb{R}^d : |y - x| \le \inf_{X\in\Phi} |y - X|\}$, in particular, $V(0) = V$ which is the virtual cell of Φ.*
- *Voronoi cells $\tilde{V}_i := \{y \in \mathbb{R}^d : |y - X_i| \le \inf_{X_j\in\Phi} |y - X_j|\}$ and their centered variants $V_i = \tilde{V}_i - X_i$ of all points (this requires some measurable numbering of points).*
- *Zero cell $V^* := V(X^*)$, where X^* is the closest point of Φ to the origin (Definition 6.2.14). It is well defined only on a subset of the probability space where X^* is unique, recall this event has stationary probability equal to 1. Also, the non-centered zero-cell $V^* = V^* + X^*$.*

It is clear that they are closed sets. It remains to show their measurability. In this regard, it is enough to prove that the virtual cell V is measurable. Observe that

$$V = \bigcap_{i\in\mathbb{N}} Z_i,$$

where $Z_i = \{y \in \mathbb{R}^d : |y| \le |y - X_i|\}$. By Lemma 9.2.3, Z_i is a random closed set. Then so is V by Proposition 9.2.2.

Lemma 9.2.6. *Let $\mathcal{F}'(\mathbb{G})$ be the space of all nonempty closed subsets of a l.c.s.h. space $\mathbb{G}$. A measure μ on $\mathcal{F}'(\mathbb{G})$ is locally finite iff*

$$\mu(\mathcal{F}_K) < \infty, \quad \textit{for all } K \in \mathcal{K}(\mathbb{G}).$$

Moreover, a locally finite measure μ on $\mathcal{F}'(\mathbb{G})$ is characterized by $\{\mu(\mathcal{F}_K) : K \in \mathcal{K}(\mathbb{G})\}$.

Proof. The proof of the first property is an immediate consequence of (i)-(ii) in Proposition 9.1.7. For a proof of the second statement, see [87, Lemma 2.3.1]. □

9.2.1 The capacity functional

We introduce now a fundamental characteristic of the distribution of a random closed set. As we shall see, it has properties analogous to these of the probability distribution function of a random vector, in particular, it uniquely characterizes the distribution of the random closed set.

Definition 9.2.7. *Given a random closed set Z on a l.c.s.h. space $\mathbb{G}$, its* capacity functional *is the mapping $T_Z : \mathcal{K}(\mathbb{G}) \to \mathbb{R}$ defined by*

$$T_Z(K) = \mathbf{P}(Z \cap K \neq \emptyset) = \mathbf{P}(Z \in \mathcal{F}_K), \quad K \in \mathcal{K}(\mathbb{G}).$$

It follows from Proposition 9.1.9 that:

Corollary 9.2.8. *The capacity functional of a random closed set on a l.c.s.h. space $\mathbb{G}$ characterizes its probability distribution.*

Proof. Let Z and Z' be two random closed sets such that $T_Z = T_{Z'}$. We have to prove that $\mathbf{P}_Z = \mathbf{P}_{Z'}$. Note that for all $K \in \mathcal{K}(\mathbb{G})$,

$$\mathbf{P}_Z\left(\mathcal{F}^K\right) = 1 - \mathbf{P}_Z(\mathcal{F}_K) = 1 - T_Z(K) = \mathbf{P}_{Z'}\left(\mathcal{F}^K\right).$$

Let $\mathcal{C} = \left\{\mathcal{F}^K : K \in \mathcal{K}(\mathbb{G})\right\}$. If follows from Proposition 9.1.9 that $\sigma(\mathcal{C}) = \mathcal{B}(\mathcal{F}(\mathbb{G}))$. Since, $\mathcal{C}$ is non-empty and stable by finite intersections ($\mathcal{F}^K \cap \mathcal{F}^L = \mathcal{F}^{K\cup L}$ for all $K, L \in \mathcal{K}(\mathbb{G})$), then the two measures $\mathbf{P}_Z$ and $\mathbf{P}_{Z'}$ which agree on $\mathcal{C}$ agree on $\sigma(\mathcal{C}) = \mathcal{B}(\mathcal{F}(\mathbb{G}))$; cf. [11, Theorem 10.3 p.163]. □

Remark 9.2.9. *The capacity functional $T_Z(B)$ can extended to sets B for which $\mathcal{F}_B \in \mathcal{B}(\mathcal{F}(\mathbb{G}))$, in particular open and closed sets $B \subset \mathbb{G}$.*

Example 9.2.10. *The support of a point process Φ on $\mathbb{R}^d$ is a random closed set (cf. Example 9.2.4) with capacity functional*

$$T_{\mathrm{supp}(\Phi)}(K) = \mathbf{P}(\mathrm{supp}(\Phi) \cap K \neq \emptyset) = \mathbf{P}(\Phi(K) \neq 0) = 1 - \nu_\Phi(K),$$

where $\nu_\Phi(K) = \mathbf{P}(\Phi(K) = 0)$ is the void probability of Φ. Recall Rényi's theorem 2.1.10, which says that the void probabilities characterize the distribution of a simple point process.

The following result states some important properties of the capacity functional of a random closed set, analogous to the classical properties of the cumulative distribution function of a real random variable, concerning its bounds, right-continuity and monotonicity.

Proposition 9.2.11. *The capacity functional T of a random closed set Z on a l.c.s.h. space $\mathbb{G}$ has the following properties:*

(i) Bounds. $0 \leq T(K) \leq 1$, $T(\emptyset) = 0$.

(ii) Upper semi-continuity. *Let $K, K_1, K_2, \ldots \in \mathcal{K}(\mathbb{G})$. If $K_n \downarrow K$ then $T(K_n) \to T(K)$.*

(iii) Complete alternation. *For $K, K_1, K_2, \ldots \in \mathcal{K}(\mathbb{G})$, let $\{S_n\}_{n\in\mathbb{N}}$ be defined recursively by $S_0(K) = 1 - T(K)$ and*

$$\begin{aligned} S_{n+1}(K; K_1, \ldots, K_n, K_{n+1}) = S_n(K; K_1, \ldots, K_n) \\ - S_n(K \cup K_{n+1}; K_1, \ldots, K_n), \end{aligned} \tag{9.2.1}$$

for all $n \geq 0$. Then, for all $n \in \mathbb{N}$, $S_n(K; K_1, \ldots, K_n) \geq 0$.

Proof. (i) The bounds are obvious. (ii) We will show that $\mathcal{F}_{K_n} \downarrow \mathcal{F}_K$ which would imply that $T(K_n) \to T(K)$ by the continuity from above property of the probability measure. Observe first that $\mathcal{F}_{K_n}$ is decreasing and that $\mathcal{F}_K \subset \bigcap_{n\in\mathbb{N}^*} \mathcal{F}_{K_n}$. On the other hand, let $F \in \bigcap_{n\in\mathbb{N}^*} \mathcal{F}_{K_n}$, then $F \cap K_n \neq \emptyset$ for all $n \in \mathbb{N}^*$. The sequence of non-empty compact sets $\{F \cap K_n\}_{n\in\mathbb{N}^*}$ is decreasing, then $\bigcap_{n\in\mathbb{N}^*} F \cap K_n \neq \emptyset$. Moreover, since $\{K_n\}_{n\in\mathbb{N}^*}$ is decreasing, then $\bigcap_{n\in\mathbb{N}^*} F \cap K_n = F \cap K$ and therefore $F \in \mathcal{F}_K$. (iii) We show by induction that

$$\begin{aligned} S_n(K; K_1, \ldots, K_n) &= \mathbf{P}(Z \cap K = \emptyset, Z \cap K_1 \neq \emptyset, \ldots, Z \cap K_n \neq \emptyset) \\ &= \mathbf{P}\left(Z \in \mathcal{F}^K_{K_1,\ldots;K_n}\right), \end{aligned}$$

which is obviously nonnegative. For $n = 0$ the identity is obvious. Assume that the above identity holds for some $n \in \mathbb{N}$,

$$\begin{aligned} &S_{n+1}(K; K_1, \ldots, K_{n+1}) \\ &\quad = S_n(K; K_1, \ldots, K_n) - S_n(K \cup K_{n+1}; K_1, \ldots, K_n) \\ &\quad = \mathbf{P}(Z \cap K = \emptyset, Z \cap K_1 \neq \emptyset, \ldots, Z \cap K_n \neq \emptyset) \\ &\quad - \mathbf{P}(Z \cap (K \cup K_{n+1}) = \emptyset, Z \cap K_1 \neq \emptyset, \ldots, Z \cap K_n \neq \emptyset) \\ &\quad = \mathbf{P}(Z \cap K = \emptyset, Z \cap K_1 \neq \emptyset, \ldots, Z \cap K_n \neq \emptyset, Z \cap (K \cup K_{n+1}) \neq \emptyset) \\ &\quad = \mathbf{P}(Z \cap K = \emptyset, Z \cap K_1 \neq \emptyset, \ldots, Z \cap K_n \neq \emptyset, Z \cap K_{n+1} \neq \emptyset). \end{aligned}$$

□

Remark 9.2.12. *The property (ii) above is in fact equivalent to the upper semi-continuity of T as a functional on $\mathcal{K}(\mathbb{G})$ with the sub-space topology induced by the Fell topology on $\mathcal{F}(\mathbb{G})$; cf. [67, Proposition 1.4.2] or [72, Proposition D.7].*

Remark 9.2.13. *Obviously, the capacity functional is monotone:*

$$T_Z(K_1) \leq T_Z(K_2), \quad \textit{for } K_1 \subset K_2 \in \mathcal{K}(\mathbb{G}).$$

The complete alternation property (9.2.1) is a stronger monotonicity property; indeed taking $n = 0$ allows one to retrieve the usual monotonicity.

In full analogy to the classical existence result regarding the distribution function of a real random variable, we have the following characterization of the probability distribution on the space $(\mathcal{F}(\mathbb{G}), \mathcal{B}(\mathcal{F}(\mathbb{G})))$.

Theorem 9.2.14. Choquet's theorem. *Let $\mathbb{G}$ ba a l.c.s.h. space and $T : \mathcal{K}(\mathbb{G}) \to \mathbb{R}$ be a mapping satisfying properties (i) to (iii) of Proposition 9.2.11. Then there exists a random closed set Z such that $T = T_Z$.*

Proof. Cf. [67, Theorem 2-2-1] or [72, §1.1.3]. The proof given by Matheron in [67, Theorem 2-2-1] is based on the routine application of the measure-theoretic arguments related to extension of measures from algebras to σ-algebras. Molchanov gives in [72, §1.1.3] a different proof, based on some arguments from harmonic analysis. □

Corollary 9.2.15. *Let $\mathbb{G}$ ba a l.c.s.h. space and T be real valued function defined on $\mathcal{K}(\mathbb{G})$. Then there exists a (necessarily unique) probability distribution μ on $(\mathcal{F}(\mathbb{G}), \mathcal{B}(\mathcal{F}(\mathbb{G})))$ such $\mu(F_K) = T$ if and only if T satisfies properties (i) to (iii) of Proposition 9.2.11.*

Proof. The direct part follows from Proposition 9.2.11. The uniqueness from Corollary 9.2.8. The converse part follows from Choquet's theorem 9.2.14. $\square$

9.2.2 Set processes

The fact that the space of closed subsets inherits crucial topological properties of $\mathbb{G}$ (cf. Theorems 9.1.5 and 9.1.8) allows us to model random collections of (closest) subsets of $\mathbb{G}$ as point processes on $\mathcal{F}(\mathbb{G})$. We call them *set processes*. However, since the whole space $\mathcal{F}(\mathbb{G})$ is a compact set, these set processes would be configurations of only finite total number of sets. In order to allow for infinite set processes, we consider them on the space of non-empty closed sets $\mathcal{F}'(\mathbb{G})$, which is a l.c.s.h. space.

Definition 9.2.16. *Let $\mathbb{G}$ be a l.c.s.h. space and let $\mathcal{F}'(\mathbb{G})$ be the the space of non-empty closed subsets of $\mathbb{G}$ equipped with the Fell subspace-topology. A point process on $\mathcal{F}'(\mathbb{G})$ is called a* set process.

Denote by $\mathbb{M}(\mathcal{F}'(\mathbb{G}))$ and $\mathcal{M}(\mathcal{F}'(\mathbb{G}))$ the space of counting measures on $\mathcal{F}'(\mathbb{G})$ and its σ-algebra, respectively, defined exactly is the same way as $\mathbb{M}(\mathbb{G})$ and $\mathcal{M}(\mathbb{G})$ were defined in §1.6 for counting measures on $\mathbb{G}$.

From the general theory of point processes, a set process is a measurable mapping from some probability space to $\mathbb{M}(\mathcal{F}'(\mathbb{G}))$ (equipped with the σ-algebra $\mathcal{M}(\mathcal{F}'(\mathbb{G}))$). Such a process admits a representation

$$\Phi_{\mathrm{f}} = \sum_{n\in\mathbb{Z}} \delta_{F_n},$$

where the sets $F_n \in \mathcal{F}'(\mathbb{G})$ are the set-atoms of Φ_{f}.

In what follows we shall focus on Poisson processes in this setting. As usual, they are characterized by intensity measures Λ_{f}, which need to be locally finite measures or, equivalently, satisfy $\Lambda_f(\mathcal{F}_K) < \infty$ for all $K \in \mathcal{K}(\mathbb{G})$; cf. Lemma 9.2.6.

Definition 9.2.17. Poisson set process. *Let $\mathcal{F}'(\mathbb{G})$ be the space of all nonempty closed subsets of a l.c.s.h. space $\mathbb{G}$ and Λ_{f} be a locally finite measure on $\mathcal{F}'(\mathbb{G})$. A* Poisson set process *of intensity measure Λ_{f} is a Poisson point process on $\mathcal{F}'(\mathbb{G})$ with intensity measure Λ_{f}.*

9.2.3 Stationarity

In this section $\mathbb{G} = \mathbb{R}^d$. For all $t \in \mathbb{R}^d$ and all $A \subset \mathbb{R}^d$, let $S_t A = A - t = \{x - t : x \in A\}$.

Definition 9.2.18. *A random closed set Z on $\mathbb{R}^d$ is said to be* stationary *if, for all $t \in \mathbb{R}^d$, $S_t Z = Z - t$ has the same probability distribution as Z.*

It follows from Corollary 9.2.8 that this holds iff

$$T_Z(K) = T_Z(S_t K), \quad \forall K \in \mathcal{K}(\mathbb{G}), t \in \mathbb{R}^d.$$

Indeed, $T_{S_t Z}(K) = \mathbf{P}((S_t Z) \cap K \neq \emptyset) = \mathbf{P}(Z \cap (S_{-t} K) \neq \emptyset) = T_Z(S_{-t} K)$.

Definition 9.2.19. *A set process $\Phi_{\mathrm{f}} = \sum_n \delta_{F_n}$ on $\mathcal{F}'(\mathbb{R}^d)$ is said to be* stationary *if $S_t \Phi_{\mathrm{f}} = \sum_n \delta_{F_n - t}$ has the same probability distribution as Φ_{f} for all $t \in \mathbb{R}^d$.*

We will often use a stationary framework $(\Omega, \mathcal{A}, \{\theta_t\}, \mathbf{P})$ and say that a random closed set Z (resp. a set process Φ_{f}) is flow-compatible if $Z \circ \theta_t = S_t Z$ (resp. $\Phi_{\mathrm{f}} \circ \theta_t = S_t \Phi_{\mathrm{f}}$). This implies that Z (resp. Φ_{f}) is stationary. The interest of this setting is exemplified by the following immediate result:

Lemma 9.2.20. *If Φ_{f} is a flow-compatible set process on $\mathcal{F}'(\mathbb{R}^d)$, then the union of its atoms Z is a flow-compatible random closed set.*

9.2.4 Characteristics of random closed set

The following characteristics of a random closed set are of particular interest. Some of them can be explicitly expressed using the capacity functional.

Definition 9.2.21. *Let Z be a random closed set on a l.c.s.h. space $\mathbb{G}$.*

(i) Its inclusion functional *is defined by*

$$\mathbf{P}(K \subset Z), \quad K \in \mathcal{K}(\mathbb{G}).$$

(ii) The n-th coverage function *is defined by*

$$p_n(x_1, \ldots, x_n) = \mathbf{P}(\{x_1, \ldots, x_n\} \subset Z), \quad n \in \mathbb{N}^*, x_1, \ldots, x_n \in \mathbb{G}.$$

(iii) For $n = 2$, the second coverage function $p_2(x_1, x_2)$ is called the covariance function *of Z.*

(iv) Let μ be a given locally finite measure on $\mathbb{G}$. The mean measure *of Z with respect to μ is defined by*

$$M_Z^{\mu}(B) = \mathbf{E}[\mu(Z \cap B)], \quad B \in \mathcal{B}(\mathbb{G}).$$

The measurability of the mapping $\omega \mapsto \mu(Z(\omega) \cap B)$ follows from Robbins' theorem *[72, Theorem 4.21 p.59].*

Remark 9.2.22. *Let Z be a random closed set on a l.c.s.h. space $\mathbb{G}$.*

(i) Obviously, $\mathbf{P}(K \subset Z) \leq T_Z(K)$ but in general the inclusion functional does not admit any explicit expression in terms of the capacity functional.

(ii) *The n-th coverage function can be seen as a discrete approximation of the inclusion functional taking $K = \{x_1, \ldots, x_n\}, n \geq 1$. It admits an explicit expression in terms of the capacity functional $T_Z(K)$; in particular,*

$$p_1(x) = T_Z(\{x\}), \quad x \in \mathbb{G},$$

$$p_2(x_1, x_2) = T_Z(\{x_1\}) + T_Z(\{x_2\}) - T_X(\{x_1, x_2\}), \quad x_1, x_2 \in \mathbb{G}, \quad (9.2.2)$$

cf. Exercise 9.3.4.

(iii) *The covariance function of Z is indeed a non-centered covariance function of the stochastic process $\{I(x) = \mathbf{1}\{x \in Z\} : x \in \mathbb{G}\}$ meaning that $\mathbf{E}[I(x_1)I(x_2)] = p_2(x_1, x_2)$.*

(iv) *Let μ be a given locally finite measure on $\mathbb{G}$. The first coverage function $p_1(x)$ is the density of the mean measure M_Z^μ with respect to μ. Indeed, for any $B \in \mathcal{B}(\mathbb{G})$,*

$$\begin{aligned} M_Z^\mu(B) &= \mathbf{E}[\mu(Z \cap B)] \\ &= \mathbf{E}\left[\int_B \mathbf{1}\{x \in Z\}\, \mu(\mathrm{d}x)\right] \\ &= \int_B \mathbf{P}\{x \in Z\}\mu(\mathrm{d}x) = \int_B p_1(x)\mu(\mathrm{d}x). \end{aligned} \quad (9.2.3)$$

The mean measure of Z with respect to the Lebesgue measure $\mu(\mathrm{d}x) = \mathrm{d}x$ is called the mean volume measure *of Z and denoted by M_Z.*

9.2.5 Characteristics of stationary random closed set

Let Z be a stationary random closed set in $\mathbb{R}^d$. Recall that its capacity functional is invariant with respect to any translation $T_Z(K) = T_Z(K + t)$ for all $K \in \mathcal{K}(\mathbb{R}^d), t \in \mathbb{R}^d$. The n-th coverage functions satisfy

$$p_n(x_1, x_2, \ldots, x_n) = p_n(0, x_2 - x_1, \ldots, x_n - x_1), \quad n \in \mathbb{N}^*, x_1, \ldots, x_n \in \mathbb{R}^d.$$

In particular, we define the *reduced covariance function*

$$C(x) := p_2(0, x), \quad x \in \mathbb{R}^d.$$

Moreover, the coverage function is a constant $p_1(x) = p$ called the *volume fraction* of Z.

These are some probability distribution functions meant to provide some information on the local geometry of the coverage process Z around a point non-covered by Z. They are usually defined for a stationary Z with respect to the origin. The generalization to a general non-stationary coverage processes on $\mathbb{R}^d$ with respect to an arbitrary point is straightforward.

Let $K \in \mathcal{K}\left(\mathbb{R}^d\right)$ be a given compact subset of $\mathbb{R}^d$ containing the origin $0 \in K$. The *contact distribution function* $H_K(r)$ of Z with respect to the test set K is defined by

$$H_K(r) = \mathbf{P}\left(rK \cap Z \neq \emptyset | 0 \in Z^c\right). \quad r \in \mathbb{R}_+,$$

that is the conditional probability that Z does not hit the dilation of the set K by the factor r, $rK := \{ry : y \in K\}$, given it does not hit 0.

We summarize the above characteristics of stationary random closed sets in the following definition.

Definition 9.2.23. *Let Z be a stationary random closed set on $\mathbb{R}^d$.*

(i) Its volume fraction *is*

$$p = \mathbf{P}\left(0 \in Z\right).$$

(ii) Its reduced covariance function *is*

$$C(x) = \mathbf{P}\left(0 \in Z, x \in Z\right), \quad x \in \mathbb{R}^d.$$

(iii) The contact distribution function *of Z with respect to the compact set K containing the origin is*

$$H_K(r) = \mathbf{P}\left(rK \cap Z \neq \emptyset | 0 \in Z^c\right). \quad 0 \in K \in \mathcal{K}\left(\mathbb{R}^d\right), r \in \mathbb{R}_+.$$

The following proposition shows that the above characteristics admit explicit expressions in terms of the capacity functional.

Proposition 9.2.24. *Let Z be a stationary random closed set on $\mathbb{R}^d$.*

(i) Its volume fraction equals

$$p = T_Z(\{0\}) = \frac{\mathbf{E}\left[|Z \cap B|\right]}{|B|}, \quad B \in \mathcal{B}\left(\mathbb{R}^d\right) \text{ such that } 0 < |B| < \infty. \tag{9.2.4}$$

(ii) Its reduced covariance function equals

$$C(x) = 2p - T_Z(\{0, x\}), \quad x \in \mathbb{R}^d. \tag{9.2.5}$$

(iii) Its contact distribution function equals

$$H_K(r) = 1 - \frac{1 - T_Z(rK)}{1 - p}, \quad 0 \in K \in \mathcal{K}\left(\mathbb{R}^d\right), r \in \mathbb{R}_+.$$

Proof. (i) For any $B \in \mathcal{B}\left(\mathbb{R}^d\right)$ such that $0 < |B| < \infty$, the expected volume of B covered by Z is equal by the definition to mean volume measure of Z on B, $M_Z(B)$, and by (9.2.3) it admits $p_1(x) = p$ as its density. Consequently

$$\frac{\mathbf{E}\left[|Z \cap B|\right]}{|B|} = \frac{M_Z(B)}{|B|} = \frac{p|B|}{|B|} = p.$$

(ii) This follows from (9.2.2). (iii)

$$\begin{aligned}H_K(r) &= 1 - \mathbf{P}\left(rK \subset Z^{\mathrm{c}} | 0 \in Z^{\mathrm{c}}\right) \\ &= 1 - \frac{\mathbf{P}\left(rK \subset Z^{\mathrm{c}}, 0 \in Z^{\mathrm{c}}\right)}{\mathbf{P}\left(0 \in Z^{\mathrm{c}}\right)} \\ &= 1 - \frac{\mathbf{P}\left(rK \subset Z^{\mathrm{c}}\right)}{1-p} \\ &= 1 - \frac{\mathbf{P}\left(Z \cap rK = \emptyset\right)}{1-p} = 1 - \frac{1 - T_Z(rK)}{1-p}.\end{aligned}$$

□

Remark 9.2.25. Isotropy *of a random closed set Z is defined as the invariance of its distribution with respect to all rotations around some fixed point, that can be considered as the origin. It is usually considered together with the stationarity; the joint property is called* motion invariance. *The distribution of a motion invariant coverage set Z is invariant with respect to all rotations around any center point. This implies the invariance of $T_Z(K)$ with respect to all rotations. In particular, the covariance function $p_2(x_1, x_2)$ depends only on the distance $|x_1 - x_2|$ and one defines the* distance covariance function $\bar{C}(r) := C(|x|) = p_2(0, x)$.

Different instances of contact distribution functions can be considered, depending on the choice of the test set K. The most popular cases are as follows:

Example 9.2.26. Spherical contact distribution function. *This is the case with the spherical test set $K = \bar{B}(0,1)$ being the closed ball centered at the origin with unit radius. In this case, the contact distribution function*

$$H_{\bar{B}(0,1)}(r) = \mathbf{P}\left(d(0, Z) \leq r | 0 \in Z^{\mathrm{c}}\right) \tag{9.2.6}$$

is the conditional cumulative distribution function of the distance from 0 to Z given that $0 \in Z^{\mathrm{c}}$.

The linear contact distribution function.

Example 9.2.27. Linear contact distribution function. *This case arises when the test set $K = [0, x]$, a unit length segment from the origin in the direction of some $x \in \mathbb{R}^d$, $|x| = 1$. In this case, the contact distribution function $H_{[0,x]}(r)$ is the conditional distribution function of the distance from 0 to Z in the direction of the vector x, given $0 \notin Z^{\mathrm{c}}$.*

If Z is isotropic (rotation invariant distribution), then the linear contact distribution function does not depend on the direction x and $H_{[0,x]}(r) = H(r)$ can be seen as the conditional distribution function of the distance from the origin to Z in a randomly chosen direction.

9.3 Exercises

9.3.1 For Section 9.2

Exercise 9.3.1. Orthant, triangle. *Prove that the following sets are random closed sets:*

(i) Random singleton $Z := \{\xi\}$ where ξ is a random variable in a l.c.s.h. space $\mathbb{G}$.

(ii) Random orthant $Z = (-\infty, X_1] \times \ldots \times (-\infty, X_d]$ where $(X_1, \ldots, X_d)$ is a random vector in $\mathbb{R}^d$.

(iii) Random triangle Z in $\mathbb{R}^d$ generated by a random vector (X_1, X_2, X_3) of its vertexes.

Solution 9.3.1. *(i) For all $K \in \mathcal{K}\left(\mathbb{R}^d\right)$*

$$\begin{aligned}\{Z \in \mathcal{F}_K\} &= \{Z \cap K \neq \emptyset\} \\ &= \{\xi \in K\}\end{aligned}$$

is a measurable event. The capacity functional is $T_Z(K) = \mathbf{P}(\xi \in K) = \mathbf{P}_\xi(K)$, with $\mathbf{P}_\xi$ the probability distribution of ξ.

(ii) Consider first the case $d = 1$. Let K be a compact in $\mathbb{R}$,

$$\{Z \in \mathcal{F}_K\} = \{(-\infty, X_1] \cap K \neq \emptyset\} = \{X_1 \geq \min K\}$$

is a measurable event.

Consider now the case $d \geq 2$. Let ρ be the euclidean distance in $\mathbb{R}^d$. Observe that for any fixed $x \in \mathbb{R}^d$, the distance between Z and x, denoted by $\rho(Z, x)$, is a measurable function $\Omega \to \mathbb{R}$ (i.e., a random variable). Then Lemma 9.2.3 allows one to conclude.

(iii) The argument is the same as for Point (ii).

Exercise 9.3.2. Random closed ball. *Assume $\mathbb{G}$ is a metric space with distance d. Let R be a nonnegative random variable and ξ be a random variable in $\mathbb{G}$. Prove that the closed ball $Z = \bar{B}(\xi, R)$ centered at ξ with radius R is a random closed set and show that its capacity functional is equal to*

$$T_Z(K) = \mathbf{P}(R \geq d(\xi, K)),$$

where $d(x, K)$ is the distance between x and the set K.

Solution 9.3.2. *Let $(\Omega, \mathbf{P}, \mathcal{A})$ be the probability space. For all $K \in \mathcal{K}(\mathbb{G})$,*

$$\left\{\omega \in \Omega : \bar{B}\left(\xi\left(\omega\right), R\left(\omega\right)\right) \cap K \neq \emptyset\right\} = \left\{\omega \in \Omega : R\left(\omega\right) \geq d\left(\xi\left(\omega\right), K\right)\right\} \in \mathcal{A},$$

where $d(x, K)$ denotes the distance between a point $x \in \mathbb{G}$ and the set $K \in \mathcal{K}$. It follows that Z is a random closed set and that its capacity functional is $T_Z(K) = \mathbf{P}(R \geq d(\xi, K))$.

Exercise 9.3.3. Level-sets. *Let $\mathbb{G}$ be a l.c.s.h. space and let $\{\xi(x)\}_{x\in\mathbb{G}}$ be a stochastic process with values in $\mathbb{R}$ and continuous sample paths. Prove that the level set $Z := \{x \in \mathbb{G} : \xi(x) \geq u\}$, with a given $u \in \mathbb{R}$, is a random closed set, with capacity functional*

$$T_Z(K) = \mathbf{P}\left(\sup_{x\in K} \xi(x) \geq u\right).$$

Solution 9.3.3. *For all $K \in \mathcal{K}(\mathbb{G})$,*

$$\{\omega \in \Omega : Z(\omega) \cap K \neq \emptyset\} = \left\{\omega \in \Omega : \sup_{x\in K} \xi(\omega)(x) \geq u\right\}$$

is a measurable event. It follows that Z is a random closed set and that its capacity functional is $T_Z(K) = \mathbf{P}(\sup_{x\in K} \xi(x) \geq u)$.

Exercise 9.3.4. *Let Z be a ranom closed set on a l.c.s.h. space $\mathbb{G}$ and consider its n-th coverage functions*

$$p_n(x_1, \ldots, x_n) = \mathbf{P}(\{x_1, \ldots, x_n\} \subset Z), \quad n \in \mathbb{N}^*, x_1, \ldots, x_n \in \mathbb{G}.$$

1. *Show that*
$$p_1(x) = T_Z(\{x\}), \quad x \in \mathbb{G}.$$

2. *Show that the covariance function may be expressed using the capacity functional as*
$$p_2(x_1, x_2) = T_Z(\{x_1\}) + T_Z(\{x_2\}) - T_X(\{x_1, x_2\}), \quad x_1, x_2 \in \mathbb{G}.$$

3. *More generally, show that the n-th coverage function may be expressed in terms of the capacity functional.*

Solution 9.3.4. *1. For $n = 1$, the coverage function $p_1(x) := \mathbf{P}(x \in Z) = T_Z(\{x\})$ coincides with the capacity and inclusion functionals with $K = \{x\}$, $x \in \mathbb{G}$.*

2. For $n = 2$,

$$\begin{aligned}
p_2(x_1, x_2) &= \mathbf{P}(\{x_1, x_2\} \subset Z) \\
&= \mathbf{P}(\{x_1 \in Z\} \cap \{x_2 \in Z\}) \\
&= \mathbf{P}(\{x_1 \in Z\}) + \mathbf{P}(\{x_2 \in Z\}) - \mathbf{P}(\{x_1 \in Z\} \cup \{x_2 \in Z\}) \\
&= p_1(x_1) + p_1(x_2) - T_X(\{x_1, x_2\}).
\end{aligned}$$

3. Recall the inclusion-exclusion formula giving the probability of the unions of events $A_1, \ldots, A_n$:

$$\mathbf{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I\subset\{1,\ldots,n\}:\mathrm{card}(I)=k} \mathbf{P}\left(\bigcap_{i\in I} A_i\right).$$

Applying the above formula with $A_i = \{x_i \in Z\}$, we get

$$T_X\left(\{x_1, \ldots, x_n\}\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \subset \{1,\ldots,n\}:\mathrm{card}(I)=k} p_k\left((x_i)_{i \in I}\right).$$

The annouced result then follows by induction on n.

Chapter 10

Coverage and germ-grain models

Coverage models serve as very general mathematical models for irregular geometrical patterns. They have many applications, traditionally in material and biological sciences but also, more recently, in communication networks, in particular wireless communications. In principle, a general random closed set considered in the previous chapter can be considered as a coverage model. However, really interesting coverage models, penetrating the whole space, are constructed via point processes of closed sets. Particularly popular coverage models on $\mathbb{R}^d$ arise via a so-called germ-grain construction. A very prominent example of coverage model, considered in this chapter, is a Boolean model. It also admits a germ-grain construction on $\mathbb{R}^d$.

For more reading on coverage models see [43] and also [24, Chapter 6].

10.1 Coverage model

A coverage model in a general l.c.s.h. space $\mathbb{G}$ is defined as the union of the set-atoms of a point process on the space of non-empty closed subsets $\mathcal{F}'(\mathbb{G})$ of $\mathbb{G}$. We have called such point processes set processes. Sometimes one assumes these set processes to have compact set-atoms.

Definition 10.1.1. *Let $\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{F_j}$ be a set process on a l.c.s.h. space $\mathbb{G}$; i.e., a point process on the space $\mathcal{F}'(\mathbb{G})$. The union of its set-atoms*

$$Z = \bigcup_{j\in\mathbb{Z}} F_j \tag{10.1.1}$$

is called a coverage model.

Proposition 10.1.2. *Let $\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{F_j}$ be a set process on a l.c.s.h. space $\mathbb{G}$. The associated coverage model Z defined by (10.1.1) is a random closed set on*

$\mathbb{G}$ with capacity functional

$$T_Z(K) = 1 - \nu_{\Phi_{\mathrm{f}}}(\mathcal{F}_K), \quad K \in \mathcal{K}(\mathbb{G}).$$

where $\nu_{\Phi_{\mathrm{f}}}(\mathcal{F}_K) = \mathbf{P}(\Phi_{\mathrm{f}}(\mathcal{F}_K) = 0)$ is the void probability of Φ_{f}. Moreover, if Φ_{f} is a Poisson set process with intensity measure Λ_{f}, then

$$T_Z(K) = 1 - e^{-\Lambda_{\mathrm{f}}(\mathcal{F}_K)}, \quad K \in \mathcal{K}(\mathbb{G}). \tag{10.1.2}$$

Proof. **Closedness**. We first prove that Z is closed. Note that for all $K \in \mathcal{K}(\mathbb{G})$,

$$\mathrm{card}\{F \in \Phi_{\mathrm{f}} : F \cap K \neq \emptyset\} = \mathrm{card}\{F \in \Phi_{\mathrm{f}} \cap \mathcal{F}_K\} = \Phi_{\mathrm{f}}(\mathcal{F}_K) < \infty, \tag{10.1.3}$$

which is finite since $\mathcal{F}_K$ is compact and Φ_{f} is a point process on $\mathcal{F}'(\mathbb{G})$. Any convergent sequence of points in Z lies eventually in a compact K of $\mathbb{G}$. By (10.1.3), they belong hence to some union of finitely many set-atoms F of Φ_{f}. As a finite union of closed sets, it is a closed set, thus containing the sequence limit.
Measurability. We now prove that Z is measurable with respect to $\mathcal{B}(\mathcal{F}'(\mathbb{G}))$. For all $K \in \mathcal{K}(\mathbb{G})$,

$$\left\{Z \in \mathcal{F}^K\right\} = \{Z \cap K = \emptyset\} = \{\Phi_{\mathrm{f}}(\mathcal{F}_K) = 0\},$$

which is measurable since Φ_{f} is a point process. Then Z is measurable.
Capacity functional. The capacity functional of Z equals

$$T_Z(K) = \mathbf{P}(Z \in \mathcal{F}_K) = 1 - \mathbf{P}\left(Z \in \mathcal{F}^K\right) = 1 - \mathbf{P}(\Phi_{\mathrm{f}}(\mathcal{F}_K) = 0).$$

Poisson property. If Φ_{f} is a Poisson set process with intensity measure Λ_{f}, then for all $K \in \mathcal{K}(\mathbb{G})$, $T_Z(K) = 1 - \mathbf{P}(\Phi_{\mathrm{f}}(\mathcal{F}_K) = 0) = 1 - e^{-\Lambda_{\mathrm{f}}(\mathcal{F}_K)}$. □

Remark 10.1.3. *The difference between the set process Φ_{f} and the associated coverage model Z lies in the fact that in Z we do not observe the shapes of individual set-atoms F_j of Φ_{f}. In particular, we do not recognize their boundaries when they overlap. This makes Z a more appropriate model for some statistical analysis.*

Every non-empty random closed set Z can be trivially represented as in (10.1.1) with the set process Φ_{f} having just one atom equal to Z. The problem becomes only slightly more complicated if we want Φ_{f} to have compact set-atoms F_j. In this case, it is enough to "decompose" Z using a countable, compact cover of $\mathbb{G}$ (it exists since $\mathbb{G}$ is a l.c.s.h. space).

The problem is non-trivial if Z is stationary and/or isotropic coverage model on $\mathbb{R}^d$ (its distribution is invariant with respect to all translations and/or rotations) and we want Φ_{f} to have the same properties; cf. [72, Theorem 8.13 p.112].

The most important example of coverage model is the Boolean coverage model, which is generated by a Poisson set process.

Definition 10.1.4. Boolean coverage model. *Let $\mathcal{F}'(\mathbb{G})$ be the space of all nonempty closed subsets of a l.c.s.h space $\mathbb{G}$ and Φ_{f} be a Poisson point process on $\mathcal{F}'(\mathbb{G})$ with intensity measure Λ_{f}. The corresponding coverage model Z is called a* Boolean coverage model.

The capacity functional of the Boolean coverage model is given by (10.1.2).

10.2 Germ-grain model

Recall that a coverage process is constructed as a union of set-atoms of a set process. A *germ-grain model* arises when the set process is first constructed from a usual point process on $\mathbb{R}^d$ marked by some random non-empty closed sets. Points of this process are called *germs*, and the marks are called *grains*.

10.2.1 Germ-grain construction

For all $A, B \subset \mathbb{R}^d$, let $\check{A} = \{-x : x \in A\}$ and $A \oplus B = \{x + y : x \in A, y \in B\}$.

Proposition 10.2.1. Germ-grain process. *Let $\tilde{\Phi} = \sum_{j\in\mathbb{Z}} \delta_{(X_j, F_j)}$ be a marked point process with ground process $\Phi = \sum_{j\in\mathbb{Z}} \delta_{X_j}$ on $\mathbb{R}^d$ and marks $\{F_j\}_{j\in\mathbb{Z}}$ in $\mathcal{F}'(\mathbb{R}^d) = \mathcal{F}(\mathbb{R}^d) \setminus \{\emptyset\}$. Let $\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{X_j + F_j}$. Then $\Phi_{\mathrm{f}}(B)$ is a random variable for any $B \in \mathcal{B}(\mathcal{F}'(\mathbb{G}))$. Moreover,*

(i) *Φ_{f} is locally finite iff*

$$\Phi_{\mathrm{f}}(\mathcal{F}_K) < \infty, \quad \text{for all } K \in \mathcal{K}(\mathbb{R}^d). \tag{10.2.4}$$

If the above condition holds $\mathbf{P}$-almost surely, then Φ_{f} is a set process.

(ii) *If $\tilde{\Phi}$ is an* i.i.d. marked *point process, then*

$$\mathbf{E}[\Phi_{\mathrm{f}}(\mathcal{F}_K)] = \mathbf{E}\left[M_\Phi\left(\check{F}_0 \oplus K\right)\right], \quad \forall K \in \mathcal{K}(\mathbb{R}^d). \tag{10.2.5}$$

(iii) *If $\tilde{\Phi}$ is a* stationary marked *point process with marks $\{F_j\}_{j\in\mathbb{Z}}$ satisfying (7.1.7), then*

$$\mathbf{E}[\Phi_{\mathrm{f}}(\mathcal{F}_K)] = \lambda \mathbf{E}^0\left[\left|\check{F}_0 \oplus K\right|\right], \quad \forall K \in \mathcal{K}(\mathbb{R}^d), \tag{10.2.6}$$

where $\lambda \in \mathbb{R}_+^*$ is the intensity of Φ and $\mathbf{E}^0$ denotes the expectation with respect to the Palm probability of Φ.

(iv) *Assume that $\tilde{\Phi}$ is a* stationary i.i.d. marked *point process and that (10.2.4) holds true. Then the set process Φ_{f} is stationary.*

(v) *If*

$$\mathbf{E}[\Phi_{\mathrm{f}}(\mathcal{F}_K)] < \infty, \quad \text{for all } K \in \mathcal{K}(\mathbb{R}^d), \tag{10.2.7}$$

then (10.2.4) holds $\mathbf{P}$-almost surely and the mean measure of Φ_{f} is locally finite and characterized by $\left\{\mathbf{E}[\Phi_{\mathrm{f}}(\mathcal{F}_K)] : K \in \mathcal{K}(\mathbb{R}^d)\right\}$.

Proof. Observe that for all $B \in \mathcal{B}(\mathcal{F}'(\mathbb{G}))$,

$$\Phi_{\mathrm{f}}(B) = \sum_{j\in\mathbb{Z}} \mathbf{1}\{(X_j + F_j) \in B\},$$

which is a random variable by Proposition 9.2.2(iii). (i) The first assertion follows from Lemma 9.2.6 and the second one follows from Corollary 1.1.8. (ii) Note that $\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{X_j+F_j}$ may be viewed as an independent displacement of Φ in the sense of Definition 2.2.13 with respect to the kernel

$$\begin{aligned} p(x, \mathcal{F}_K) &= \mathbf{P}((x + F_0) \in \mathcal{F}_K) \\ &= \mathbf{P}((x + F_0) \cap K \neq \emptyset) \\ &= \mathbf{P}\left(x \in \check{F}_0 \oplus K\right), \quad x \in \mathbb{R}^d. \end{aligned}$$

Applying Equation (2.2.5), we get for all $K \in \mathcal{K}\left(\mathbb{R}^d\right)$,

$$\begin{aligned} \mathbf{E}[\Phi_{\mathrm{f}}(\mathcal{F}_K)] &= \int_{\mathbb{R}^d} \mathbf{P}\left(x \in \check{F}_0 \oplus K\right) M_\Phi(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathbf{E}\left[\mathbf{1}\left\{x \in \check{F}_0 \oplus K\right\}\right] M_\Phi(\mathrm{d}x) \\ &= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\left\{x \in \check{F}_0 \oplus K\right\} M_\Phi(\mathrm{d}x)\right] = \mathbf{E}\left[M_\Phi\left(\check{F}_0 \oplus K\right)\right]. \end{aligned}$$

(iii) It follows from Proposition 7.2.4 that

$$\begin{aligned} \mathbf{E}[\Phi_f(\mathcal{F}_K)] &= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}(x + F_0 \circ \theta_x \cap K \neq \emptyset)\Phi(\mathrm{d}x)\right] \\ &= \lambda \mathbf{E}^0\left[\int_{\mathbb{R}^d} \mathbf{1}(x + F_0 \cap K \neq \emptyset)\mathrm{d}x\right] = \lambda \mathbf{E}^0\left[|K \oplus \check{F}_0|\right]. \end{aligned}$$

This shows (10.2.6). The remaining part follows in the same lines as in (ii). (iv) Let $t \in \mathbb{R}^d$ be fixed. The i.i.d. marked point processes $\tilde{\Phi} = \sum_{j\in\mathbb{Z}} \delta_{(X_j,F_j)}$ and $S_t\tilde{\Phi} = \sum_{j\in\mathbb{Z}} \delta_{(X_j+t,F_j)}$ have the same distribution by Example 7.1.2. Thus $\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{X_j+F_j}$ and $S_t\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{X_j+t+F_j}$ have also the same distribution. (v) Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathbb{R}^d$ and let $K_n = \bar{B}(x_n, 1)$ the closed ball of center x_n and radius 1. For each $n \in \mathbb{N}$, since $\mathbf{E}[\Phi_{\mathrm{f}}(\mathcal{F}_{K_n})] < \infty$, then $\Phi_{\mathrm{f}}(\mathcal{F}_{K_n}) < \infty$, $\mathbf{P}$-almost surely; that is, there exists some event A_n such that $\Phi_{\mathrm{f}}(\omega)(\mathcal{F}_{K_n}) < \infty$ for all $\omega \in A_n$. Let $\omega \in \bigcap_{n\in\mathbb{N}} A_n$. Any $K \in \mathcal{K}\left(\mathbb{R}^d\right)$ may be covered by a finite number of open balls $B(x_n, 1)$. Let I be the set of such x_n. Then $K \subset \bigcup_{n\in I} K_n$. Thus $\mathcal{F}_K \subset \bigcup_{n\in I} \mathcal{F}_{K_n}$. Therefore, $\Phi_{\mathrm{f}}(\omega)(\mathcal{F}_K) \leq \sum_{n\in I} \Phi_{\mathrm{f}}(\omega)(\mathcal{F}_{K_n}) < \infty$. Then (10.2.4) holds $\mathbf{P}$-almost surely. It follows from (i) that Φ_{f} is a set process. The announced properties of $M_{\Phi_{\mathrm{f}}}$ follow from Lemma 9.2.6. □

Here is a sufficient condition for (10.2.7) to hold. We call *diameter* of a compact set of $\mathbb{R}^d$ the diameter of the smallest ball containing the compact.

Lemma 10.2.2. *Let $\tilde{\Phi} = \sum_{j\in\mathbb{Z}} \delta_{(X_j,F_j)}$ be a stationary marked point process with ground process $\Phi = \sum_{j\in\mathbb{Z}} \delta_{X_j}$ on $\mathbb{R}^d$ and marks $\{F_j\}_{j\in\mathbb{Z}}$ in $\mathcal{K}'\left(\mathbb{R}^d\right)$ and let $\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{X_j+F_j}$. A sufficient condition for (10.2.7) to hold is that the diameter D_0 of the typical grain F_0 be such that $\mathbf{E}^0[D_0^d] < \infty$. In the particular case where $\{F_j\}_{j\in\mathbb{Z}}$ are i.i.d. marks, the sufficient condition reads $\mathbf{E}[D_0^d] < \infty$.*

Proof. For all compact sets K, there exists a positive r and a ball of radius r that contains K. Hence $\check{F}_0 \oplus K$ is included in a ball of radius $D_0/2 + r$. Hence the assumption $\mathbf{E}^0[D_0^d] < \infty$ implies that $\mathbf{E}^0[\left|\check{F}_0 \oplus K\right|] < \infty$. Hence by (10.2.6), the condition (10.2.7) holds true. The proof of the last statement follows the same lines. □

Definition 10.2.3. *Consider the setting of Proposition 10.2.1 and assume that condition (10.2.4) holds true.*

(i) *The set process $\Phi_{\mathrm{f}} = \sum_{j\in\mathbb{Z}} \delta_{X_j+F_j}$ is called a* germ-grain process. *The points of this process are called the* germs, *and its marks are called the* grains. *The ground process $\Phi = \sum_{j\in\mathbb{Z}} \delta_{X_j}$ is called* germ process.

(ii) *The coverage model Z defined by (10.1.1) corresponding to a germ-grain process Φ_{f} is called a* germ-grain coverage model.

(iii) *If the germ-grain process Φ_{f} is Poisson (set process), then Z is called a* Boolean germ-grain coverage model.

Corollary 10.2.4. Little's law for stationary germ-grain coverage models. *Consider the setting of Proposition 10.2.1(iii) and let N be the number of set-atoms $X_j + F_j$ of Φ_{f} covering the origin. Then*

$$\mathbf{E}\left[N\right] = \lambda \mathbf{E}^0\left[|F_0|\right].$$

Proof. This follows from (10.2.6) with $K = \{0\}$. □

Corollary 10.2.5. Boolean germ-grain coverage model. *Consider the setting of Proposition 10.2.1(ii) and assume moreover that the germ process Φ is a Poisson point process on $\mathbb{R}^d$ and that (10.2.7) holds. Then the germ-grain process Φ_{f} is Poisson with intensity measure characterized by (10.2.5). In particular, $\Phi_{\mathrm{f}}\left(\mathcal{F}_K\right)$ is a Poisson random variable with mean (10.2.5).*

Proof. Recall that, by Proposition 10.2.1(v), Condition (10.2.4) holds **P**-almost surely and the mean measure of Φ_{f} is locally finite. When Φ is a Poisson point process, it follows from the displacement theorem 2.2.17 that Φ_{f} is a Poisson point process on $\mathcal{F}\left(\mathbb{R}^d\right)$ with intensity measure characterized by (10.2.5). □

Example 10.2.6. Stationary Boolean germ-grain coverage model. *Consider a Poisson germ process $\Phi = \sum_{j\in\mathbb{Z}} \delta_{X_j}$ with i.i.d. grains $\{F_j\}_{j\in\mathbb{Z}}$ such that (10.2.7) holds. Then $Z = \bigcup_{j\in\mathbb{Z}} (X_j + F_j)$ is a* stationary Boolean germ-grain coverage model.

The volume fraction of Z equals

$$p = T_Z(\{0\}) = 1 - e^{-\mathbf{E}\left[\Phi_{\mathrm{f}}\left(\mathcal{F}_{\{0\}}\right)\right]} = 1 - e^{-\lambda \mathbf{E}[|F_0|]},$$

where the above three equalities follow from (9.2.4), (10.1.2), and (10.2.6), respectively, and λ is the intensity of the germ process. The reduced covariance function of Z equals,

$$\begin{aligned} C(x) &= 2p - T_Z(\{0, x\}) \\ &= 2p - 1 + e^{-\mathbf{E}\left[\Phi_{\mathrm{f}}\left(\mathcal{F}_{\{0,x\}}\right)\right]} \\ &= 2p - 1 + e^{-\lambda \mathbf{E}\left[\left|\check{F}_0 \oplus \{0,x\}\right|\right]} \\ &= 2p - 1 + e^{-\lambda \mathbf{E}\left[\left|\check{F}_0 \cup \left(\check{F}_0 + x\right)\right|\right]} \\ &= 2p - 1 + e^{-\lambda \mathbf{E}\left[\left|\check{F}_0\right| + \left|\check{F}_0 + x\right| - \left|\check{F}_0 \cap \left(\check{F}_0 + x\right)\right|\right]} \\ &= 2p - 1 + (1-p)^2 e^{\lambda \mathbf{E}[|F_0 \cap (F_0 - x)|]}, \end{aligned}$$

for any $x \in \mathbb{R}^d$. The first three equalities follow respectively from (9.2.5), (10.1.2), and (10.2.6).

Example 10.2.7. Stationary Boolean germ-grain coverage model with spherical grains. *Consider a stationary Boolean germ-grain coverage model Z with i.i.d. spherical grains $F_j = \bar{B}(0, R_j)$ satisfying $\mathbf{E}\left[R_0^d\right] < \infty$. This is an example of a motion invariant coverage process. The capacity functional of Z satisfies*

$$T_Z\left(\bar{B}(0,a)\right) = 1 - e^{-\lambda \kappa_d \sum_{k=0}^{d} \binom{d}{k} a^{d-k} \mathbf{E}\left[R_0^k\right]}, \quad a \in \mathbb{R}_+,$$

where $\kappa_d = \left|\bar{B}(0,1)\right|$ and λ is the intensity of the germ process.

Indeed, let $\hat{\Phi} = \sum_{j \in \mathbb{Z}} \delta_{(X_j, R_j)}$ which is an i.i.d. marked point process associated to the germ process $\Phi = \sum_{j \in \mathbb{Z}} \delta_{X_j}$ in the sense of Definition 2.2.18. The mean measure of $\hat{\Phi}$ given by (2.2.8)

$$M_{\hat{\Phi}}(\mathrm{d}x \times \mathrm{d}r) = \lambda \mathrm{d}x \times p_{R_0}(\mathrm{d}r),$$

which is clearly locally finite. Then it follows from Theorem 2.2.21 that $\hat{\Phi}$ is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+$. Let $\tilde{\Phi}$ be a thinning of $\hat{\Phi}$ with respect to the probability of retention

$$p(X_j, R_j) = \mathbf{P}(a + R_j \geq |X_j|).$$

We deduce from Corollary 2.2.7 that $\tilde{\Phi}$ is a Poisson point process with intensity measure

$$\begin{aligned} M_{\tilde{\Phi}}(\mathrm{d}x \times \mathrm{d}r) &= p(x,r) M_{\hat{\Phi}}(\mathrm{d}x \times \mathrm{d}r) \\ &= \mathbf{P}(a + r \geq |x|) \lambda \mathrm{d}x \times p_{R_0}(\mathrm{d}r). \end{aligned}$$

Note that

$$\begin{aligned} T_Z\left(B\left(0,a\right)\right) &= 1-\mathbf{P}\left(B\left(0,a\right)\cap Z=\emptyset\right) \\ &= 1-\mathbf{P}\left(\tilde{\Phi}\left(\mathbb{R}^d\times\mathbb{R}_+\right)=0\right) \\ &= 1-e^{-M_{\tilde{\Phi}}\left(\mathbb{R}^d\times\mathbb{R}_+\right)}. \end{aligned}$$

Finally,

$$\begin{aligned} M_{\tilde{\Phi}}\left(\mathbb{R}^d\times\mathbb{R}_+\right) &= \lambda\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}^d}\mathbf{P}\left(a+r\geq|x|\right)\mathrm{d}x\right)p_{R_0}\left(\mathrm{d}r\right) \\ &= \lambda\kappa_d\int_{\mathbb{R}_+}\left(a+r\right)^d p_{R_0}\left(\mathrm{d}r\right) \\ &= \lambda\kappa_d\sum_{k=0}^{d}\binom{d}{k}a^{d-k}\mathbf{E}\left[R_0^k\right]. \end{aligned}$$

The Boolean germ-grain model is one of the most popular models in stochastic geometry. It features independent grains. We shall present models with dependent grains in Section 10.3.

10.2.2 Inverse construction

We saw above that an independently marked point process on $\mathbb{R}^d$ with i.i.d. marks in $\mathcal{F}'(\mathbb{G})$ allows one to construct a set process.

The aim of what follows is to show that conversely, to all stationary set processes Φ_{f} with atoms in $\mathcal{K}'(\mathbb{R}^d)$ (the set of nonempty compact sets in $\mathbb{R}^d$), one can associate a stationary marked point process of $\mathbb{R}^d$ with marks in $\mathcal{K}'(\mathbb{R}^d)$. which in a sense characterizes Φ_{f}.

This construction relies on the notion of *center* of a compact set $C\in\mathcal{K}'(\mathbb{R}^d)$.

Definition 10.2.8. *For all $C\in\mathcal{K}'(\mathbb{R}^d)$, the center $\sigma(C)$ of C is the center of the smallest ball that contains C.*

We will give some properties of the center function $\sigma:\mathcal{K}'(\mathbb{R}^d)\to\mathbb{R}^d$ to be used later. In this regard, we define the *Hausdorff distance* δ on $\mathcal{K}'(\mathbb{R}^d)$ by

$$\delta\left(A,B\right):=\max\left(\sup_{y\in B}\rho(y,A),\sup_{x\in A}\rho(x,B)\right),\quad A,B\in\mathcal{K}'(\mathbb{R}^d),\tag{10.2.8}$$

where $\rho\left(z,C\right)=\inf_{w\in C}|z-w|$ is the distance between $z\in\mathbb{R}^d$ and $C\in\mathcal{K}'(\mathbb{R}^d)$; cf. [87, p 7]. In what follows, the continuity of functions on $\mathcal{K}'(\mathbb{R}^d)$ will be with respect to the Hausdorff distance rather than with respect to Fell topology. For measurability, there is no difference [87, Theorem 2.4.1].

Lemma 10.2.9. *The center function is continuous on $\mathcal{K}'(\mathbb{R}^d)$ with respect to the Hausdorff distance (10.2.8).*

Proof. Cf. [87, Lemma 4.1.1]. □

It follows from the above lemma that the map σ is measurable from $\mathcal{K}'(\mathbb{R}^d)$ to $\mathbb{R}^d$. Moreover, observe that for all $t \in \mathbb{R}^d$ and all $C \in \mathcal{K}'(\mathbb{R}^d)$, $\sigma(C+t) = t + \sigma(C)$.

Let $\mathcal{K}'_0(\mathbb{R}^d)$ be the subset of $K \in \mathcal{K}'(\mathbb{R}^d)$ such that $c(K) = 0$. The mapping

$$\phi : \mathbb{R}^d \times \mathcal{K}'_0(\mathbb{R}^d) \to \mathcal{K}'(\mathbb{R}^d); (t, C) \mapsto t + C$$

is bijective and continuous with respect to the Hausdorff distance; cf. [87, p 101]. Let $\psi : \mathbb{M}\left(\mathcal{K}'(\mathbb{R}^d)\right) \to \mathbb{M}\left(\mathbb{R}^d \times \mathcal{K}'_0(\mathbb{R}^d)\right)$ be the mapping

$$\Phi_{\mathrm{f}} = \sum_{n\in\mathbb{Z}} \delta_{K_n} \mapsto \psi(\Phi_{\mathrm{f}}) = \sum_{n\in\mathbb{Z}} \delta_{c(K_n), K_n - c(K_n)} = \sum_{n\in\mathbb{Z}} \delta_{\phi^{-1}(K_n)}. \tag{10.2.9}$$

Proposition 10.2.10. *Let $\Phi_{\mathrm{f}} = \sum_{n\in\mathbb{Z}} \delta_{K_n}$ be a stationary point process on $\mathcal{K}'(\mathbb{R}^d)$ with non-zero locally finite mean measure, $\mathcal{K}'_0(\mathbb{R}^d) = \left\{K \in \mathcal{K}'(\mathbb{R}^d) : c(K) = 0\right\}$, and ψ be the function defined by (10.2.9). Then $\tilde{\Phi} = \psi(\Phi_{\mathrm{f}})$ is a stationary marked point process on $\mathbb{R}^d \times \mathcal{K}'_0\left(\mathbb{R}^d\right)$. Its mean measure satisfies*

$$M_{\tilde{\Phi}}(A \times B) = \lambda \ell^d(A) Q(B), \quad A \in \mathcal{B}\left(\mathbb{R}^d\right), B \in \mathcal{B}\left(\mathcal{K}'\left(\mathbb{R}^d\right)\right),$$

where $\lambda \in \mathbb{R}_+^*$ is the intensity of the ground process and Q is the Palm distribution of the mark, which is a probability distribution on $\mathcal{K}'_0\left(\mathbb{R}^d\right)$.

Proof. **Finiteness property.** Let $\Phi_{\mathrm{f}} = \sum_{n\in\mathbb{Z}} \delta_{K_n}$, then $\tilde{\Phi} = \sum_{n\in\mathbb{Z}} \delta_{c(K_n), K_n - c(K_n)}$. We first show that $\Phi = \sum_{n\in\mathbb{Z}} \delta_{c(K_n)}$ is almost surely locally finite. Indeed, let $C_0^d = [0,1)^d$ and let $\{z_j\}_{j\in\mathbb{N}}$ be an enumeration of points of $\mathbb{Z}^d$, then

$$\begin{aligned}
\mathbf{E}\left[\Phi\left(C_0^d\right)\right] &= M_{\Phi_{\mathrm{f}}}\left(\left\{K \in \mathcal{K}'\left(\mathbb{R}^d\right) : c(K) \in C_0^d\right\}\right) \\
&= M_{\Phi_{\mathrm{f}}}\left(\left\{K \in \mathcal{K}'\left(\mathbb{R}^d\right) : \exists j \in \mathbb{N} : K \cap \left(z_j + C_0^d\right) \neq \emptyset, c(K) \in C_0^d\right\}\right) \\
&\leq \sum_{j\in\mathbb{N}} M_{\Phi_{\mathrm{f}}}\left(\left\{K \in \mathcal{K}'\left(\mathbb{R}^d\right) : K \cap \left(z_j + C_0^d\right) \neq \emptyset, c(K) \in C_0^d\right\}\right) \\
&= \sum_{j\in\mathbb{N}} M_{\Phi_{\mathrm{f}}}\left(\left\{K \in \mathcal{K}'\left(\mathbb{R}^d\right) : K \cap C_0^d \neq \emptyset, c(K) \in C_0^d - z_j\right\}\right) \\
&= M_{\Phi_{\mathrm{f}}}\left(\left\{K \in \mathcal{K}'\left(\mathbb{R}^d\right) : K \cap C_0^d \neq \emptyset\right\}\right) \leq M_{\Phi_{\mathrm{f}}}\left(\mathcal{F}_{C_0^d}\right) < \infty.
\end{aligned}$$

Then $\Phi\left(C_0^d\right) < \infty$ almost surely, which implies that Φ is almost surely locally finite. **Stationarity**. Observe that, for all $t \in \mathbb{R}^d$,

$$\begin{aligned}
S_t\tilde{\Phi} &= \sum_{n\in\mathbb{Z}} \delta_{c(K_n)-t, K_n - c(K_n)} \\
&= \sum_{n\in\mathbb{Z}} \delta_{c(K_n - t), K_n - t - c(K_n - t)} = \psi(S_t\Phi_{\mathrm{f}}).
\end{aligned}$$

Since $S_t\Phi_{\mathrm{f}}$ has the same distribution as Φ_{f} then $S_t\tilde{\Phi} = \psi(S_t\Phi_{\mathrm{f}})$ has the same distribution as $\tilde{\Phi} = \psi(\Phi_{\mathrm{f}})$. Therefore $\tilde{\Phi}$ is a stationary marked point process on $\mathbb{R}^d \times \mathcal{K}'\left(\mathbb{R}^d\right)$. Then applying the general result in Section 7.3.1, we get the announced decomposition of the mean measure $M_{\tilde{\Phi}}$. □

10.3 Further examples

10.3.1 Hard-core coverage models

Example 10.3.1. Hard-core coverage models. *General hard-core coverage models are generated by set processes with non-overlapping set-atoms. Special cases of germ-grain hard-core models are generated by the Matérn hard-core point processes Φ_1 and Φ_2 in $\mathbb{R}^d$ considered in Examples 3.4.1 and 3.4.2 respectively. Recall, these point processes respect some exclusion distance h between points. Consequently, assuming spherical germs $F_j = \bar{B}(0, h/2)$ of radius $h/2$, one obtains hard-core germ-grain coverage models. In the stationary case, the corresponding volume fraction equals*

$$p_i = \lambda_i \kappa_d \left(\frac{h}{2}\right)^d, \quad i = 1, 2, \tag{10.3.1}$$

where λ_i is the intensity of Φ_i $(i = 1, 2)$. In particular, invoking (3.4.2) and (3.4.3) respectively, we get

$$\max_{\lambda \in \mathbb{R}_+^*} p_1 = \frac{1}{e 2^d}$$

and

$$\lim_{\lambda \to \infty} p_2 = \sup_{\lambda \in \mathbb{R}_+^*} p_2 = \frac{1}{2^d}, \tag{10.3.2}$$

where λ is the intensity of the underlying Poisson point process.

We will now prove (10.3.1). Indeed, by (9.2.4), for all $A \in \mathcal{B}\left(\mathbb{R}^d\right)$ such that $0 < |A| < \infty$,

$$p_i = \frac{1}{|A|} \mathbf{E}\left[\left|\bigcup_{X_j \in \Phi_i} \bar{B}(X_j, h/2) \cap A\right|\right] = \frac{1}{|A|} \mathbf{E}\left[\sum_{X_j \in \Phi_i} \left|\bar{B}(X_j, h/2) \cap A\right|\right].$$

Taking $A = B(0, R)$ and denoting $A^+ = \bar{B}(0, R + h/2)$ and $A^- = B(0, R - h/2)$, we have the obvious bounds

$$\sum_{X_j \in \Phi_i \cap A^-} \left|\bar{B}(X_j, h/2)\right| \leq \sum_{X_j \in \Phi_i} \left|\bar{B}(X_j, h/2) \cap A\right| \leq \sum_{X_j \in \Phi_i \cap A^+} \left|\bar{B}(X_j, h/2).\right|$$

On the other hand, by the C-L-M-M theorem 6.1.28, for $\varepsilon \in \{-, +\}$,

$$\mathbf{E}\left[\sum_{X_j \in \Phi_i \cap A^\varepsilon} \left|\bar{B}(X_j, h/2)\right|\right] = \lambda_i \left|A^\varepsilon\right| \mathbf{E}^0\left[\left|\bar{B}(0, h/2)\right|\right] = \lambda_i \kappa_d \left(\frac{h}{2}\right)^d \left|A^\varepsilon\right|$$

Thus

$$\lambda_i \kappa_d \left(\frac{h}{2}\right)^d \frac{|A^-|}{|A|} \leq p_i \leq \lambda_i \kappa_d \left(\frac{h}{2}\right)^d \frac{|A^+|}{|A|}.$$

Letting $R \to \infty$ proves (10.3.1).

Remark 10.3.2. *Note that the asymptotic volume fraction of the Matérn II hard-core model given by (10.3.2) equals the lower bound of all saturated hard-core configurations; cf. Exercise 10.4.4.*

10.3.2 Shot-noise coverage models

Note that the Voronoi tessellation generated by a a simple point process Φ in $\mathbb{R}^d$ can also be seen as a germ-grain model generated by the points of the process, with grains being the centered Voronoi cells. The coverage properties in this case are trivial. However, selecting a subset of the Voronoi cells leads to a non-trivial coverage process; cf. Example 6.2.19. The following modifications of the Voronoi cells lead also to non-trivial coverage models.

Example 10.3.3. Johnson-Mehl coverage model. *This is a family of coverage models parameterized by $r > 0$, having Voronoi grains F_j restricted to the ball $\bar{B}(0, r)$; that is*

$$F_j^{(r)} := F_j \cap \bar{B}(0, r), \quad j \in \mathbb{Z}.$$

The parameter r might be considered as the growth time. Initially each grain grows spherically in all directions but as soon as two grains touch each other at some location, they stop growing in the "blocked" direction. Johnson-Mehl is "almost" a hard-core model with the d-dimensional volume of the overlapping of set-atoms equal to 0. As $r \to \infty$, $F_j^{(r)} \to F_j$.

Example 10.3.4. Shot-noise coverage model. *This is a germ-grain coverage model on $\mathbb{R}^d$ in which the sets $X_j + F_j$ are defined as the regions of the space where the "impact"of point X_j exceeds the cumulative (additive) effect of the impacts of all other points. In the simplest scenario, the impact is modeled by a deterministic, decreasing function of the distance. The cumulative effect is called the shot-noise process associated with the point process and the impact function (also called the* response function*). This germ-grain coverage process was initially proposed to model wireless coverage cells, with the shot-noise modeling the interference in radio communications. When playing with the model parameters, such processes can exhibit a wide range of coverage patterns approaching Boolean models on one side and Voronoi tessellations and Johnson-Mehl models on the other, including hard-core scenarios. For more reading on this model see [5], [24, §6.5.4], [6, Chapter 7]. Some more advanced quantitative coverage results are presented in [13, Chapters 5-7].*

10.4 Exercises

Exercise 10.4.1. *Let Φ be a homogeneous Poisson point process on $\mathbb{R}^2$ with intensity $\lambda \in \mathbb{R}_+^*$. We associate to each point $X \in \Phi$ a closed ball of center X and radius $r \in \mathbb{R}_+^*$ denoted by $\bar{B}(X, r)$. Let $f(y)$ be the number of balls covering $y \in \mathbb{R}^2$.*

1. *Find the probability distribution of $f(y)$.*

2. *Show that the expectation of the surface within the square* $[0,T]^2$ *that is not covered by a ball equals* $\bar{S} = T^2 e^{-\lambda\pi r^2}$.

3. *Show that the average length of the boundary of this surface equals* $\bar{L} = \lambda e^{-\lambda\pi r^2} 2\pi r T^2$.

Solution 10.4.1. *Note that*

$$\begin{aligned} f(y) &= \sum_{X\in\Phi} \mathbf{1}\{|y-X| \le a\} \\ &= \sum_{X\in\Phi} \mathbf{1}\left\{X \in \bar{B}(y,r)\right\} = \Phi\left(\bar{B}(y,r)\right). \end{aligned}$$

1. $f(y) = \Phi\left(\bar{B}(y,r)\right)$ *is a Poisson random variable of mean* $\lambda\pi r^2$.

2. *The expectation of the surface within the square* $[0,T]^2$ *that is not covered by a ball equals*

$$\begin{aligned} \bar{S}(r) &= \mathbf{E}\left[\int_{[0,T]^2} \mathbf{1}\{f(y)=0\}\,\mathrm{d}y\right] \\ &= \int_{[0,T]^2} \mathbf{P}(f(y)=0)\,\mathrm{d}y \\ &= \int_{[0,T]^2} e^{-\lambda\pi r^2}\mathrm{d}y = T^2 e^{-\lambda\pi r^2}. \end{aligned}$$

3. *Let* y *be a point on the boundary. Since from Lemma 6.2.6,* $\mathbf{P}$*-almost surely,* Φ *has no two points equidistant from* y*, then* y *is at distance* r *of a single point of* Φ*. Thus the average length of the boundary equals*

$$\bar{L}(r) = \mathbf{E}\left[\sum_{X\in\Phi}\int_{(0,T)^2\cap\partial\bar{B}(X,r)} \mathbf{1}\left\{\Phi\left(\bar{B}(y,r)\right)=1\right\}\mathrm{d}y\right] = \mathbf{E}\left[\sum_{X\in\Phi} f(X,\omega)\right],$$

where

$$f(x,\omega) := \int_{(0,T)^2\cap\partial\bar{B}(x,r)} \mathbf{1}\left\{\Phi\left(\bar{B}(y,r)\right)=1\right\}\mathrm{d}y.$$

Note that

$$\begin{aligned} f(x,\theta_{-x}\omega) &:= \int_{(0,T)^2\cap\partial\bar{B}(x,r)} \mathbf{1}\left\{\Phi\circ\theta_{-x}\left(\bar{B}(y,r)\right)=1\right\}\mathrm{d}y \\ &= \int_{(0,T)^2\cap\partial\bar{B}(x,r)} 1\left\{S_{-x}\Phi\left(\bar{B}(y,r)\right)=1\right\}\mathrm{d}y \\ &= \int_{(0,T)^2\cap\partial\bar{B}(x,r)} 1\left\{\Phi\left(\bar{B}(y-x,r)\right)=1\right\}\mathrm{d}y \\ &= \int_{(-x,T-x)^2\cap\partial\bar{B}(0,r)} \mathbf{1}\left\{\Phi\left(\bar{B}(z,r)\right)=1\right\}\mathrm{d}z, \end{aligned}$$

where the last equality follows from the change of variable $z = y - x$. Using the C-L-M-M theorem 6.1.28, we get

$$
\begin{aligned}
\bar{L}(r) &= \mathbf{E}\left[\sum_{X\in\Phi} f(X,\omega)\right] \\
&= \lambda \int_{\mathbb{R}^2} \mathbf{E}^0[f(x,\theta_{-x}\omega)]\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^2} \mathbf{E}^0\left[\int_{(-x,T-x)^2\cap\partial\bar{B}(0,r)} \mathbf{1}\left\{\Phi\left(\bar{B}(z,r)\right)=1\right\}\mathrm{d}z\right]\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^2}\int_{(-x,T-x)^2\cap\partial\bar{B}(0,r)} \mathbf{P}^0\left(\Phi\left(\bar{B}(z,r)\right)=1\right)\mathrm{d}z\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^2}\int_{(-x,T-x)^2\cap\partial\bar{B}(0,r)} \mathbf{P}\left(\Phi\left(\bar{B}(z,r)\right)=0\right)\mathrm{d}z\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^2}\int_{(-x,T-x)^2\cap\partial\bar{B}(0,r)} e^{-\lambda\pi r^2}\mathrm{d}z\mathrm{d}x \\
&= \lambda e^{-\lambda\pi r^2}\int_{\partial\bar{B}(0,r)}\left(\int_{\mathbb{R}^2}\mathbf{1}\left\{z\in(-x,T-x)^2\right\}\mathrm{d}x\right)\mathrm{d}z,
\end{aligned}
$$

where the fourth equality follows from Slivnyak's theorem 3.2.4. Finally, for any fixed $z \in \mathbb{R}^2$,

$$
\int_{\mathbb{R}^2}\mathbf{1}\left\{z\in(-x,T-x)^2\right\}\mathrm{d}x = \int_{\mathbb{R}^2}\mathbf{1}\left\{-x\in(-z,T-z)^2\right\}\mathrm{d}x = T^2.
$$

Then

$$
\bar{L}(r) = \lambda e^{-\lambda\pi r^2}\int_{\partial\bar{B}(0,r)} T^2\mathrm{d}z = T^2\lambda 2\pi r e^{-\lambda\pi r^2},
$$

(which may be seen as a particular case of [93, Equation (3.2.6)]).

Remark. *Note that $\bar{L}(r) = -\bar{S}'(r)$ which is plausible since $\bar{S}(r+\mathrm{d}r) - \bar{S}(r)$ is approximately a thin surface of thickness $\mathrm{d}r$ surrounding $\bar{S}(r)$.*

Exercise 10.4.2. SNR coverage by a Poisson process of access points. *Consider transmitters (access points) on the plane modeled by a homogeneous Poisson point process $\Phi = \sum_{n\in\mathbb{Z}}\delta_{X_n}$ on $\mathbb{R}^2$ of intensity λ, transmitting with random, independent powers P_n of some given distribution. Suppose that a location on the plane is covered by some access point if the power P_{rec} received at this location from the considered access point divided by the noise power W is larger than some threshold W: SNR $:= P_{\mathrm{rec}}/W \geq T$. Assume the propagation model (3.4.4).*

1. *Show that the covered region is*

$$
Z = \bigcup_{n\in\mathbb{Z}}\bar{B}(X_n,R_n), \quad \text{where } R_n := \frac{1}{K}\left(\frac{P_n}{WT}\right)^{1/\beta}.
$$

2. *Calculate the fraction of the plane covered by at least one access point*

$$p = \frac{\mathbf{E}\left[|Z \cap K|\right]}{|K|}, \quad \text{where } K \text{ is a compact in } \mathbb{R}^2.$$

3. *Calculate the mean coverage number; i.e., the expected number of access points covering a given location.*

4. *Calculate the probability that the distance from some given location, given it is not covered, to a nearest location covered by at least one access point is larger than r, $r \geq 0$.*

5. *The locations of the users are modelled by a point process Φ_{u} on $\mathbb{R}^2$, independent of Φ.*

   (a) *Calculate the density of users covered by the network*

   $$\frac{\mathbf{E}\left[\Phi_{\mathrm{u}}\left(Z \cap K\right)\right]}{\mathbf{E}\left[\Phi_{\mathrm{u}}\left(K\right)\right]}, \quad \text{where } K \text{ is a compact in } \mathbb{R}^2.$$

   (b) *Assume that Φ_{u} is stationary with intensity μ. Calculate the expected number of users per surface unit*

   $$\frac{\mathbf{E}\left[\Phi_{\mathrm{u}}\left(Z \cap K\right)\right]}{|K|}, \quad \text{where } K \text{ is a compact in } \mathbb{R}^2.$$

Solution 10.4.2. *1. A location $y \in \mathbb{R}^2$ is covered by (i.e., can decode the signal from) X_n if*

$$\frac{P_n(K\,|y - X_n|)^{-\beta}}{W} \geq T \Leftrightarrow y \in \bar{B}\left(X_n, R_n\right),$$

where $R_n := \frac{1}{K}\left(\frac{P_n}{WT}\right)^{1/\beta}$. The total region covered by all the access points is

$$Z = \bigcup_{n \in \mathbb{Z}} \bar{B}\left(X_n, R_n\right).$$

We recognize the Boolean model *with random spherical grains.*

2. The expected fraction of the volume of a compact K covered by Z equals

$$p = \frac{\mathbf{E}\left[|Z \cap K|\right]}{|K|}.$$

Since Φ is stationary, then

$$p = \frac{1}{|K|}\mathbf{E}\left[\int_K \mathbf{1}\{x \in Z\}\,\mathrm{d}x\right] = \frac{1}{|K|}\int_K \mathbf{P}\left(x \in Z\right)\mathrm{d}x = \mathbf{P}\left(0 \in Z\right).$$

Reminder: The capacity functional *is defined by*

$$T_Z\left(K\right) = \mathbf{P}\left(Z \cap K \neq \emptyset\right), \quad K \text{ compact set,}$$

which is the probability that some part of K is covered by Z. Since Φ is a Poisson point process, then

$$T_Z(K) = 1 - e^{-\lambda \mathbf{E}\left[\left|K \oplus \bar{B}(0,R_1)\right|\right]}.$$

(Observe that $K \oplus \bar{B}(0,R) = \left\{y \in \mathbb{R}^2 : d(y,K) \le R\right\}$.) Thus

$$p = \mathbf{P}(0 \in Z) = T_Z(\{0\}) = 1 - e^{-\lambda\pi\mathbf{E}\left[R_1^2\right]}.$$

3. Reminder: The number of access points covering some part of K equals

$$N_Z(K) = \operatorname{card}\left\{X_n \in \Phi : \bar{B}(X_n, R_n) \cap K \neq \emptyset\right\}, \quad K \text{ compact set.}$$

Then $N_Z(K)$ is a Poisson random variable with mean $\lambda\left|K \oplus \bar{B}(0,R_1)\right|$.

In particular, the number of access points covering location $y \in \mathbb{R}^2$ is a Poisson random variable with mean $\lambda\left|\bar{B}(y,R_1)\right| = \lambda\pi\mathbf{E}\left[R_1^2\right]$

4. Reminder: The complementary spherical contact distribution function *(cf. Equation (9.2.6))*

$$G(r) = \frac{\mathbf{P}(d(0,Z) > r)}{\mathbf{P}(0 \in Z^{\mathrm{c}})} = \mathbf{P}(d(0,Z) > r | 0 \in Z^{\mathrm{c}}).$$

Then

$$\begin{aligned} G(r) &= \frac{1 - T_Z\left(\bar{B}(0,r)\right)}{1-p} \\ &= \frac{e^{-\lambda\mathbf{E}\left[\left|\bar{B}(0,r)\oplus\bar{B}(0,R_1)\right|\right]}}{e^{-\lambda\pi\mathbf{E}\left[R_1^2\right]}} = \frac{e^{-\lambda\mathbf{E}\left[(r+R_1)^2\right]}}{e^{-\lambda\pi\mathbf{E}\left[R_1^2\right]}}. \end{aligned}$$

5.a. Observe that

$$\begin{aligned} \mathbf{E}\left[\Phi_{\mathrm{u}}(Z \cap K)\right] &= \mathbf{E}\left[\int_{\mathbb{R}^2} \mathbf{1}\{x \in K\}\mathbf{1}\{x \in Z\}\Phi_{\mathrm{u}}(\mathrm{d}x)\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\int_{\mathbb{R}^2} \mathbf{1}\{x \in K\}\mathbf{1}\{x \in Z\}\Phi_{\mathrm{u}}(\mathrm{d}x)\middle|\Phi_{\mathrm{u}}\right]\right] \\ &= \mathbf{E}\left[\int_{\mathbb{R}^2} \mathbf{1}\{x \in K\}\mathbf{P}(x \in Z|\Phi_{\mathrm{u}})\Phi_{\mathrm{u}}(\mathrm{d}x)\right] \\ &= \mathbf{E}\left[\int_{\mathbb{R}^2} \mathbf{1}\{x \in K\}p\Phi_{\mathrm{u}}(\mathrm{d}x)\right] = p\mu|B|. \end{aligned}$$

Then the density of users covered by the network

$$\frac{\mathbf{E}\left[\Phi_{\mathrm{u}}(Z \cap K)\right]}{\mathbf{E}\left[\Phi_{\mathrm{u}}(K)\right]} = p.$$

5.b. We deduce from the above equation that

$$\frac{\mathbf{E}\left[\Phi_{\mathrm{u}}(Z \cap K)\right]}{|K|} = \mu p = \mu\left(1 - e^{-\lambda\pi\mathbf{E}\left[R_1^2\right]}\right).$$

Exercise 10.4.3. Boolean model. *Consider a germ grain coverage model* $Z = \bigcup_{j\in\mathbb{Z}} (X_j + F_j)$. *A germ* X_j *is said to be* isolated *if*

$$(X_j + F_j) \cap \bigcup_{k\in\mathbb{Z}\setminus\{j\}} (X_k + F_k) = \emptyset.$$

Consider a stationary Boolean germ-grain coverage model Z *with spherical grains as in Example 10.2.7. Assume that the intensity* λ *of the the germ process* Φ *is non-null and let* $\mathbf{P}^0$ *be the Palm probability associated to* Φ. *Show that the probability that a typical grain is isolated equals*

$$\begin{aligned}\mathbf{P}^0 (X_0 \text{ is isolated}) &= 1 - \mathbf{E}\left[T_Z\left(\bar{B}(0,R)\right)\right]\\ &= \mathbf{E}\left[e^{-\lambda\kappa_d \sum_{k=0}^{d}\binom{d}{k} R^{d-k}\mathbf{E}\left[R_0^k\right]}\right],\end{aligned}$$

where R *is a random variable with the same probability distribution as* R_0 *and independent from* $R_0, R_1, \ldots$. *Show that*

$$\mathbf{P}\left(\operatorname{card}\{X_j \text{ isolated}\} = \infty\right) = 1.$$

Solution 10.4.3. *By Slivnyak's theorem,*

$$\begin{aligned}\mathbf{P}^0 (X_0 \text{ is isolated}) &= \mathbf{P}\left(\bar{B}(0,R) \cap Z = \emptyset\right)\\ &= 1 - \mathbf{P}\left(\bar{B}(0,R) \cap Z \neq \emptyset\right)\\ &= 1 - \mathbf{E}\left[\mathbf{P}\left(\bar{B}(0,R) \cap Z \neq \emptyset | R\right)\right]\\ &= 1 - \mathbf{E}\left[T_Z\left(\bar{B}(0,R)\right)\right]\\ &= \mathbf{E}\left[e^{-\lambda\kappa_d \sum_{k=0}^{d}\binom{d}{k} R^{d-k}\mathbf{E}\left[R_0^k\right]}\right],\end{aligned}$$

where the last equality follows from Example 10.2.7. Note in particular that $\mathbf{P}^0 (X_0 \text{ is isolated}) > 0$.

Let $f(x,\omega) = \mathbf{1}\{x \text{ is isolated}\}$, *then*

$$\begin{aligned}\mathbf{E}\left[\operatorname{card}\{X_j \text{ isolated}\}\right] &= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\{x \text{ is isolated}\}\,\Phi(\mathrm{d}x)\right]\\ &= \mathbf{E}\left[\int_{\mathbb{R}^d} f(x,\omega)\,\Phi(\mathrm{d}x)\right]\\ &= \lambda\int_{\mathbb{R}^d} \mathbf{E}^0\left[f(x,\theta_{-x}\omega)\right]\mathrm{d}x\\ &= \lambda\int_{\mathbb{R}^d} \mathbf{E}^0\left[f(0,\omega)\right]\mathrm{d}x\\ &= \mathbf{P}^0 (X_0 \text{ is isolated})\left(\lambda\int_{\mathbb{R}^d}\mathrm{d}x\right) = \infty,\end{aligned}$$

where the third equality is due to the C-L-M-M theorem 6.1.28 and the last one is due the fact that $\mathbf{P}^0 (X_0 \text{ is isolated}) > 0$.

For any $k \in \mathbb{N}$, note that

$$A_k = \{\omega \in \Omega : \mathrm{card}\,\{X_j(\omega) \ \textit{isolated}\} = k\}$$

is θ_x-invariant for all $x \in \mathbb{R}^d$. Then $\mathbf{P}(A_k) \in \{0,1\}$ since the i.i.d. marked point process $\tilde{\Phi} = \sum_{j\in\mathbb{Z}} \delta_{(X_j,F_j)}$ is ergodic by Corollary 8.4.3.
Therefore $\mathrm{card}\,\{X_j(\omega) \ \textit{isolated}\}$ is almost surely constant, since its expectation is infinite, we deduce that $\mathrm{card}\,\{X_j(\omega) \ \textit{isolated}\} = \infty$ almost surely.

Exercise 10.4.4. Lower bound of volume fraction of hard-core models. *Let $\mu = \sum_{j\in\mathbb{Z}} \delta_{x_j}$ be a counting measure in $\mathbb{R}^d$ exhibiting the exclusion distance $h \in \mathbb{R}_+^*$; i.e.*

$$|x_j - x_k| > h \quad \textit{for all } j,k \in \mathbb{Z}. \tag{10.4.1}$$

μ is called a hard-core configuration of points*. Let us center a closed ball of radius $h/2$ at any point of μ and consider the disjoint union*

$$Z := \bigcup_{j\in\mathbb{Z}} \bar{B}(x_j, h/2).$$

We define the volume fraction of Z as the asymptotic fraction of the volume of a large window occupied by Z; that is

$$p := \liminf_{n\to\infty} \frac{|Z \cap W_n|}{|W_n|},$$

where the $W_n := \left[-n^{1/d}/2, n^{1/d}/2\right]^d$ is the window of volume n. A hard-core configuration of points μ of is called saturated*, when no ball can be added to μ without violating the hard-core condition (10.4.1). Show that the volume fraction p of a saturated configuration μ of points in $\mathbb{R}^d$ is not smaller than $1/2^d$.*

Solution 10.4.4. *Observe that*

$$\begin{aligned}
p &= \liminf_{n\to\infty} \frac{1}{n} \sum_{x_j\in\mu} \left|\bar{B}(x_j, h/2) \cap W_n\right| \\
&= \liminf_{n\to\infty} \frac{1}{n} \sum_{x_j\in\mu\cap W_n} |\bar{B}(x_j, h/2)| \\
&= \frac{1}{2^d} \liminf_{n\to\infty} \frac{1}{n} \sum_{x_j\in\mu\cap W_n} |\bar{B}(x_j, h)| \\
&= \frac{1}{2^d} \liminf_{n\to\infty} \frac{1}{n} \sum_{x_j\in\mu} |\bar{B}(x_j, h) \cap W_n| \\
&\geq \frac{1}{2^d} \liminf_{n\to\infty} \frac{1}{n} \left| \bigcup_{x_j\in\mu} \bar{B}(x_j, h) \cap W_n \right| \\
&= \frac{1}{2^d} \frac{|W_n|}{n} = \frac{1}{2^d},
\end{aligned}$$

where we ignore the boundary effects in the second and fourth equalities (which may justified as in Example 10.3.1) and where the last but one equality follows from the assumption that μ is saturated: indeed any location $y \in \mathbb{R}^d$ in the space is within the distance at most h from some point of μ; consequently, $\bigcup_{x_j \in \mu} \bar{B}(x_j, h) = \mathbb{R}^d$.

Exercise 10.4.5. *Consider the setting of Proposition 10.2.1(iii) and assume that $\mathbf{P}$-almost surely, the origin is covered by at most one set-atom $X_j + F_j$ of Φ_{f}. (This assumption is trivially true when Φ_{f} is a hard core set process. This is also true for the Voronoi tessellation and hence the Johnson-Mehl model since the stationary point processes do not have points equidistant to the origin.) Show that the volume fraction p of Z is equal to*

$$p = \lambda \mathbf{E}^0[|F_0|].$$

Solution 10.4.5. *Let N be the number of set-atoms $X_j + F_j$ of Φ_{f} covering the origin. Then*

$$N = \mathbf{1}\left\{\exists j \in \mathbb{Z} : 0 \in X_j + F_j\right\}.$$

Thus $p = \mathbf{E}[N]$. Invoking Corollary 10.2.4 allows one to conclude.

Chapter 11

Line processes and tessellations

11.1 Line processes in $\mathbb{R}^2$

11.1.1 Parameterization of lines in $\mathbb{R}^2$

Consider a straight line D in $\mathbb{R}^2$ (which may be identified with the complex plane $\mathbb{C}$). Let $z \in \mathbb{C}$ be the projection of the origin 0 on D, which admits the polar coordinate representation

$$z = re^{i\theta}, \quad \text{for some } r \in \mathbb{R}, \theta \in [0, \pi).$$

If $z = 0$, we take $\theta \in [0, \pi)$ to be the argument of the normal vector to D and $r = 0$. Note that the Cartesian equation of D writes

$$x \cos\theta + y \sin\theta = r, \quad x, y \in \mathbb{R}. \tag{11.1.1}$$

The mapping Δ associating to each $(r, \theta) \in \mathbb{R} \times [0, \pi)$ the line with the above Cartesian equation, called *line parameterization*, is clearly bijective.

Lemma 11.1.1. *Let Δ be the mapping associating to each $(r, \theta) \in \mathbb{R} \times [0, \pi)$ the line with the Cartesian equation (11.1.1). Let $\mathcal{F}'(\mathbb{R}^2)$ be the set of non-empty closed subsets of $\mathbb{R}^2$ with the Fell sub-topology.*

(i) The mapping $\Delta : \mathbb{R} \times [0, \pi) \to \mathcal{F}'(\mathbb{R}^2)$ is measurable.

(ii) Moreover, for all relatively compact $\mathcal{H} \subset \mathcal{F}'(\mathbb{R}^2)$, $\Delta^{-1}(\mathcal{H})$ is relatively compact.

Proof. (i) By Proposition 9.1.9, it is enough to prove the measurability of $\Delta^{-1}(\mathcal{F}_K)$ for all compact K. Observe first that $\mathcal{F}_K$ is compact by Proposition 9.1.7(i). Then $\Delta^{-1}(\mathcal{F}_K)$ is compact; cf. [73, Lemma 2.16]. (ii) Let

$\mathcal{H} \subset \mathcal{F}'\left(\mathbb{R}^2\right)$ be relatively compact. By Proposition 9.1.6(iv), there is some compact K such that $\bar{\mathcal{H}} \subset \mathcal{F}_K$. Then $\Delta^{-1}(\mathcal{H}) \subset \Delta^{-1}\left(\bar{\mathcal{H}}\right) \subset \Delta^{-1}(\mathcal{F}_K)$ which implies that $\Delta^{-1}(\mathcal{H})$ is relatively compact. □

Definition 11.1.2. *For any point process $\Phi = \sum_{n\in\mathbb{Z}} \delta_{(r_n,\theta_n)}$ on $\mathbb{R}\times[0,\pi)$, the image of Φ by Δ, which is by Definition 2.2.8 $\Phi_\Delta = \Phi\circ\Delta^{-1} = \sum_{n\in\mathbb{Z}} \delta_{\Delta(r_n,\theta_n)}$, is a point process on $\mathcal{F}'\left(\mathbb{R}^2\right)$ concentrated on the set of straight lines in $\mathbb{R}^2$. Φ_Δ is called a* line process *with* underlying *point process Φ on $\mathbb{R}\times[0,\pi)$.*

The mean measure of a line process Φ_Δ is given by

$$\begin{aligned} M_{\Phi_\Delta}(\mathcal{F}_K) &= M_\Phi \circ \Delta^{-1}(\mathcal{F}_K) \\ &= \int_{\mathbb{R}\times[0,\pi)} \mathbf{1}\left\{(r,\theta)\in\Delta^{-1}(\mathcal{F}_K)\right\} M_\Phi(\mathrm{d}r\times\mathrm{d}\theta) \\ &= \int_{\mathbb{R}\times[0,\pi)} \mathbf{1}\left\{\Delta(r,\theta)\cap K\neq\emptyset\right\} M_\Phi(\mathrm{d}r\times\mathrm{d}\theta). \end{aligned} \tag{11.1.2}$$

11.1.2 Stationarity

The above line parameterization Δ depends on the origin of the coordinate system. Changing this origin is equivalent to make a translation in $\mathbb{R}^2$. For a line process Φ_Δ to be stationary, its distribution $\mathbf{P}_{\Phi_\Delta}$ should be invariant under translations in $\mathbb{R}^2$.

A translation in $\mathbb{R}^2$ is characterized by its vector u which may be represented in the complex plane as

$$u = \rho e^{i\varphi}, \quad \text{for some } \rho\in\mathbb{R}, \varphi\in[0,\pi).$$

Note that the translation of a line $\Delta(r,\theta)$ does not modify the angle θ whereas the distance between the origin and the line is changed to $r+\rho\cos(\varphi-\theta)$. Thus a translation acts as a mapping from the *cylinder* $\mathbb{R}\times[0,\pi)$ to itself defined as follows

$$(r,\theta)\mapsto(r+\rho\cos(\varphi-\theta),\theta)$$

and called a *cylinder shear*.

Therefore, for a line process Φ_Δ to be stationary, the distribution of the underlying point process Φ on $\mathbb{R}\times[0,\pi)$ should be invariant with respect to cylinder shears.

Proposition 11.1.3.

(i) Let G be a finite measure on $[0,\pi)$, then the measure $\mathrm{d}r\times G(\mathrm{d}\theta)$ on $\mathbb{R}\times[0,\pi)$ is invariant with respect to cylinder shears.

(ii) Let μ be a locally finite measure on $\mathbb{R}\times[0,\pi)$ invariant with respect to cylinder shears, then there exists a finite measure G on $[0,\pi)$, such that

$$\mu(\mathrm{d}r\times\mathrm{d}\theta) = \mathrm{d}r\times G(\mathrm{d}\theta). \tag{11.1.3}$$

Proof. (i) For all measurable mappings $f:\mathbb{R}\times[0,\pi)\to\mathbb{R}_+$, and all $\rho\in\mathbb{R},\varphi\in[0,\pi)$,

$$\begin{aligned}
&\int_{\mathbb{R}\times[0,\pi)} f\left(r+\rho\cos\left(\varphi-\theta\right),\theta\right)\mathrm{d}r\times G\left(\mathrm{d}\theta\right)\\
&\qquad=\int_{[0,\pi)}\left(\int_{\mathbb{R}} f\left(r+\rho\cos\left(\varphi-\theta\right),\theta\right)\mathrm{d}r\right)G\left(\mathrm{d}\theta\right)\\
&\qquad=\int_{[0,\pi)}\left(\int_{\mathbb{R}} f\left(v,\theta\right)\mathrm{d}v\right)G\left(\mathrm{d}\theta\right)\\
&\qquad=\int_{\mathbb{R}\times[0,\pi)} f\left(r,\theta\right)\mathrm{d}r\times G\left(\mathrm{d}\theta\right),
\end{aligned}$$

where the second equality uses the change of variable $v:=r+\rho\cos\left(\varphi-\theta\right)$. (ii) Cf. [31, Lemma 15.3.I]. □

Corollary 11.1.4. *Consider a line process Φ_Δ such that:*

(i) $\mathbf{P}_{\Phi_\Delta}$ is invariant under translations in $\mathbb{R}^2$;

(ii) and the mean measure M_Φ of the underlying point process Φ on $\mathbb{R}\times[0,\pi)$ is locally finite.

Then M_Φ is in the form (11.1.3).

Definition 11.1.5. *A line process Φ_Δ is said to be* stationary *if it satisfies Properties (i) and (ii) of Corollary 11.1.4.*

Example 11.1.6. *Let $\Phi=\sum_{n\in\mathbb{Z}}\delta_{(r_n,\theta_n)}$ be a Poisson point process on $\mathbb{R}\times[0,\pi)$ with intensity measure of the form (11.1.3). Equivalently Φ is an i.i.d. marked stationary Poisson point process with intensity $\lambda=G([0,\pi))$ and mark distribution $G(\mathrm{d}\theta)/G([0,\pi))$. Then $\Phi_\Delta=\sum_{n\in\mathbb{Z}}\delta_{\Delta(r_n,\theta_n)}$ is a Poisson point process on $\mathcal{F}'\left(\mathbb{R}^2\right)$ with intensity measure (11.1.2) called a* Poisson line process*.*

In order to prove that Φ_Δ is Poisson, it is enough to show that $M_\Phi\circ\Delta^{-1}$ is locally finite and invoke Proposition 2.2.9. Indeed, for any $K\in\mathcal{K}\left(\mathbb{R}^2\right)$, there exists some $R\in\mathbb{R}_+$ such that $K\subset B\left(0,R\right)$, then starting from (11.1.2)

$$\begin{aligned}
M_\Phi\circ\Delta^{-1}\left(\mathcal{F}_K\right)&=\int_{\mathbb{R}\times[0,\pi)}\mathbf{1}\left\{\Delta\left(r,\theta\right)\cap K\neq\emptyset\right\}\mathrm{d}r\times G\left(\mathrm{d}\theta\right)\\
&\leq\int_{\mathbb{R}\times[0,\pi)}\mathbf{1}\left\{\Delta\left(r,\theta\right)\cap B\left(0,R\right)\neq\emptyset\right\}\mathrm{d}r\times G\left(\mathrm{d}\theta\right)\\
&=\int_{\mathbb{R}\times[0,\pi)}\mathbf{1}\left\{|r|<R\right\}\mathrm{d}r\times G\left(\mathrm{d}\theta\right)\\
&=2RG\left([0,\pi)\right)<\infty.
\end{aligned}$$

11.1.3 Associated random measures

Proposition 11.1.7. *Let $\Phi_{\Delta} = \sum_{n\in\mathbb{Z}} \delta_{\Delta(r_n,\theta_n)}$ be a stationary line process such that M_{Φ} has the form (11.1.3). Then*

$$\tilde{\Phi}(B) = \sum_{n\in\mathbb{Z}} \ell^1 (\Delta(r_n,\theta_n) \cap B), \quad B \in \mathcal{B}(\mathbb{R}^2)$$

defines a stationary random measure on $\mathbb{R}^2$ called a line measure. *Its intensity equals $G([0,\pi))$.*

Proof. Note first, that for any $\omega \in \Omega$, $\tilde{\Phi}(\omega)(\cdot)$ is a countable sum of measures, thus it is a measure. For each $B \in \mathcal{B}(\mathbb{R}^2)$, the mapping $\mathbb{R}\times[0,\pi) \to \mathbb{R}_+; (r,\theta) \mapsto \ell^1(\Delta(r,\theta)\cap B)$ is measurable. Then by Theorem 1.2.5, $\tilde{\Phi}(B)$ is a well defined random variable. Campbell's averaging formula (1.2.2) implies

$$\begin{aligned}
\mathbf{E}\left[\tilde{\Phi}(B)\right] &= \int_{\mathbb{R}\times[0,\pi)} \ell^1(\Delta(r,\theta)\cap B)\, M_{\Phi}(\mathrm{d}r\times\mathrm{d}\theta) \\
&= \int_{[0,\pi)} \left(\int_{\mathbb{R}} \ell^1(\Delta(r,\theta)\cap B)\,\mathrm{d}r\right) G(\mathrm{d}\theta) \\
&= \int_{[0,\pi)} \ell^2(B)\, G(\mathrm{d}\theta) = \ell^2(B)\, G([0,\pi)). \qquad (11.1.4)
\end{aligned}$$

When $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{R}^2)$, the above quantity is finite, then $\tilde{\Phi}(B)$ is almost surely finite. Thus, the measure $\tilde{\Phi}(\omega)(\cdot)$ is locally finite for almost all $\omega \in \Omega$. Then $\tilde{\Phi}$ is a random measure on $\mathbb{R}^2$ by Proposition 1.1.7. The stationarity of $\tilde{\Phi}$ follows from that of the line process Φ_{Δ}. Indeed, for all $B \in \mathcal{B}(\mathbb{R}^2)$,

$$\tilde{\Phi}(B+t) = \sum_{n\in\mathbb{Z}} \ell^1(\Delta(r_n,\theta_n)\cap(B+t)) = \sum_{n\in\mathbb{Z}} \ell^1((\Delta(r_n,\theta_n)-t)\cap B).$$

Since $\sum_{n\in\mathbb{Z}} \delta_{\Delta(r_n,\theta_n)-t}$ has the same distribution as $\Phi_{\Delta} = \sum_{n\in\mathbb{Z}} \delta_{\Delta(r_n,\theta_n)}$, then, for all $B_1,\ldots,B_j \in \mathcal{B}(\mathbb{R}^2)$, $\left(\tilde{\Phi}(B_1+t),\ldots,\tilde{\Phi}(B_j+t)\right)$ has the same distribution as $\left(\tilde{\Phi}(B_1),\ldots,\tilde{\Phi}(B_j)\right)$. This implies that $\tilde{\Phi}$ is stationary. The intensity of $\tilde{\Phi}$ is deduced from Equation (11.1.4). □

The above proposition justifies the following definition.

Definition 11.1.8. *Let Φ_{Δ} be a stationary line process such that M_{Φ} has the form (11.1.3). Then*

$$\lambda = G([0,\pi)) \quad \text{and} \quad Q(\mathrm{d}\theta) = \frac{G(\mathrm{d}\theta)}{\lambda} \text{ on } [0,\pi)$$

are respectively called the intensity *and the* directional distribution *of Φ_{Δ}. If $Q(\mathrm{d}\theta) = \frac{\mathrm{d}\theta}{\pi}$, then Φ_{Δ} is an* isotropic line process.

Proposition 11.1.9. *Let Φ_Δ be an isotropic line process. Then its distribution as well as that of the corresponding line measure $\tilde{\Phi}$ are invariant with respect to rotations.*

Proof. Cf. [73, Proposition 2.2]. □

Proposition 11.1.10. *Let Φ_Δ be a stationary line process in $\mathbb{R}^2$ with intensity λ and directional distribution Q. Let V be a line in $\mathbb{R}^2$ with parameters $(r_V, \theta_V) \in \mathbb{R} \times [0, \pi)$, and let Φ_V be the point process on V counting its intersections with Φ_Δ.*

(i) Then Φ_V is a stationary point process on V with intensity

$$\lambda_V = \lambda \int_{[0,\pi)} |\sin(\theta_V - \theta)|\, Q(\mathrm{d}\theta). \tag{11.1.5}$$

(ii) If moreover Φ_Δ is Poisson, then so is Φ_V.

(iii) If the line process Φ_Δ is isotropic, then

$$\lambda_V = \frac{2\lambda}{\pi}.$$

Proof. (i) Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{(r_n,\theta_n)}$ be the underlying point process in $\mathbb{R} \times [0, \pi)$, so that $\Phi_\Delta = \sum_{n\in\mathbb{Z}} \delta_{\Delta(r_n,\theta_n)}$. Observe that

$$\begin{aligned}
\mathbf{E}\left[\Phi\left\{(r_V, \theta_V)\right\}\right] &= \int_{\mathbb{R}\times[0,\pi)} \mathbf{1}\{r = r_V\}\, \mathbf{1}\{\theta = \theta_V\}\, M_\Phi(\mathrm{d}r \times \mathrm{d}\theta) \\
&= \lambda \int_{\mathbb{R}\times[0,\pi)} \mathbf{1}\{r = r_V\}\, \mathbf{1}\{\theta = \theta_V\}\, \mathrm{d}r \times Q(\mathrm{d}\theta) \\
&= \lambda \int_{[0,\pi)} \left(\int_{\mathbb{R}} \mathbf{1}\{r = r_V\}\, \mathrm{d}r\right) \mathbf{1}\{\theta = \theta_V\}\, Q(\mathrm{d}\theta) = 0.
\end{aligned}$$

Then almost surely $\Phi\{(r_V, \theta_V)\} = 0$, which implies $\Phi_\Delta\{V\} = 0$. Then the atoms of Φ_Δ are almost surely different from V. Thus

$$\Phi_V = \sum_{n\in\mathbb{Z}} \mathbf{1}\{\Delta(r_n, \theta_n) \cap V \neq \emptyset\}\, \delta_{\Delta(r_n,\theta_n)\cap V}.$$

We may assume without loss of generality that $r_V = 0$. Then the algebraic distance from the coordinate origin to the intersection $\Delta(r, \theta) \cap V$ equals

$$u(r, \theta) = \frac{r}{|\cos(\theta_V - \theta - \pi/2)|} = \frac{r}{|\sin(\theta_V - \theta)|}.$$

Let

$$u = \mathbb{R} \times [0, \pi) \to \mathbb{R} \cup \{i\mathbb{R}\}\,; (r, \theta) \mapsto \begin{cases} \frac{r}{|\sin(\theta_V - \theta)|} & \theta_V \neq \theta \\ r \times i & \theta_V = \theta, \end{cases}$$

where i is the imaginary unit. Clearly, u is measurable. Moreover, u^{-1} of a bounded set is bounded. Thus, then image of Φ by u given in Definition 2.2.8; that is

$$\Phi \circ u^{-1} = \sum_{n\in\mathbb{Z}} \delta_{u(r_n,\theta_n)}$$

is a point process in $\mathbb{R} \cup \{i\mathbb{R}\}$. Identifying V with the real set $\mathbb{R}$, we may write

$$\Phi_V = \sum_{n\in\mathbb{Z}} \mathbf{1}\left\{u\left(r_n,\theta_n\right) \in \mathbb{R}\right\} \delta_{u(r_n,\theta_n)},$$

which is a thinned version of $\Phi \circ u^{-1}$, then Φ_V is a point process in V. Campbell's averaging formula (1.2.2) implies

$$\begin{aligned}
\mathbf{E}\left[\Phi_V\left(B\right)\right] &= \int_{\mathbb{R}\times[0,\pi)} \mathbf{1}\left\{\Delta\left(r,\theta\right) \cap B \neq \emptyset\right\} M_\Phi\left(\mathrm{d}r \times \mathrm{d}\theta\right) \\
&= \lambda \int_{[0,\pi)} \left(\int_{\mathbb{R}} \mathbf{1}\left\{\Delta\left(r,\theta\right) \cap B \neq \emptyset\right\} \mathrm{d}r\right) Q\left(\mathrm{d}\theta\right) \\
&= \lambda \int_{[0,\pi)} \left(\left|\sin\left(\theta_V - \theta\right)\right| \int_{\mathbb{R}} \mathbf{1}\left\{x \in B\right\} \mathrm{d}x\right) Q\left(\mathrm{d}\theta\right) \\
&= \lambda \int_{[0,\pi)} \left(\left|\sin\left(\theta_V - \theta\right)\right| \ell^1\left(B\right)\right) Q\left(\mathrm{d}\theta\right) = \lambda_V \ell^1\left(B\right),
\end{aligned}$$

where the third equality is due to the change of variable $r \to x = \frac{r}{|\sin(\theta_V - \theta)|}$ and where λ_V is given by Equation (11.1.5). (ii) Observe that $\Phi \circ u^{-1}$ is Poisson by Proposition 2.2.9. Since Φ_V is a thinning of $\Phi \circ u^{-1}$, then Φ_V is Poisson by Proposition 2.2.6. (iii) If the line process Φ_Δ is isotropic, then Equation (11.1.5) gives

$$\lambda_V = \frac{\lambda}{\pi} \int_{[0,\pi)} \left|\sin\left(\theta_V - \theta\right)\right| \mathrm{d}\theta = \frac{2\lambda}{\pi}.$$

□

Proposition 11.1.11. *Let Φ_Δ be a stationary line process in $\mathbb{R}^2$ with intensity λ and directional distribution Q. Let B be a closed bounded convex set in $\mathbb{R}^2$, and let $Y(B)$ be the number of distinct lines of Φ_Δ intersecting B. If the line process is isotropic, then*

$$\mathbf{E}[Y(B)] = \frac{\lambda L(B)}{\pi},$$

where $L(B)$ is the perimeter of B.

Proof. Cf. [31, Proposition 15.3.IV]. □

Proposition 11.1.12. *Let $\Phi = \sum_{n\in\mathbb{Z}} \delta_{d_n}$ be a simple and stationary line process on $\mathbb{R}^2$ with finite intensity λ and directional distribution Q. For all $m, n \in \mathbb{Z}$, with $m \neq n$, let $v_{m,n}$ be the intersection of d_m and d_n. Then $\Phi_0 = \sum_{m,n\in\mathbb{Z}} \mathbf{1}\left\{v_{m,n} \neq \emptyset\right\} \delta_{v_{m,n}}$ is a stationary point process on $\mathbb{R}^2$. If moreover Φ is Poisson, then Φ_0 has a finite intensity.*

Proof. For all $\omega \in \Omega$, $\Phi_0(\omega)(\cdot)$ is a countable sum of measures, thus it is a measure on $\mathbb{R}^2$. The measurability of the map $\omega \mapsto \Phi_0(\omega)$ follows from the measurability of the intersection in the Fell topology [87, Theorem 12.2.6]. Let B be a square in $\mathbb{R}^2$ and let $Y(B)$ be the number of distinct lines of Φ intersecting B. Clearly,

$$\Phi_0(B) \leq Y(B)^2.$$

On the other hand, $Y(B)$ is smaller than the number of lines of Φ intersecting the sides of the square B. Moreover, by Proposition 11.1.10(i), the expectation of this number is smaller than $L(B)\lambda\pi$, where $L(B)$ is the perimeter of B. Thus

$$\mathbf{E}[Y(B)] \leq L(B)\lambda\pi < \infty.$$

Then $\mathbf{P}$-almost surely, $Y(B) < \infty$, and therefore $\Phi_0(B) < \infty$. Thus, for almost all $\omega \in \Omega$, $\Phi_0(\omega)(\cdot)$ is locally finite, i.e., Φ_0 is a point process on $\mathbb{R}^2$. Observe that

$$\begin{aligned} S_t\Phi_0 &= \sum_{m,n\in\mathbb{Z}} \mathbf{1}\{v_{m,n} \neq \emptyset\}\, \delta_{v_{m,n}-t} \\ &= \sum_{m,n\in\mathbb{Z}} \mathbf{1}\{(d_m - t) \cap (d_n - t) \neq \emptyset\}\, \delta_{(d_m-t)\cap(d_n-t)}. \end{aligned}$$

The stationarity of Φ_0 then follows from that of Φ. Let us now show that if Φ is a stationary Poisson line process, then Φ_0 cannot have an infinite intensity. Let B be a square in $\mathbb{R}^2$ and let $Y(B)$ be the number of distinct lines of Φ intersecting B. If Φ is Poisson, then $Y(B)$ is a sum of Poisson random variables by Proposition 11.1.10(ii). Thus $\mathbf{E}[\Phi_0(B)] \leq \mathbf{E}\left[Y(B)^2\right] < \infty$ since the variance of a Poisson random variable with finite mean is finite. $\square$

Note that Φ_0 defined in Proposition 11.1.12 may be the null measure. For instance, if $Q = \delta_0$, then all the lines of Φ are parallel, thus $\Phi_0 = 0$. The intensity of Φ_0 will be further discussed in Subsection 11.2.2.

Note also that Φ_0 is *not* a homogeneous Poisson point process (there is an infinite number of points on some lines; cf. Exercise 4.6.1).

11.2 Planar tessellations

11.2.1 Voronoi tessellation

Let Φ_2 be a simple stationary and ergodic point process on $\mathbb{R}^2$ with intensity $\lambda_2 \in \mathbb{R}_+^*$ (the choice of the index 2 will become clear later). Recall that the *Voronoi cell* of each $x \in \mathbb{R}^2$ is defined by Equation (6.2.2); that is

$$\tilde{V}(x, \Phi_2) = \left\{ y \in \mathbb{R}^2 : |y - x| \leq \inf_{Z\in\Phi_2} |y - Z| \right\},$$

which is a convex closed set. Observe that $y \in \tilde{V}(x, \Phi_2)$ iff $\Phi_2(B(y, |y - x|)) = 0$.

By Corollary 8.3.8, **P**-almost surely, for all $X \in \Phi_2$, $\tilde{V}(X, \Phi_2)$ is a bounded polygon in $\mathbb{R}^2$, then it has well defined edges (which are indeed segments) and well defined vertices. Let Φ_0 and Φ_1 be the point processes whose atoms are respectively the vertices and edge centers of the Voronoi cells of Φ_2; (the measurability of the maps $\omega \mapsto \Phi_0(\omega)$ and $\omega \mapsto \Phi_1(\omega)$ needs to be proved; cf. Exercise 11.3.4). The subscripts of these processes are chosen relatively to the dimension of the objects they are counting: Φ_2 counts the Voronoi cells of dimension 2, Φ_1 counts the segments of dimension 1, and Φ_0 the vertices of dimension 0. Each Φ_k is a stationary simple point process on $\mathbb{R}^2$ with intensity denoted by $\lambda_k \in \mathbb{R}_+^*$ ($k \in \{0, 1, 2\}$).

Lemma 11.2.1. *Let Φ_2 be a simple stationary point process on $\mathbb{R}^2$ with intensity $\lambda_2 \in \mathbb{R}_+^*$, let Φ_0 and Φ_1 be the point processes whose atoms are respectively the vertices and edge centers of the Voronoi cells of Φ_2, and let λ_0 and λ_1 be their respective intensities. Then*

$$2\lambda_1 = \lambda_2 \mathbf{E}_2^0 [M], \tag{11.2.1}$$

*where $\mathbf{E}_2^0$ is the expectation with respect to the Palm probability of Φ_2, and $M := \Phi_1\left(\tilde{V}(0, \Phi_2)\right)$. If moreover, **P**-almost surely, there are no 4 points of Φ_2 lying on a circle of $\mathbb{R}^2$, then*

$$3\lambda_0 = \lambda_2 \mathbf{E}_2^0 [M]. \tag{11.2.2}$$

Proof. Applying Corollary 6.3.21 with $f \equiv 1$ we get

$$\lambda_2 \mathbf{E}_2^0 \left[\Phi_1\left(\tilde{V}(0, \Phi_2)\right)\right] = \lambda_1 \mathbf{E}_1^0 \left[\int_{\mathbb{R}^d} \mathbf{1}\left\{0 \in \tilde{V}(x, \Phi_2)\right\} \Phi_2(\mathrm{d}x)\right].$$

Under $\mathbf{P}_1^0$, there are two Voronoi cells of Φ_2 having the origin on their edge, then the expectation in the right-hand side of the above equation equals 2. This proves (11.2.1). Applying again Corollary 6.3.21 for the point processes Φ_0 and Φ_2 with $f \equiv 1$ we get

$$\lambda_2 \mathbf{E}_2^0 \left[\Phi_0\left(\tilde{V}(0, \Phi_2)\right)\right] = \lambda_0 \mathbf{E}_0^0 \left[\int_{\mathbb{R}^d} \mathbf{1}\left\{0 \in \tilde{V}(x, \Phi_2)\right\} \Phi_2(\mathrm{d}x)\right].$$

Since for all $X \in \Phi_2$, $\tilde{V}(X, \Phi_2)$ is a bounded polygon in $\mathbb{R}^2$, it has an equal number of edges and vertices, then the left-hand side of the above equation equals $\lambda_2 \mathbf{E}_2^0 [M]$. The points of Φ_0 are the locations of space which are equidistant to three (or more) points of Φ_2. Assume that **P**-almost surely, there are no 4 points of Φ_2 lying on a circle of $\mathbb{R}^2$. Then all points of Φ_0 are almost surely equidistant to exactly three points of Φ_2. Thus, under $\mathbf{P}_0^0$, there are three Voronoi cells of Φ_2 having the origin as vertex, then the expectation in the right-hand side of the above equality equals 3. This proves (11.2.2). □

Alternative proof of Lemma 11.2.1. Define a directed bipartite graph with a directed edge from each $X \in \Phi_2$ to all points of Φ_1 in the Voronoi cell $\tilde{V}(X, \Phi_2)$.

This graph is translation invariant. We get (11.2.1) from the mass transport formula (6.1.19). We get (11.2.2) in the same way when considering the directed bipartite graph with a directed edge from each $X \in \Phi_2$ to all points of Φ_0 in the Voronoi cell $\tilde{V}(X, \Phi_2)$. □

Remark 11.2.2. *Observe that* $\mathbf{P}$*-almost surely, there are no* 4 *points of a homogeneous Poisson process lying on a circle of* $\mathbb{R}^2$*; cf. Exercise 11.3.1.*

Theorem 11.2.3. Mean value formulas for planar Voronoi tessellations. *Under the conditions of Lemma 11.2.1; and assuming that, almost surely, there are no* 4 *points of* Φ_2 *lying on a circle of* $\mathbb{R}^2$*, then*

$$\begin{cases} \mathbf{E}_2^0[M] = 6 \\ \lambda_0 = 2\lambda_2 \\ \lambda_1 = 3\lambda_2, \end{cases}$$

with $\lambda_2 = \lambda$*.*

Proof. We apply Corollary 6.3.21 for the point processes Φ_0 and Φ_2 with

$$f(x, \omega) = \mathbf{1}\left\{0 \in \Phi_2, x \in \Phi_0, x \in \tilde{V}(0, \Phi_2)\right\} \alpha(x),$$

where $\alpha(x)$ is the internal angle at the vertex x of the Voronoi cell $\tilde{V}(0, \Phi_2)$. Then

$$\begin{aligned} &\lambda_2 \mathbf{E}_2^0\left[\int_{\tilde{V}(0,\Phi_2)} f(x, \omega)\, \Phi_1(\mathrm{d}x)\right] \\ &\quad = \lambda_0 \mathbf{E}_0^0\left[\int_{\mathbb{R}^d} \mathbf{1}\left\{0 \in \tilde{V}(x, \Phi_2)\right\} f(-x, \theta_x \omega)\, \Phi_2(\mathrm{d}x)\right]. \end{aligned}$$

Observe that the integral in the left-hand side of the above equation is the sum of internal angles of a polygon having M vertices; that is $(M-2)\pi$. The integral in the right-hand side is the sum of internal angles seen from 0; that is 2π. Thus

$$\lambda_2 \mathbf{E}_2^0[M] - 2\lambda_2 = 2\lambda_0.$$

Thus, using Lemma 11.2.1, we get

$$\begin{cases} \lambda_0 = \frac{1}{3}\lambda_2 \mathbf{E}_2^0[M] \\ \lambda_1 = \frac{1}{2}\lambda_2 \mathbf{E}_2^0[M] \\ \lambda_2 \mathbf{E}_2^0[M] - 2\lambda_2 = 2\lambda_0. \end{cases}$$

Solving the above system, we get the stated result. □

Alternative proof of Theorem 11.2.3. For any finite planar graph in the plane without any edge intersections, Euler's formula states that

$$v - e + f = 2,$$

where v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely large region). Then

$$\lambda_0 - \lambda_1 + \lambda_2 = 0. \tag{11.2.3}$$

Thus, using Lemma 11.2.1, we get

$$\begin{cases} \lambda_0 = \frac{1}{3}\lambda_2 \mathbf{E}_2^0[M] \\ \lambda_1 = \frac{1}{2}\lambda_2 \mathbf{E}_2^0[M] \\ \lambda_1 - \lambda_0 = \lambda_2. \end{cases}$$

Solving the above system, we get the stated result. □

11.2.2 Crofton tessellation

The *Crofton tessellation* features a simple and stationary line process on $\mathbb{R}^2$

$$\Phi = \sum_{n\in\mathbb{Z}} \delta_{d_n}.$$

One sees $Z = \bigcup_{n\in\mathbb{Z}} d_n$ as a planar random graph of $\mathbb{R}^2$ with vertices the intersections of the lines of Φ and edges the segments of Z connecting these vertices. Let $\{v_m\}_{m\in\mathbb{Z}}$ be the set of vertices of this graph (this is the support of the point process Φ_0 introduced in Proposition 11.1.12), $\{S_n\}_{n\in\mathbb{Z}}$ be the set of its edges, which are subsets of the lines of Φ, and $\{C_p\}_{p\in\mathbb{Z}}$ be the set of its 2-dimensional facets, which are closed convex sets of $\mathbb{R}^2$. Let

$$\Phi_0 = \sum_{m\in\mathbb{Z}} \delta_{v_m}, \quad \Phi_1 = \sum_{n\in\mathbb{Z}} \delta_{S_n}, \quad \Phi_2 = \sum_{p\in\mathbb{Z}} \delta_{C_p}$$

be the associated point processes when they exist.

For studying this graph, we assume that

1. Φ_0 is a θ_t-compatible point process on $\mathbb{R}^2$ with positive and finite intensity λ_0, and

2. Φ_1 (resp. Φ_2) is a θ_t-compatible point process on $\mathcal{F}'\left(\mathbb{R}^2\right)$ with locally finite and non-zero mean measure Λ_1 (resp. Λ_2) having its support concentrated on the space of compact segments (resp. of compact and convex sets) of $\mathbb{R}^2$.

Under these assumptions, denote by $\tilde{\Phi}_k$ the point process on $\mathbb{R}^2$ of centers (cf. Definition 10.2.8) of Φ_k and by λ_k its intensity, $k = 1, 2$. From Proposition 10.2.10, λ_1 and λ_2 are positive and finite. It also follows from this proposition that the point process $\tilde{\Phi}_1$ has the centered segments $\{S_n - \sigma(S_n)\}_{n\in\mathbb{Z}}$ as marks, whereas the point process $\tilde{\Phi}_2$ has the centered compact and convex sets $\{C_p - \sigma(C_p)\}_{p\in\mathbb{Z}}$ as marks. The latter are the *Crofton cells* of the line process Φ. See Figure 11.1.

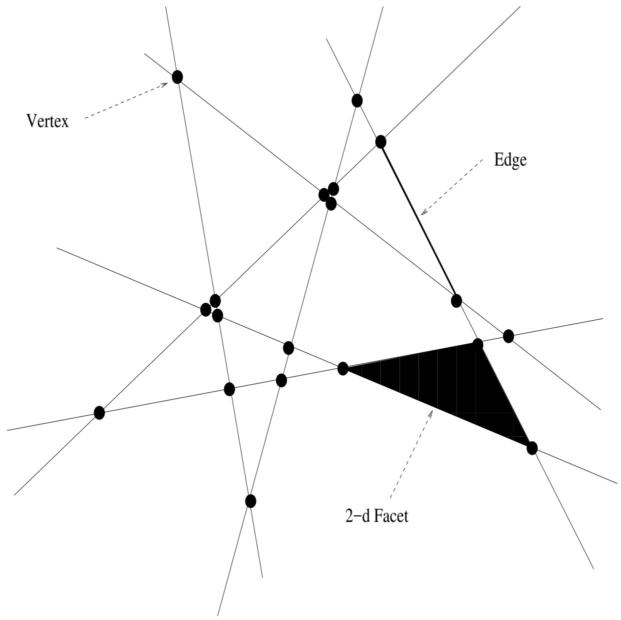

Figure 11.1: Crofton cells on $\mathbb{R}^2$.

Example 11.2.4. *Let us now check the above assumptions in the case when the line process is moreover Poisson, provided Φ_0 is not the empty measure. Indeed, it was shown in Proposition 11.1.12 that in this case, Φ_0 is a stationary point process with finite intensity λ_0. We will assume without proof that the maps $\omega \mapsto \Phi_1(\omega)$ and $\omega \mapsto \Phi_2(\omega)$ are measurable (cf. Exercise 11.3.4 for a similar question).*

The fact that Φ_2 is locally finite in this case follows from the inequality

$$\Phi_2(\mathcal{F}_K) \leq 2^{\Phi(\mathcal{F}_K)},$$

which holds for all compacts K of $\mathbf{R}^2$ (there are no more cells hitting K than ways to be on either side of each line hitting K). This counting measure is obviously compatible with the translations of Φ_0. The fact that the mean measure of Φ_2 is also locally finite follows from the above bound together with the property that the number of random lines that hit K, $\Phi(\mathcal{F}_K)$, is a Poisson random variable. The assumption that Φ_0 is not the empty measure implies that Q is not concentrated on a single atom. This in turn implies that each cell is a.s. compact.

The fact that Φ_1 is locally finite in the Poisson case is based on similar ideas and leverage the fact that the number of segments hitting K is bounded above by the square of the number of lines hitting K.

Below, we also assume that **P**-a.s., no three lines of Φ have a common intersection point.

Let M be the number of edges of the 2-dimensional facet containing 0. Let $\mathbf{E}_2^0$ denote the expectation with respect to the Palm probability of $\tilde{\Phi}_2$. Define a directed graph with a directed edge from each 2-dimensional facet $X \in \tilde{\Phi}_2$ to to all points of $\tilde{\Phi}_1$ on the edges of X. This graph is translation invariant. From the mass transport formula (6.1.19),

$$2\lambda_1 = \lambda_2 \mathbf{E}_2^0 [M] .$$

Define now a directed graph with a directed edge from each $X \in \tilde{\Phi}_2$ to all his vertices in Φ_0. Applying again the mass transport formula (6.1.19), and using the fact that each vertex has a.s. degree 4, we get

$$4\lambda_0 = \lambda_2 \mathbf{E}_2^0 [M] .$$

Combining the above two results with Euler's formula (11.2.3), we get

$$\begin{cases} \lambda_0 = \frac{1}{4}\lambda_2 \mathbf{E}_2^0 [M] \\ \lambda_1 = \frac{1}{2}\lambda_2 \mathbf{E}_2^0 [M] \\ \lambda_1 - \lambda_0 = \lambda_2. \end{cases}$$

Solving the above system, we get

$$\begin{cases} \mathbf{E}_2^0 [M] = 4 \\ \lambda_1 = 2\lambda_0 \\ \lambda_2 = \lambda_0. \end{cases}$$

In the isotropic case, a second factorial moment measure argument gives

$$\begin{aligned} \lambda_0 &= \frac{1}{|B(0,1)|} \mathbf{E} \left[\Phi_0 \left(B(0,1) \right) \right] \\ &= \frac{1}{\pi} \mathbf{E} \left[\sum_{m \in \mathbb{Z}} \mathbf{1} \left\{ v_m \in B(0,1) \right\} \right] \\ &= \frac{1}{\pi} \mathbf{E} \left[\sum_{k,n \in \mathbb{Z}, k \neq n} \mathbf{1} \left\{ d_n \cap B(0,1) \neq \emptyset \right\} \mathbf{1} \left\{ d_k \cap B(0,1) \neq \emptyset \right\} \right] \\ &= \frac{1}{\pi} \int_{\mathbb{R} \times [0,\pi)} \int_{\mathbb{R} \times [0,\pi)} \mathbf{1} \left\{ r_1 \leq 1 \right\} \mathbf{1} \left\{ r_2 \leq 1 \right\} \lambda \mathrm{d}r_1 \lambda \mathrm{d}r_2 \frac{\mathrm{d}\theta_1}{\pi} \frac{\mathrm{d}\theta_2}{\pi} \\ &= \frac{\lambda^2}{\pi}, \end{aligned}$$

with λ the intensity of the isotropic line process and the second equality is due to (2.3.18).

11.3 Exercises

Exercise 11.3.1. *Prove that* $\mathbf{P}$*-almost surely, there are no* 4 *points of a homogeneous Poisson process lying on a circle of* $\mathbb{R}^2$*.*

Solution 11.3.1. *The argument follows in the same lines as the proof of Lemma 6.2.6.*

$$
\begin{aligned}
&\mathbf{P}\left(\{\exists\,(X_1,\dots,X_4)\in\Phi^{(4)} \text{ lying on a circle}\}\right)\\
&\le \mathbf{E}[\mathrm{card}(\{(X_1,\dots,X_4)\in\Phi^{(4)} \text{ lying on a circle}\})]\\
&= \mathbf{E}\left[\int_{(\mathbb{R}^2)^4}\mathbf{1}\{(x_1,\dots,x_4)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}x_1\right)\dots\Phi\left(\mathrm{d}x_4\right)\right]\\
&= \lambda\mathbf{E}^0\left[\int_{(\mathbb{R}^2)^4}\mathbf{1}\{(x_1,x_2,x_3,x)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\left(\Phi\circ\theta_{-x}\right)\left(\mathrm{d}x_1\right)\dots\left(\Phi\circ\theta_{-x}\right)\left(\mathrm{d}x_3\right)\mathrm{d}x\right]\\
&= \lambda\int_{\mathbb{R}^2}\mathbf{E}^0\left[\int_{(\mathbb{R}^2)^3}\mathbf{1}\{(y_1,y_2,y_3)\in\left(\mathbb{R}^2\right)^{(3)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}y_1\right)\dots\Phi\left(\mathrm{d}y_3\right)\right]\mathrm{d}x\\
&= \lambda\int_{\mathbb{R}^2}\mathbf{E}\left[\int_{(\mathbb{R}^2)^3}\mathbf{1}\{(y_1,y_2,y_3)\in\left(\mathbb{R}^2\right)^{(3)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}y_1\right)\dots\Phi\left(\mathrm{d}y_3\right)\right]\mathrm{d}x,
\end{aligned}
$$

where the third line is due to the C-L-M-M theorem 6.1.28, for the fourth line we make the change of variable $x_i \to y_i = x_i - x$, and the fifth line is due to Slivnyak theorem 6.1.31(iii). By the same arguments, we have

$$
\begin{aligned}
&\mathbf{E}\left[\int_{(\mathbb{R}^2)^3}\mathbf{1}\{(y_1,y_2,y_3,0)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}y_1\right)\dots\Phi\left(\mathrm{d}y_3\right)\right]\\
&= \lambda\mathbf{E}^0\left[\int_{(\mathbb{R}^2)^3}\mathbf{1}\{(y_1,y_2,y_3,0)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\left(\Phi\circ\theta_{-y}\right)\left(\mathrm{d}y_1\right)\left(\Phi\circ\theta_{-y}\right)\left(\mathrm{d}y_2\right)\mathrm{d}y\right]\\
&= \lambda\int_{\mathbb{R}^2}\mathbf{E}^0\left[\int_{(\mathbb{R}^2)^2}\mathbf{1}\{(z_1,z_2,0,-y)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}z_1\right)\Phi\left(\mathrm{d}z_2\right)\right]\mathrm{d}y\\
&= \lambda\int_{\mathbb{R}^2}\mathbf{E}\left[\int_{(\mathbb{R}^2)^2}\mathbf{1}\{(z_1,z_2,0,-y)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}z_1\right)\Phi\left(\mathrm{d}z_2\right)\right]\mathrm{d}y.
\end{aligned}
$$

Again, for any $y\in\mathbb{R}^2$,

$$
\begin{aligned}
&\mathbf{E}\left[\int_{(\mathbb{R}^2)^2}\mathbf{1}\{(z_1,z_2,0,-y)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}z_1\right)\Phi\left(\mathrm{d}z_2\right)\right]\\
&= \lambda\mathbf{E}^0\left[\int_{(\mathbb{R}^2)^2}\mathbf{1}\{(z_1,z,0,-y)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\left(\Phi\circ\theta_{-z}\right)\left(\mathrm{d}z_1\right)\mathrm{d}z\right]\\
&= \lambda\int_{\mathbb{R}^2}\mathbf{E}^0\left[\int_{\mathbb{R}^2}\mathbf{1}\{(t,0,-z,-y-z)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}t\right)\right]\mathrm{d}z\\
&= \lambda\int_{\mathbb{R}^2}\mathbf{E}\left[\int_{\mathbb{R}^2}\mathbf{1}\{(t,0,-z,-y-z)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\Phi\left(\mathrm{d}t\right)\right]\mathrm{d}z\\
&= \lambda\int_{\mathbb{R}^2}\lambda\left[\int_{\mathbb{R}^2}\mathbf{1}\{(t,0,-z,-y-z)\in\left(\mathbb{R}^2\right)^{(4)} \text{ lies on a circle}\}\mathrm{d}t\right]\mathrm{d}z = 0.
\end{aligned}
$$

Exercise 11.3.2. Mean Voronoi edge length per unit area. *Let* $\Phi = \sum_{n\in\mathbb{Z}} \delta_{X_n}$ *be a stationary point process in* $\mathbb{R}^2$ *with intensity* $\lambda \in \mathbb{R}_+^*$*. Let* $\Phi_{\mathrm{f}} = \sum_{n\in\mathbb{Z}} \delta_{s_n}$ *and* $\Phi_1 = \sum_{n\in\mathbb{Z}} \delta_{Y_n}$ *denote the point processes of edges and edge centers respectively of the Voronoi tessellation of* Φ *(*Y_n *is the center of the segment* s_n*). Let* λ_1 *be the intensity of* Φ_1*. For all* $B \in \mathcal{B}\left(\mathbb{R}^2\right)$*, let*

$$L(B) = \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \ell^1(s_n \cap B)\right], \quad M(B) = \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{Y_n \in B\right\} \ell^1(s_n)\right].$$

The expectation with respect to the Palm probability of Φ *and* Φ_1 *are denoted respectively by* $\mathbf{E}^0$ *and* $\mathbf{E}_1^0$*.*

1. *Comment on the differences between*

   $$\sum_{n\in\mathbb{Z}} \ell^1(s_n \cap B)$$

   and

   $$\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{Y_n \in B\right\} \ell^1(s_n).$$

   Show that the sequence $\{s_n - Y_n\}_{n\in\mathbb{Z}}$ *is a sequence of compatible marks of* Φ_1 *with values in* $\mathcal{K}_0'\left(\mathbb{R}^2\right)$ *and use this to evaluate* $M(B)$*.*

2. *Show that* $L(B)$ *is a translation invariant measure on* $\mathbb{R}^2$*.*

3. *Show that* $L(B) = \lambda_1 \ell^2(B)\mathbf{E}_1^0[\ell^1(s_0)]$*. Comment the result.*

Solution 11.3.2.

1. *In* $\sum_{n\in\mathbb{Z}} \ell^1(s_n \cap B)$ *the lengths of certain parts of the segments* s_n *are added up; this may include parts of segments whose center is not in* B*. In* $\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{Y_n \in B\right\} \ell^1(s_n)$*, only whole segment lengths are added up, with as condition for counting them the fact that the location of their center be in* B*.*

   Note that $\Phi_{\mathrm{f}} = \sum_{n\in\mathbb{Z}} \delta_{s_n}$ *is a stationary set process, then* $\{s_n - Y_n\}_{n\in\mathbb{Z}}$ *are compatible marks of* Φ_1 *by Proposition 10.2.10. Then it follows from Proposition 7.2.4, that*

   $$M(B) = \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\left\{Y_n \in B\right\} \ell^1(s_n)\right] = \lambda^1 \ell^2(B)\mathbf{E}_1^0[\ell^1(s_0)].$$

2. *Let*

   $$\tilde{\Phi}\left(\omega\right)\left(B\right) = \sum_{n\in\mathbb{Z}} \ell^1(s_n \cap B).$$

   Note first, that for any $\omega \in \Omega$*,* $\tilde{\Phi}\left(\omega\right)\left(\cdot\right)$ *is a countable sum of measures, thus it is a measure.* $L(\cdot) = \mathbf{E}\left[\tilde{\Phi}\left(\cdot\right)\right]$ *inherits the property of finite additivity from that of* $\tilde{\Phi}$*, moreover, if the sequence of Borel sets* $B_n \uparrow B$*, then by monotone convergence* $L\left(B_n\right) \uparrow L\left(B\right)$*. Then* $L(\cdot)$ *is a measure on* $\mathbb{R}^2$*.*

Observe that, for all $B \in \mathcal{B}(\mathbb{R}^2)$,

$$\tilde{\Phi}(B+t) = \sum_{n\in\mathbb{Z}} \ell^1(s_n \cap (B+t)) = \sum_{n\in\mathbb{Z}} \ell^1((s_n - t)\cap B).$$

Since Φ_{f} *is stationary, then* $\mathbf{E}\left[\tilde{\Phi}(B+t)\right] = \mathbf{E}\left[\tilde{\Phi}(B)\right]$, *thus* $L(\cdot)$ *is translation invariant.*

3. *Let* $Z_n = s_n - Y_n$ *and*

$$g(Y_n, Z_n) = \ell^1((Y_n + Z_n)\cap B).$$

Then, by Proposition 7.2.4

$$\begin{aligned}
L(B) &= \mathbf{E}\left[\sum_{n\in\mathbb{Z}} g(Y_n, Z_n)\right] \\
&= \lambda_1 \int_{\mathbb{R}^2} \mathbf{E}_1^0[g(y, Z_0)]\mathrm{d}y \\
&= \lambda_1 \mathbf{E}_1^0\left[\int_{\mathbb{R}^2} \ell^1((y+Z_0)\cap B)\mathrm{d}y\right] \\
&= \lambda_1 \mathbf{E}_1^0\left[\int_{\mathbb{R}^2}\left(\int_{Z_0} \mathbf{1}\{y+t\in B\}\,\mathrm{d}t\right)\mathrm{d}y\right] \\
&= \lambda_1 \mathbf{E}_1^0\left[\int_{Z_0}\left(\int_{\mathbb{R}^2} \mathbf{1}\{y+t\in B\}\,\mathrm{d}y\right)\mathrm{d}t\right] = \lambda_1 \ell^2(B)\mathbf{E}_1^0[\ell^1(s_0)].
\end{aligned}$$

So

$$\mathbf{E}\left[\sum_{n\in\mathbb{Z}} \ell^1(s_n\cap B)\right] = \mathbf{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}\{Y_n\in B\}\,\ell^1(s_n)\right],$$

which shows that the differences between these two sums which were stressed in 1 compensate in mean.

Exercise 11.3.3. Mean perimeter of the typical Voronoi cell. *The setting is the same as in Exercise 11.3.2. Assume moreover that* $\mathbf{P}$*-almost surely, there are no 4 points of* Φ *lying on a circle of* $\mathbb{R}^2$. *Let* $V = V(\Phi)$ *denote the virtual cell w.r.t.* Φ *defined by (6.2.1). Under* $\mathbf{P}^0$, *the perimeter of the Voronoi cell of* 0 *is defined by*

$$U(V) = \sum_{Y_n\in\Phi_1} \mathbf{1}_{Y_n\in V}\ell^1(s_n).$$

It was proved in Theorem 11.2.3 that the mean number of edges of the typical Voronoi cell is 6. Hence, it ought to be true that $\mathbf{E}^0[U(V)] = 6\overline{\ell^1(s)}$ *with* $\overline{\ell^1(s)}$ *the "mean edge length". The aim of this exercise is to make this statement precise by proving that*

$$\mathbf{E}^0[U(V)] = 6\mathbf{E}_1^0[\ell^1(s_0)]. \qquad (11.3.1)$$

Solution 11.3.3. *The mean number of points of Φ_1 which are on the topological boundary of V is positive. Hence we cannot apply Neveu's formula (6.3.4) (which would erroneously give that $\mathbf{E}^0[U(V)] = 3\mathbf{E}_1^0[l^1(s_0)]$). Rather than this, we use the mass transport formula (6.1.16) with Φ and $\Phi' = \Phi_1$, and with*

$$g(y,\omega) = \mathbf{1}_{y \in V(\Phi(\omega))} \ell^1(s \circ \theta_y),$$

which gives

$$\begin{aligned}
\lambda \mathbf{E}^0[U(V)] &= \lambda_1 \mathbf{E}_1^0 \left[\int_{\mathbb{R}^d} \ell^1(s_0) \mathbf{1}_{-x \in V(\Phi(\theta_x \omega))} \Phi(\mathrm{d}x) \right] \\
&= \lambda_1 \mathbf{E}_1^0 \left[\ell^1(s_0) \operatorname{card} \left(\{ X_n \in \Phi : -X_n \in V(\Phi(\theta_{X_n})) \} \right) \right] \\
&= \lambda_1 \mathbf{E}_1^0 \left[\ell^1(s_0) \operatorname{card} \Big(\{ X_n \in \Phi : 0 \in \tilde{V}(X_n) \} \Big) \right].
\end{aligned}$$

where the last equality is due to (6.2.3) with $\tilde{V}(X_n)$ being the Voronoi cell associated to X_n. From the foregoing assumptions, the cardinality in the last equation is 2 a.s. Hence

$$\mathbf{E}^0[U(V)] = 2 \frac{\lambda_1}{\lambda} \mathbf{E}_1^0 \left[\ell^1(s_0) \right],$$

which concludes the proof since $\lambda_1 = 3\lambda$ by Theorem 11.2.3.

Exercise 11.3.4. Planar Voronoi tessellation; Measurability. *Let Φ_2 be a simple stationary and ergodic point process on $\mathbb{R}^2$ with intensity $\lambda_2 \in \mathbb{R}_+^*$. Prove the measurability of the maps $\omega \mapsto \Phi_0(\omega)$ and $\omega \mapsto \Phi_1(\omega)$ counting respectively the vertices and the edge centers of the Voronoi cells associated to Φ_2; cf. Section 11.2.1.*

Solution 11.3.4. *A vertex of the Voronoi tessellation may be characterized as follows. For all 3-tuples of points of Φ_2 which are not aligned, let D denote their circumscribed circle. The center of D is an atom of Φ_0 iff the interior of this disc does not contain any point of Φ_2. All the operations in this construction are measurable. This proves that the map $\omega \mapsto \Phi_0(\omega)$ is measurable.*

The edge centers of the Voronoi cells may be characterized as follows. For any pair of points (X, Y) of Φ_2, consider their middle point I. Then I is an atom of Φ_1 iff I belongs to the Voronoi cells of X and Y. Again the operations in this construction are measurable. Then the map $\omega \mapsto \Phi_1(\omega)$ is measurable.

Exercise 11.3.5. Voronoi-protected shot-noise [3] *Let Φ be some homogoneous Poisson point processes on $\mathbb{R}^2$ with intensity λ. Let l be some nonnegative function from $\mathbb{R}^+$ to $\mathbb{R}^+$. The Voronoi-protected shot-noise is the value at the origin of the shot-noise created by all points of Φ outside the cell containing the origin for the reponse function l, that is*

$$J = \sum_{X \in \Phi \setminus X^*} l(||X||),$$

where X^* is the point of Φ that is the closest to the origin. Give an integral expression for the Laplace transform of J.

Assume now that the point process Φ is independently marked with i.i.d exponential marks of mean 1 and let

$$K = \sum_{X \in \Phi \setminus X^*} F_X l(||X||),$$

where F_X is the exponential mark of X. Assume in addition that $l(r) = r^{-\beta}$ for some $\beta > 0$. Give the conditions under which J is non-degenerate.

Give an integral expression for the c.d.f of the signal to protected shot noise ratio $\frac{F_{X^*} l(||X^{*|}|)}{K}$.

Solution 11.3.5. *Conditionally on $||X^*|| = r$, the point process $\Phi \setminus X^*$ is a non homogenous Poisson point process with intensity 0 in the ball of center 0 and radius r and λ outside this ball. Hence, it follows from the expression of the Laplace transform of a Poisson point process that*

$$\mathbb{E}[e^{-sJ} \mid ||X^*|| = r] = \exp\left(-2\pi\lambda \int_{u=r}^{\infty} (1 - e^{-sl(u)}) u \mathrm{d}u\right).$$

Hence

$$\mathbb{E}[e^{-sJ}] = \int_{\mathbb{R}^+} \exp\left(-2\pi\lambda \int_{u=r}^{\infty} (1 - e^{-sl(u)}) u \mathrm{d}u\right) \exp\left(-\pi\lambda r^2\right) 2\pi\lambda r \mathrm{d}r.$$

By the same argument applied to the independently marked Poisson point process,

$$\mathbb{E}[e^{-sK} \mid ||X^*|| = r] = \exp\left(-2\pi\lambda \int_{u=r}^{\infty} \left(1 - \frac{1}{1 + su^{-\beta}}\right) u \mathrm{d}u\right).$$

It is easy to see that the random variable K is finite iff $\beta > 2$. Using the fact that F_{X^*} is an independent exponential random variable, by a direct conditioning argument, for all $t \geq 0$,

$$\begin{aligned}
\mathbb{P}[F_{X^*} l(||X^*||) > tK] &= \int_{\mathbb{R}^+} \mathbb{E}[e^{-tr^{\beta}K} \mid ||X^*|| = r] \exp\left(-\pi\lambda r^2\right) 2\pi\lambda r \mathrm{d}r \\
&= \exp\left(-2\pi\lambda \int_{u=r}^{\infty} \left(1 - \frac{1}{1 + tr^{\beta}u^{-\beta}}\right) u \mathrm{d}u\right) \exp\left(-\pi\lambda r^2\right) 2\pi\lambda r \mathrm{d}r \\
&= \frac{1}{1 + \rho(t, \beta)},
\end{aligned}$$

with

$$\rho(t, \beta) = t^{2/\beta} \int_{t^{-2/\beta}}^{\infty} \frac{1}{1 + u^{\beta/2}} \mathrm{d}u,$$

So the c.d.f. at t is $\rho(t, \beta)/(1 + \rho(t, \beta))$ and does not depend on λ.

Chapter 12

Complements

12.1 Strong Markov property of Poisson point process

The *strong Markov property* of Poisson point process extends the following property, which is a simple consequence of the independence property from its definition 2.1.1. Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ and consider a measurable function $f : \mathbb{M}(\mathbb{G}) \to \mathbb{R}$, then for any $B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})$

$$\mathbf{E}[f(\Phi)] = \mathbf{E}[f(\Phi|_B + \Phi'|_{B^{\mathrm{c}}})], \tag{12.1.1}$$

where Φ' is an independent copy of Φ, $B^{\mathrm{c}} = \mathbb{G}\backslash B$ is the complementary of B, and $\mu|_B$ denotes the restriction of the measure μ to B; i.e. $\mu|_B(\cdot) = \mu(\cdot \cap B)$.

The strong Markov property says that the above statement holds when B is not necessarily constant but a random stopping set with respect to Φ. This latter notion can be formalized as follows.

Definition 12.1.1. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and let $S : \mathbb{M}(\mathbb{G}) \to \mathcal{F}(\mathbb{G})$ be a measurable function ($\mathbb{M}(\mathbb{G})$ and $\mathcal{F}(\mathbb{G})$ being equipped with the σ-algebras $\mathcal{M}(\mathbb{G})$ and $\mathcal{B}(\mathcal{F}(\mathbb{G}))$ respectively) with values in $\mathcal{K}(\mathbb{G})$. Then the random compact set $S(\Phi)$ is called a* stopping set *with respect to Φ if for any $K \in \mathcal{K}(\mathbb{G})$, the event $\{S(\Phi) \subset K\}$ is $\Phi|_K$-measurable, i.e.; belongs to the σ-field generated by $\{\Phi|_K(B) : B \in \mathcal{B}_{\mathrm{c}}(\mathbb{G})\}$; that is*

$$\{S(\Phi) \subset K\} \in \sigma(\Phi|_K), \quad \forall K \in \mathcal{K}(\mathbb{G}),$$

where $\sigma(\Phi|_K)$ is the σ-algebra generated by $\Phi|_K$.

In more simple words, S is a stopping set if one can say whether the event $\{S(\Phi) \subset K\}$ holds or not knowing only the points of Φ in K.

Here is an example of a stopping set.

Example 12.1.2. n-th smallest random ball. *Let Φ be a point process on a l.c.s.h. space $\mathbb{G}$ and let $x_0 \in \mathbb{G}$ be fixed. Consider some metric on $\mathbb{G}$; cf.*

Section 1.1. Let $R_n = R_n(\Phi)$ be the distance from x_0 to the n-th closest point of Φ. Consider the random (closed) ball $\bar{B}(x_0, R_n)$. In order to prove that $\bar{B}(x_0, R_n)$ is a stopping set let us perform the following mental experiment. Given a realization of Φ and a compact set K, let us start 'growing' a ball $\bar{B}(x_0, r)$ centered at the origin increasing its radius r from x_0 until the moment when either (i) it accumulates n or more points or (ii) it hits the complement K^c of K. If (i) happens, then $\bar{B}(x_0, R_n) \subset K$. If (ii) happens, then $\bar{B}(x_0, R_n) \subset K$. In each of these cases, we have not used any information about points of Φ in K^c; so $\bar{B}(x_0, R_n)$ is a stopping set with respect to Φ.

The following result extends (12.1.1) to the case when B is a stopping set.

Theorem 12.1.3. Strong Markov property of Poisson point process. *Let Φ be a Poisson point processes on a l.c.s.h. space $\mathbb{G}$ and let $S(\Phi)$ be a stopping set with respect to Φ. Then for all measurable functions $f : \mathbb{M}(\mathbb{G}) \to \mathbb{R}_+$*

$$\mathbf{E}[f(\Phi)] = \mathbf{E}[f((\Phi|_{S(\Phi)}) \cup (\Phi'|_{S(\Phi)^c}))],$$

where Φ' is an independent copy of Φ.

Proof. The equality (12.1.1) shows that the Poisson point process Φ is a Markov stochastic process; cf. [85, Example p.67]. Therefore, by [85, Theorem 4 p.92], Φ possesses the strong Markov property with respect to all compact stopping sets. $\square$

Remark 12.1.4. *The strong Markov property of Poisson point processes on $\mathbb{R}_+$ is proved in [19, Theorem 1.1 p.370].*

Example 12.1.5. Ordering the points of a Poisson point process according to their distance. *Let Φ be a Poisson point process on a l.c.s.h. space $\mathbb{G}$ with intensity measure Λ and let $x_0 \in \mathbb{G}$ be fixed. Consider some metric on $\mathbb{G}$; cf. Section 1.1. Let $R_n = R_n(\Phi)$ be the distance from x_0 to the n-th closest point of Φ. We assume that, $\mathbf{P}$-almost surely, Φ has no two points equidistant from x_0 (which is the case for example when $\mathbb{G} = \mathbb{R}^d$ and the intensity measure Λ has a density with respect to the Lebesgue measure; cf. Exercise 3.5.4); so that the sequence $\{R_n\}_{n\geq 1}$ is strictly increasing. One can conclude from the strong Markov property of the Poisson point process that this sequence is a Markov process with transition probability*

$$\mathbf{P}\{R_n > t | R_{n-1} = s\} = \begin{cases} e^{-\Lambda(\bar{B}(x_0,t)) + \Lambda(\bar{B}(x_0,s))}, & \text{if } t > s \geq 0, \\ 1, & \text{if } 0 \leq t \leq s. \end{cases}$$

Part IV

Appendix

Chapter 13

Transforms of random variables

13.A Random variables

We will consider random variables taking their values either in $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{N}$. For each case, a specific transform will be defined (characteristic function, Laplace transform or generating function respectively) and we will give the relation between the moments of the random variable and these transforms.

13.A.1 Characteristic function

Let X be a real-valued random variable. Its *characteristic function* is the function $\Psi_X : \mathbb{R} \to \mathbb{C}$ defined by

$$\Psi_X(t) = \mathbf{E}\left[e^{\boldsymbol{i}tX}\right], \quad t \in \mathbb{R},$$

where $\boldsymbol{i}$ is the imaginary unit complex number. It is shown in [36, Theorem XV.3.1 p.508] that the characteristic function of a random variable characterizes its distribution; i.e., if two real-valued random variables X and Y have the same characteristic function, then $X \stackrel{\text{dist.}}{=} Y$.

Lemma 13.A.1. *Let X be a real-valued random variable. Assume that $\mathbf{E}[|X|^n] < \infty$ for some $n \in \mathbb{N}^*$. Then, the following results hold true.*

(i) For all $k \leq n$

$$\Psi_X^{(k)}(t) = \boldsymbol{i}^k \mathbf{E}\left[X^k e^{\boldsymbol{i}tX}\right], \quad t \in \mathbb{R} \tag{13.A.1}$$

and in particular

$$\mathbf{E}\left[X^k\right] = \frac{\Psi_X^{(k)}(0)}{\boldsymbol{i}^k}. \tag{13.A.2}$$

(ii) Moreover,

$$\Psi_X(t) = 1 + \sum_{k=1}^{n} \frac{\mathbf{E}\left[X^k\right]}{k!} (\boldsymbol{i}t)^k + \frac{\varepsilon_n(t)}{n!} t^n, \quad t \in \mathbb{R}, \tag{13.A.3}$$

where $|\varepsilon_n(t)| \le 3\mathbf{E}[|X|^n]$ *and* $\lim_{t\to 0} \varepsilon_n(t) = 0$.

Proof. (i) Assume that $\mathbf{E}[|X|^n] < \infty$. Since, for all $k \le n$, $|x|^k \le 1 + |x|^n$, $\forall x \in \mathbb{R}$ we deduce that

$$\mathbf{E}\left[|X|^k\right] \le 1 + \mathbf{E}[|X|^n], \quad \forall k \le n \tag{13.A.4}$$

(we may alternatively invoke Lyapunov's inequality $\mathbf{E}\left[|X|^k\right]^{1/k} \le \mathbf{E}[|X|^n]^{1/n}$) and therefore $\mathbf{E}\left[|X|^k\right] < \infty$, $\forall k \le n$. We will prove (13.A.1) by induction on k. By definition of the characteristic function, Equation (13.A.1) holds for $k = 0$. Assume that Equation (13.A.1) holds true for some $k \in \{0, 1, \ldots, n-1\}$. Then, for all $h \in \mathbb{R}$,

$$\frac{\Psi^{(k)}(t+h) - \Psi^{(k)}(t)}{h} = \boldsymbol{i}^k \mathbf{E}\left[X^k e^{\boldsymbol{i}tX} \frac{e^{\boldsymbol{i}hX} - 1}{h}\right].$$

Note that $\lim_{h\to 0} \frac{e^{\boldsymbol{i}hX}-1}{h} = \boldsymbol{i}X$. Moreover,

$$\left|X^k e^{\boldsymbol{i}tX} \frac{e^{\boldsymbol{i}hX} - 1}{h}\right| \le \left|X^k \frac{e^{\boldsymbol{i}hX} - 1}{h}\right| \le |X|^{k+1},$$

(which follows from $|e^{\boldsymbol{i}a} - 1|^2 = 2(1 - \cos a) \le a^2$) whose expectation is finite. It follows from the *dominated convergence theorem* [11, Theorem 16.4 p.209] that

$$\begin{aligned}
\Psi^{(k+1)}(t) &= \lim_{h\to 0} \frac{\Psi^{(k)}(t+h) - \Psi^{(k)}(t)}{h} \\
&= \boldsymbol{i}^k \mathbf{E}\left[X^k e^{\boldsymbol{i}tX} \lim_{h\to 0} \frac{e^{\boldsymbol{i}hX} - 1}{h}\right] \\
&= \boldsymbol{i}^{k+1} \mathbf{E}[X^{k+1} e^{\boldsymbol{i}tX}],
\end{aligned}$$

which completes the proof of (13.A.2) by induction. (ii) We aim now to prove (13.A.3). It follows from Taylor-Lagrange theorem, for $x \in \mathbb{R}$,

$$e^{\boldsymbol{i}x} = \cos x + \boldsymbol{i}\sin x = \sum_{k=0}^{n-1} \frac{(\boldsymbol{i}x)^k}{k!} + \frac{(\boldsymbol{i}x)^n}{n!}\left[\cos(a_x x) + \boldsymbol{i}\sin(b_x x)\right],$$

for some $a_x, b_x \in [-1, 1]$. Therefore, for given $t \in \mathbb{R}$,

$$\begin{aligned}
e^{\boldsymbol{i}tX} &= 1 + \sum_{k=1}^{n-1} \frac{(\boldsymbol{i}tX)^k}{k!} + \frac{(\boldsymbol{i}tX)^n}{n!}\left[\cos(a_X tX) + \boldsymbol{i}\sin(b_X tX)\right] \\
&= 1 + \sum_{k=1}^{n} \frac{(\boldsymbol{i}tX)^k}{k!} + \frac{(\boldsymbol{i}tX)^n}{n!}\left[\cos(a_X tX) + \boldsymbol{i}\sin(b_X tX) - 1\right].
\end{aligned}$$

Taking the expectation, we get

$$\mathbf{E}[e^{\boldsymbol{i}tX}] = 1 + \sum_{k=1}^{n} \frac{\mathbf{E}[X^k]}{k!}(\boldsymbol{i}t)^k + \frac{\varepsilon_n(t)}{n!}t^n,$$

where

$$\varepsilon_n(t) = \boldsymbol{i}^n \mathbf{E}[X^n(\cos(a_X tX) + \sin(b_X tX) - 1)].$$

Clearly $|\varepsilon_n(t)| \le 3\mathbf{E}[|X|^n]$. Moreover, observe that

$$\begin{cases} \lim_{t\to 0} X^n(\cos(a_X tX) + \sin(b_X tX) - 1) = 0, \\ |X^n(\cos(a_X tX) + \sin(b_X tX) - 1)| \le 3\,|X|^n\,, \\ \mathbf{E}\left[|X|^n\right] < \infty. \end{cases}$$

It follows from the dominated convergence theorem that $\lim_{t\to 0} \varepsilon_n(t) = 0$. □

Remark 13.A.2. Bibliographic notes. *Lemma 13.A.1 is proved in [89, Theorem 1 p. 278].*

Corollary 13.A.3. *Let X be a real-valued random variable. If $\mathbf{E}[|X^n|] < \infty$ for some $n \in \mathbb{N}^*$, then Ψ_X is C^n (i.e., n times differentiable and its n-th derivative is continuous) on $\mathbb{R}$.*

Proof. It follows from (13.A.1) that

$$\left|\Psi_X^{(k)}(t)\right| \le \mathbf{E}\left[\left|X^k\right|\right] < \infty, \quad k \in \{0, 1, \dots, n\}, t \in \mathbb{R}.$$

Then, Ψ_X is n times differentiable on $\mathbb{R}$. Using (13.A.1) and the dominated convergence theorem, it follows that $\Psi_X^{(n)}$ is continuous on $\mathbb{R}$. Then Ψ_X is C^n on $\mathbb{R}$. □

A converse of the above Corollary is formulated in the following lemma.

Lemma 13.A.4. *Let X be a real-valued random variable. If Ψ_X is $2k$ times differentiable at 0 for some $k \in \mathbb{N}^*$, then $\mathbf{E}\left[X^{2k}\right] < \infty$.*

Proof. Cf. [89, p. 278]. □

Example 13.A.5. *Let X be an exponentially distributed random variable with parameter λ. Its characteristic function is*

$$\Psi_X(t) = \frac{\lambda}{\lambda - \boldsymbol{i}t} = \left(1 - \boldsymbol{i}\frac{t}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \boldsymbol{i}^k \frac{t^k}{\lambda^k}.$$

Then, for all $k \in \mathbb{N}$, Ψ_X is $2k$ times differentiable at 0, then $\mathbf{E}\left[X^{2k}\right] < \infty$ by Lemma 13.A.4. Thus by (13.A.4), for all $n \le 2k$, $\mathbf{E}[X^n] < \infty$ and it follows from Lemma 13.A.1 that

$$\mathbf{E}[X^n] = \frac{n!}{\lambda^n}.$$

Definition 13.A.6. Real analytic function *[59, Definition 1.1.3]. A function $f:\mathbb{R}\to\mathbb{C}$ is said to be* real analytic *on $U\subset\mathbb{R}$ if it may be represented by a convergent power series in the neighborhood of any $x\in U$.*

Proposition 13.A.7. Infinite expansion of the characteristic function. *Let X be a real-valued random variable such that $\mathbf{E}\left[|X|^n\right]<\infty$ for all $n\in\mathbb{N}^*$ and let R_{Ψ_X} be the radius of convergence of the series $z\mapsto\sum_{n=1}^{\infty}\frac{\mathbf{E}[X^n]}{n!}z^n$, called the* radius of convergence *of the characteristic function. Then the following results hold true.*

(i)

$$R_{\Psi_X}=\left[e\limsup_{n\to\infty}\frac{\left|\mathbf{E}\left[X^n\right]\right|^{1/n}}{n}\right]^{-1}. \tag{13.A.5}$$

Moreover,

$$\limsup_{n\to\infty}\frac{\left|\mathbf{E}\left[X^n\right]\right|^{1/n}}{n}=\limsup_{n\to\infty}\frac{\mathbf{E}\left[|X|^n\right]^{1/n}}{n}=\limsup_{n\to\infty}\frac{\mathbf{E}\left[X^{2n}\right]^{\frac{1}{2n}}}{2n}. \tag{13.A.6}$$

(ii) If $R_{\Psi_X}>0$, then

$$\Psi_X(t)=1+\sum_{n=1}^{\infty}\frac{\mathbf{E}\left[X^n\right]}{n!}(\boldsymbol{i}t)^n,\quad t\in\mathbb{R},|t|<R_{\Psi_X}. \tag{13.A.7}$$

(iii) If $R_{\Psi_X}>0$, then Ψ_X is real analytic on $\mathbb{R}$.

(iv) If $R_{\Psi_X}>0$, then the distribution of X is characterized by its moments. (That is, if Y is a real-valued random variable such that $\mathbf{E}\left[Y^n\right]=\mathbf{E}\left[X^n\right]$ for all $n\in\mathbb{N}^*$, then $Y\stackrel{\text{dist.}}{=}X$.)

Proof. (i) By the Cauchy-Hadamard theorem [59, Lemma 1.1.6],

$$R_{\Psi_X}=\left[\limsup_{n\to\infty}\left(\frac{\left|\mathbf{E}\left[X^n\right]\right|}{n!}\right)^{1/n}\right]^{-1}.$$

Using the Stirling's equivalence formula $n!\sim n^ne^{-n}\sqrt{2\pi n}$, we get $(n!)^{1/n}\sim n/e$ which combined with the above equality proves (13.A.5). Let

$$\mu_n=\mathbf{E}\left[X^n\right],\ \text{and}\ \nu_n=\mathbf{E}\left[|X|^n\right],\quad n\in\mathbb{N}^*.$$

Since $|\mu_n|\le\nu_n$, then

$$\limsup_{n\to\infty}\frac{|\mu_n|^{\frac{1}{n}}}{n}\le\limsup_{n\to\infty}\frac{\nu_n^{\frac{1}{n}}}{n}. \tag{13.A.8}$$

On the other hand, by Lyapunov's inequality, $\mathbf{E}\left[|X|^k\right]^{1/k} \leq \mathbf{E}\left[|X|^n\right]^{1/n}$ for any $k \leq n$. Then

$$\nu_{2n-1}^{\frac{1}{2n-1}} \leq \nu_{2n}^{\frac{1}{2n}} \leq \nu_{2n+1}^{\frac{1}{2n+1}}.$$

Thus

$$\limsup_{n\to\infty} \frac{\nu_{2n}^{\frac{1}{2n}}}{2n} = \limsup_{n\to\infty} \frac{\nu_{2n+1}^{\frac{1}{2n+1}}}{2n+1} = \limsup_{n\to\infty} \frac{\nu_n^{\frac{1}{n}}}{n}.$$

Therefore,

$$\limsup_{n\to\infty} \frac{\nu_n^{\frac{1}{n}}}{n} = \limsup_{n\to\infty} \frac{\nu_{2n}^{\frac{1}{2n}}}{2n} = \limsup_{n\to\infty} \frac{\mu_{2n}^{\frac{1}{2n}}}{2n} \leq \limsup_{n\to\infty} \frac{|\mu_n|^{\frac{1}{n}}}{n}. \tag{13.A.9}$$

Combining (13.A.8) and (13.A.9) proves (13.A.6). (The above proof is from [81, p.31].) (ii) By the Cauchy-Hadamard theorem and (13.A.6), the radius of convergence of the series $z \mapsto \sum_{n=1}^{\infty} \frac{\mathbf{E}[|X|^n]}{n!} z^n$ equals R_{Ψ_X}.Then for any $t \in \mathbb{R}$ such that $|t| < R_{\Psi_X}$,

$$\lim_{n\to\infty} \frac{\mathbf{E}\left[|X|^n\right]}{n!} |t|^n = 0. \tag{13.A.10}$$

On the other hand, it follows from Lemma 13.A.1 that, for any $n \in \mathbb{N}^*$,

$$\Psi_X(t) = 1 + \sum_{k=1}^{n-1} \frac{\mathbf{E}\left[X^k\right]}{k!} (\boldsymbol{i}t)^k + \frac{\varepsilon_n(t)}{n!} t^n, \quad t \in \mathbb{R},$$

where $|\varepsilon_n(t)| \leq 3\mathbf{E}\left[|X|^n\right]$. If $|t| < R_{\Psi_X}$, then

$$\lim_{n\to\infty} \left|\frac{\varepsilon_n(t)}{n!} t^n\right| \leq 3 \lim_{n\to\infty} \frac{\mathbf{E}\left[|X|^n\right]}{n!} |t|^n = 0,$$

and expansion (13.A.7) follows. (One may also invoke [81, Theorem 3.5].) (iii) By [36], for any $x \in \mathbb{R}$,

$$\left| e^{\boldsymbol{i}x} - \sum_{k=0}^{n-1} \frac{(\boldsymbol{i}x)^k}{k!} \right| \leq \frac{|x|^n}{n!}.$$

Therefore, for any $\zeta, t \in \mathbb{R}$,

$$\left| e^{\boldsymbol{i}\zeta X} \left(e^{\boldsymbol{i}tX} - \sum_{k=0}^{n-1} \frac{(\boldsymbol{i}tX)^k}{k!} \right) \right| \leq \frac{|tX|^n}{n!}.$$

Taking the expectation and using (13.A.1), we get

$$\left| \Psi_X(\zeta + t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \Psi_X^{(k)}(\zeta) \right| \leq \frac{\mathbf{E}\left[|X|^n\right]}{n!} |t|^n$$

The above inequality together with (13.A.10) imply that, for any $\zeta \in \mathbb{R}$,

$$\Psi_X(\zeta + t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Psi_X^{(k)}(\zeta), \quad t \in \mathbb{R}, |t| < R_{\Psi_X}.$$

Thus Ψ_X is may be represented by a convergent power series in the neighborhood of ζ. This being true for any $\zeta \in \mathbb{R}$, then Ψ_X is real analytic on $\mathbb{R}$. (iv) Since Ψ_X is real analytic on $\mathbb{R}$, then, by [59, Corollary 1.2.4], Ψ_X is completely determined by $\left\{\Psi_X^{(k)}(0) : k \in \mathbb{N}^*\right\}$. But by (13.A.1), $\Psi_X^{(k)}(0) = \boldsymbol{i}^k \mathbf{E}\left[X^k\right]$ for all $k \in \mathbb{N}^*$. Thus Ψ_X is completely determined by the moments. Recalling that the characteristic function characterizes the distribution [36, Theorem XV.3.1 p.508] finishes the proof. □

Remark 13.A.8. Moment problem. *The solution in Proposition 13.A.7(iv) of the so-called* moment problem *is given in [81, Theorem 3.6] or [11, Theorem 30.1 p.388] and also in [36, p.514] in conjunction with (13.A.6).*

Corollary 13.A.9. *Let X and Y be a real-valued random variable such that $\mathbf{E}[|X|^n] < \infty$ and $\mathbf{E}[|Y|^n] < \infty$ for all $n \in \mathbb{N}^*$ and such that $R_{\Psi_X} > 0$ and $R_{\Psi_Y} > 0$. If $\Psi_X(t) = \Psi_Y(t)$ over an open subset of $\mathbb{R}$, then $X \overset{\text{dist.}}{=} Y$.*

Proof. By Proposition 13.A.7(iii), Ψ_X and Ψ_Y are real analytic on $\mathbb{R}$. Since $\Psi_X(t) = \Psi_Y(t)$ over an open subset of $\mathbb{R}$, then, by [59, Corollary 1.2.5], $\Psi_X(t) = \Psi_Y(t)$ for all $t \in \mathbb{R}$. Since the characteristic function of a random variable characterizes its distribution [36, Theorem XV.3.1 p.508], it follows that $X \overset{\text{dist.}}{=} Y$. □

Remark 13.A.10. Bibliographic notes. *Given a real-valued random variable X, the properties of the function $z \mapsto \mathbf{E}\left[e^{zX}\right]$ for complex z are studied in [81, Chapter 3].*

13.A.2 Generating function

Let X be a random variable with values in $\mathbb{N}$. Its *generating function*, denoted by $\mathcal{G}_X$, is defined by

$$\mathcal{G}_X(z) = \mathbf{E}\left[z^X\right] = \sum_{k=0}^{\infty} \mathbf{P}(X = k) z^k, \quad z \in \mathbb{C} \text{ such that } \mathbf{E}\left[|z|^X\right] < \infty. \tag{13.A.11}$$

The above power series has a radius of convergence, denoted by $R_{\mathcal{G}_X}$, satisfying

$$R_{\mathcal{G}_X} \geq 1. \tag{13.A.12}$$

This follows from the fact that $\sum_{k=0}^{\infty} \mathbf{P}(X = k) = 1$.

Definition 13.A.11. *Let X be a random variable with values in $\mathbb{N}$. The radius of convergence of the series $z \mapsto \sum_{k=0}^{\infty} \mathbf{P}(X=k)z^k$ is denoted by $R_{\mathcal{G}_X}$ and called the* radius of convergence *of the generating function of the random variable X.*

By the Cauchy-Hadamard theorem,

$$R_{\mathcal{G}_X} = \left(\limsup_{k\to\infty} \left[\mathbf{P}(X=k)\right]^{1/k} \right)^{-1}. \tag{13.A.13}$$

Example 13.A.12. Geometric random variable. *Let X be a geometric random variable with parameter p; i.e., $\mathbf{P}(X=k) = p(1-p)^k$ for any $k \in \mathbb{N}$. It follows from (13.A.13) that*

$$R_{\mathcal{G}_X} = \frac{1}{1-p}. \tag{13.A.14}$$

We introduce the notation

$$z^{(n)} := z(z-1)\dots(z-n+1), \quad z \in \mathbb{C}, n \in \mathbb{N}^*$$

called the n-th *factorial power*. The quantity $\mathbf{E}\left[X^{(n)}\right]$ is called the n-th *factorial moment* of X.

Lemma 13.A.13. *Let X be a random variable with values in $\mathbb{N}$.*

(i) For any $k \in \mathbb{N}^*$,

$$\mathbf{E}\left[X^{(k)}\right] = \lim_{x\uparrow 1} \mathcal{G}_X^{(k)}(x). \tag{13.A.15}$$

(ii) When the radius of convergence of the infinite series (13.A.11) $R_{\mathcal{G}_X} > 1$, the left hand side of the above equation equals $\mathcal{G}_X^{(k)}(1)$ which is finite; moreover

$$\mathcal{G}_X(x) = 1 + \sum_{k=1}^{\infty} \mathbf{E}\left[X^{(k)}\right] \frac{(x-1)^k}{k!} < \infty, \quad x \in \mathbb{R}, |x-1| < R_{\mathcal{G}_X} - 1, \tag{13.A.16}$$

and the series in the right-hand side converges absolutely.

Proof. Consider the series

$$\mathcal{G}_X(x) = \sum_{n=0}^{\infty} \mathbf{P}(X=n)x^n, \quad x \in \mathbb{R}, |x| < R_{\mathcal{G}_X}.$$

By [59, Proposition 1.1.8], differentiating k times the above series, we get

$$\mathcal{G}_X^{(k)}(x) = \sum_{n=k}^{\infty} \mathbf{P}(X=n) n^{(k)} x^{n-k} < \infty, \quad x \in \mathbb{R}, |x| < R_{\mathcal{G}_X}. \tag{13.A.17}$$

(i) Recall that $R_{\mathcal{G}_X} \geq 1$. Observe from (13.A.17) that $\mathcal{G}_X^{(k)}(x)$ increases with $x \in [0,1)$, then $\lim_{x\uparrow 1} \mathcal{G}_X^{(k)}(x)$ exists (possibly infinite). Therefore, by Abel's theorem [19, Theorem 1.3 p.419]

$$\lim_{x\uparrow 1} \mathcal{G}_X^{(k)}(x) = \sum_{n=0}^{\infty} \mathbf{P}(X=n) n^{(k)} = \mathbf{E}\left[X^{(k)}\right].$$

(ii) Assume that $R_{\mathcal{G}_X} > 1$. It follows from [59, Corollary 1.2.3] that $\mathcal{G}_X(x)$ is analytic at 1. Moreover, by [59, Proposition 1.2.2], $\mathcal{G}_X(x)$ may be represented by the following absolutely convergent series in the neighborhood of 1

$$\mathcal{G}_X(x) = 1 + \sum_{k=1}^{\infty} \mathcal{G}_X^{(k)}(1) \frac{(x-1)^k}{k!}, \quad x \in \mathbb{R}, |x-1| < R_{\mathcal{G}_X} - 1.$$

On the other hand, letting $x = 1$ in (13.A.17), we obtain

$$\mathcal{G}_X^{(k)}(1) = \sum_{n=k}^{\infty} \mathbf{P}(X=n) n^{(k)} = \mathbf{E}\left[X^{(k)}\right] < \infty.$$

□

Corollary 13.A.14. *Let X be a random variable with values in $\mathbb{N}$. If $R_{\mathcal{G}_X} > 2$, then*

$$\mathbf{P}(X=0) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathbf{E}\left[X^{(k)}\right],$$

and the series in the right-hand side converges absolutely.

Proof. Applying (13.A.11) with $z = 0$ gives $\mathbf{P}(X=0) = \mathcal{G}_X(0)$. If $R_{\mathcal{G}_X} > 2$, then the announced expansion follows from (13.A.16) with $x = 0$. □

Here are alternative conditions for the expansion (13.A.16) to hold true.

Lemma 13.A.15. *Let X be a random variable with values in $\mathbb{N}$. The expansion (13.A.16) holds true for all real $x \in [1, \infty)$.*

Proof. We have

$$\begin{aligned} \mathcal{G}_X(x) &= \mathbf{E}\left[(x-1+1)^X\right] \\ &= \mathbf{E}\left[1 + \sum_{k=1}^{X} \binom{X}{k} (x-1)^k\right] \\ &= \mathbf{E}\left[1 + \sum_{k=1}^{\infty} X^{(k)} \frac{(x-1)^k}{k!}\right]. \end{aligned}$$

If $x \in [1, \infty)$ the monotone convergence theorem justifies the inversion of the order of the expectation and series in the above expression which completes the proof. □

Lemma 13.A.16. *Let X be a random variable with values in $\mathbb{N}$. Let $R_{\mathcal{G}_X}$ be the radius of convergence of the infinite series (13.A.11) and $R'_{\mathcal{G}_X}$ be the radius of convergence of $z \mapsto \sum_{k=1}^{\infty} \mathbf{E}\left[X^{(k)}\right] \frac{z^k}{k!}$. Then $R_{\mathcal{G}_X} > 1$ iff $R'_{\mathcal{G}_X} > 0$. If either of these conditions holds true, then*

$$R'_{\mathcal{G}_X} = R_{\mathcal{G}_X} - 1.$$

Proof. (i) Assume that $R_{\mathcal{G}_X} > 1$. By Lemma 13.A.13(ii) for all $x \in \mathbb{R}$ such that $|x-1| < R_{\mathcal{G}_X} - 1$,

$$\mathcal{G}_X(x) = 1 + \sum_{k=1}^{\infty} \mathbf{E}\left[X^{(k)}\right] \frac{(x-1)^k}{k!} < \infty.$$

Then for all $y \in [0, R_{\mathcal{G}_X} - 1)$,

$$\sum_{k=1}^{\infty} \mathbf{E}\left[X^{(k)}\right] \frac{y^k}{k!} < \infty,$$

which shows that $R'_{\mathcal{G}_X} \geq R_{\mathcal{G}_X} - 1 > 0$. (ii) Inversely assume that $R'_{\mathcal{G}_X} > 0$. For any $x \in \left[0, R'_{\mathcal{G}_X}\right)$, taking the expectation in the identity

$$(x+1)^X = 1 + \sum_{k=1}^{\infty} X^{(k)} \frac{x^k}{k!},$$

we get

$$\mathbf{E}\left[(x+1)^X\right] = 1 + \sum_{k=1}^{\infty} \mathbf{E}\left[X^{(k)}\right] \frac{x^k}{k!} < \infty.$$

Then $R_{\mathcal{G}_X} \geq x + 1$. Thus $R_{\mathcal{G}_X} \geq R'_{\mathcal{G}_X} + 1 > 1$. (iii) Therefore $R_{\mathcal{G}_X} > 1$ iff $R'_{\mathcal{G}_X} > 0$. If either of these conditions hold, then the other one also holds true; and therefore by (i) $R'_{\mathcal{G}_X} \geq R_{\mathcal{G}_X} - 1$ and by (ii) $R_{\mathcal{G}_X} \geq R'_{\mathcal{G}_X} + 1$; thus $R'_{\mathcal{G}_X} = R_{\mathcal{G}_X} - 1$. □

Lemma 13.A.17. *Compound random variables. Let $X = \sum_{n=1}^{N} Z_n$ where N is an integer valued random variable and $Z, Z_1, Z_2, \ldots$ are i.i.d integer valued random variables independent from N, then*

$$\mathcal{G}_X(x) = \mathcal{G}_N(\mathcal{G}_Z(x)), \quad x \in [0,1]. \tag{13.A.18}$$

In particular,

$$\mathbf{E}[X] = \mathbf{E}[Z_1]\mathbf{E}[N] \tag{13.A.19}$$

called Wald's identity.

Proof. Let $S_n = \sum_{k=1}^{n} Z_k$ and $S_0 = 0$ then $X = S_N = \sum_{n=0}^{\infty} \mathbf{1}\{N = n\} S_n$. Thus for all $x \in [0,1]$

$$\begin{aligned}
\mathcal{G}_X(x) &= \mathbf{E}\left[x^{S_N}\right] \\
&= \mathbf{E}\left[x^{\sum_{n=1}^{\infty} \mathbf{1}\{N=n\} S_n}\right] \\
&= \mathbf{E}\left[\sum_{n=1}^{\infty} \mathbf{1}\{N = n\} x^{S_n}\right] \\
&= \sum_{n=1}^{\infty} \mathbf{E}\left[\mathbf{1}\{N = n\} x^{S_n}\right] \\
&= \sum_{n=1}^{\infty} \mathbf{P}(N = n)\, \mathbf{E}\left[x^{S_n}\right] \\
&= \sum_{n=1}^{\infty} \mathbf{P}(N = n)\, \mathcal{G}_Z(x)^n = \mathcal{G}_N(\mathcal{G}_Z(x)),
\end{aligned}$$

where the fourth equality follows from the monotone convergence theorem. Differentiating the above equality and letting $x \uparrow 1$ we get (13.A.19). □

13.A.3 Moments versus factorial moments

In order to get the relation between moments and factorial moments of a random variable we need the following preliminary result.

Lemma 13.A.18. *For any $z \in \mathbb{C}, n \in \mathbb{N}^*$*

$$z^{(n)} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} z^k, \tag{13.A.20}$$

$$z^n = \sum_{k=1}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} z^{(k)}, \tag{13.A.21}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ and $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}$ are Stirling numbers of the first and second kind respectively.

Proof. Cf. [1, §24.1.3 and §24.1.4]. □

Let X be a random variable with values on $\mathbb{N}$. From (13.A.20) and (13.A.21) we deduce the following relations between its moments and factorial moments

$$\mathbf{E}\left[X^{(n)}\right] = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} \mathbf{E}\left[X^k\right], \tag{13.A.22}$$

$$\mathbf{E}[X^n] = \sum_{k=1}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \mathbf{E}\left[X^{(k)}\right]. \tag{13.A.23}$$

Example 13.A.19. *Let* $m_n = \mathbf{E}[X^n]$ *and* $m_{(n)} = \mathbf{E}\left[X^{(n)}\right]$. *Then*

$$m_1 = m_{(1)},$$

$$\begin{aligned} m_2 &= m_{(1)} + m_{(2)}, \\ m_{(2)} &= -m_1 + m_2, \end{aligned}$$

$$\begin{aligned} m_3 &= m_{(1)} + 3m_{(2)} + m_{(3)}, \\ m_{(3)} &= 2m_1 - 3m_2 + m_3, \end{aligned}$$

$$\begin{aligned} m_4 &= m_{(1)} + 7m_{(2)} + 6m_{(3)} + m_{(4)}, \\ m_{(4)} &= -6m_1 + 11m_2 - 6m_3 + m_4. \end{aligned}$$

Corollary 13.A.20. *If* $\mathbf{E}[|X|^n] < \infty$, *then* $\mathbf{E}\left[|X|^{(k)}\right] < \infty$, $\forall k \leq n$.

Proof. If $\mathbf{E}[|X|^n] < \infty$, then $\mathbf{E}\left[|X|^k\right] < \infty$, $\forall k \leq n$. Then, Equation (13.A.22) shows that $\mathbf{E}\left[|X|^{(k)}\right] < \infty$, $\forall k \leq n$. □

Lemma 13.A.21. *Let* X *be a random variable with values in* $\mathbb{N}$. *If* $\mathbf{E}[X^n] < \infty$ *for some* $n \in \mathbb{N}^*$, *then*

$$\mathcal{G}_X(x) = 1 + \sum_{k=1}^{n} \mathbf{E}\left[X^{(k)}\right] \frac{(x-1)^k}{k!} + \frac{(x-1)^n}{n!} \varepsilon_n(x), \quad \forall x \in \mathbb{R}, |x| \leq 1, \tag{13.A.24}$$

where $|\varepsilon_n(x)| \leq 2\mathbf{E}\left[X^{(n)}\right]$ *and* $\lim_{x\uparrow 1} \varepsilon_n(x) = 0$. *Moreover,* $\mathcal{G}_X$ *is* C^n *on* $[0,1]$.

Proof. For $x \in \mathbb{R}$, $n, N \in \mathbb{N}$,

$$x^N = [1 + (x-1)]^N = \sum_{k=0}^{n} \frac{N^{(k)}}{k!} (x-1)^k + R_n(N, x). \tag{13.A.25}$$

By Taylor-Young theorem

$$\lim_{x \to 1} \frac{1}{(x-1)^n} R_n(N, x) = 0. \tag{13.A.26}$$

Taking the primitive of (13.A.25) with respect to x we get

$$\frac{x^{N+1} - 1}{N+1} = \sum_{k=0}^{n} \frac{N^{(k)}}{k!} \frac{(x-1)^{k+1}}{k+1} + \int_1^x R_n(N, t) \mathrm{d}t.$$

Then

$$\begin{aligned} x^{N+1} &= 1 + (N+1) \sum_{k=0}^{n} \frac{N^{(k)}}{k!} \frac{(x-1)^{k+1}}{k+1} + (N+1) \int_1^x R_n(N, t) \mathrm{d}t \\ &= \sum_{k=0}^{n+1} \frac{(N+1)^{(k)}}{k!} (x-1)^k + (N+1) \int_1^x R_n(N, t) \mathrm{d}t. \end{aligned}$$

Thus
$$R_{n+1}(N+1,x) = (N+1)\int_1^x R_n(N,t)\mathrm{d}t.$$
We prove by induction on n that
$$\mathbf{P}(n): \quad |R_n(N,x)| \le \frac{N^{(n)}}{n!}(x-1)^n, \quad \forall x \in \mathbb{R}, |x| \le 1.$$
Firstly, since $|x| \le 1$, $|R_0(N,x)| = \left|x^N - 1\right| \le \left|x^N\right| + 1 \le 2$, thus $\mathbf{P}(0)$ holds true. Assume $\mathbf{P}(n)$, then
$$\begin{aligned}|R_{n+1}(N+1,x)| &\le (N+1)\int_1^x \frac{2N^{(n)}}{n!}(t-1)^n\,\mathrm{d}t \\ &= \frac{2(N+1)^{(n+1)}}{(n+1)!}(x-1)^{n+1}.\end{aligned}$$
Thus $\mathbf{P}(n+1)$ is true. Applying (13.A.25) with $N = X$ and then taking the expectation gives
$$\mathcal{G}_X(x) = 1 + \sum_{k=1}^{n} \frac{\mathbf{E}\left[X^{(k)}\right]}{k!}(x-1)^k + \mathbf{E}\left[R_n(X,x)\right].$$
Observe that
$$|\mathbf{E}\left[R_n(X,x)\right]| \le \mathbf{E}\left[|R_n(X,x)|\right] \le \frac{2\mathbf{E}\left[X^{(n)}\right]}{n!}|x-1|^n.$$
Let
$$\varepsilon_n(x) = \frac{n!}{(x-1)^n}\mathbf{E}\left[R_n(X,x)\right],$$
then
$$\mathcal{G}_X(x) = 1 + \sum_{k=1}^{n} \frac{\mathbf{E}\left[X^{(k)}\right]}{k!}(x-1)^k + \frac{(x-1)^n}{n!}\varepsilon_n(x),$$
where $|\varepsilon_n(x)| \le 2\mathbf{E}\left[X^{(n)}\right]$. Moreover,
$$\frac{1}{(x-1)^n}R_n(X,x) \le \frac{2N^{(n)}}{n!}.$$
Then, the dominated convergence theorem and (13.A.26) imply
$$\lim_{x \uparrow 1} \varepsilon_n(x) = 0,$$
which completes the proof of (13.A.24). We already know that, $\mathcal{G}_X$ is C^∞ on $[0,1)$. and that for all $k \in \mathbb{N}$,
$$\mathbf{E}\left[X^{(k)}\right] = \lim_{x \uparrow 1} \mathcal{G}_X^{(k)}(x).$$
The expansion (13.A.24) shows that $\mathcal{G}_X$ is n times differentiable at 1^- and that
$$\mathcal{G}_X^{(n)}(1^-) = \mathbf{E}\left[X^{(k)}\right].$$
Thus $\mathcal{G}_X$ is C^n on $[0,1]$. □

13.A.4 Ordinary cumulants

Let X be a real random variable and let $\Psi_X(t) = \mathbf{E}\left[e^{\boldsymbol{i}tX}\right], t \in \mathbb{R}$ be its characteristic function. Assume that $\mathbf{E}[|X^n|] < \infty$ for some $n \in \mathbb{N}^*$. Then Ψ_X is C^n on $\mathbb{R}$. Moreover, $\Psi_X(0) = 1 > 0$, then the function

$$\zeta_X(t) = \log \Psi_X(t)$$

called the *cumulant function*, is well-defined on a neighborhood of 0. (where for all $z \in \mathbb{C}^*$, $\log z = \log|z| + \boldsymbol{i} \arg(z)$ denotes the principal value of the logarithm). Moreover, ζ_X is C^n on a neighborhood of 0 by Corollary 13.A.3. The n-th *cumulant* of X is defined by

$$c_n = \frac{1}{\boldsymbol{i}^n} \zeta_X^{(n)}(0).$$

Example 13.A.22. *Let X be an exponentially distributed random variable with parameter λ. Its characteristic function is*

$$\Psi_X(t) = \frac{\lambda}{\lambda - \boldsymbol{i}t} = \left(1 - \boldsymbol{i}\frac{t}{\lambda}\right)^{-1},$$

then its cumulant function equals

$$\zeta_X(t) = \log \Psi_X(t) = -\log\left(1 - \boldsymbol{i}\frac{t}{\lambda}\right) = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\boldsymbol{i}t}{\lambda}\right)^k.$$

Thus the cumulants are

$$c_k = \frac{(k-1)!}{\lambda^k}, \quad \forall k \in \mathbb{N}.$$

Example 13.A.23. *Let X be a Poisson random variable with parameter λ. Its characteristic function is*

$$\Psi_X(t) = \exp(\lambda(\exp(\boldsymbol{i}t) - 1)),$$

then its cumulant function is

$$\zeta_X(t) = \log \Psi_X(t) = \lambda(\exp(\boldsymbol{i}t) - 1) = \sum_{k=1}^{\infty} \frac{\lambda}{k!}(\boldsymbol{i}t)^k.$$

Thus the cumulants equal

$$c_k = \lambda, \quad \forall k \in \mathbb{N}. \tag{13.A.27}$$

Example 13.A.24. *Let X be a Gaussian random variable with mean μ and variance σ^2. Its characteristic function is*

$$\Psi_X(t) = \exp(\boldsymbol{i}\mu t - \frac{1}{2}\sigma^2 t^2).$$

Then

$$\zeta_X(t) = \boldsymbol{i}\mu t - \frac{1}{2}\sigma^2 t^2.$$

Thus, $c_1 = \mu$, c_2, $c_k = 0$, $\forall k \geq 3$.

Example 13.A.25. *Let X be a random variable with the Gamma distribution with probability density function*

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbf{1}_{\{x>0\}}.$$

Its characteristic function is

$$\Psi_X(t) = \left(1 - \boldsymbol{i}\frac{t}{\beta}\right)^{-\alpha}.$$

Then,

$$\zeta_X(t) = -\alpha \log\left(1 - \boldsymbol{i}\frac{t}{\beta}\right) = \alpha \sum_{k=1}^{\infty} \frac{1}{k}\left(\boldsymbol{i}\frac{t}{\beta}\right)^k.$$

Thus

$$c_n = \alpha \frac{(n-1)!}{\beta^n}.$$

In particular, with $\alpha = 1$ and $\beta = \lambda$ we retrieve the result for an exponentially distributed random variable with parameter λ given in Example 13.A.22.

Remark 13.A.26. *Assume that $m_n = \mathbf{E}[X^n] < \infty$ for some $n \in \mathbb{N}^*$. Then, by Lemma 13.A.1*

$$\Psi_X(t) = 1 + \sum_{k=1}^{n} m_k \frac{(\boldsymbol{i}t)^k}{k!} + o(t^n), \quad t \in \mathbb{R}_+.$$

Moreover, ζ_X is C^n on $\mathbb{R}_+$, then, By Taylor-Young theorem it admits an expansion

$$\zeta_X(t) = \sum_{k=1}^{n} c_k \frac{(\boldsymbol{i}t)^k}{k!} + o(t^n), \quad t \in \mathbb{R}_+.$$

Since $\zeta_X(t) = \log \Psi_X(t)$ and $\log(1-x) = -\sum_{k=1}^{n} \frac{x^k}{k} + o(x^n)$, it follows that c_k are universal (i.e., independent of X) polynomial of $m_1 \ldots m_k$. Inversely, it follows from the expansion $e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$ that m_k is a universal polynomial of $c_1 \ldots c_k$. We will give explicit expressions of these polynomials in Section 13.A.5.

13.A.5 Factorial cumulants

Let X be a random variable with values in $\mathbb{N}$ and let $\mathcal{G}_X(x)$ be its generating function. Note that $\mathcal{G}_X(x) > 0$, $\forall x \in \mathbb{R}_+^*$, then the function

$$\mathfrak{F}_X(x) = \log \mathcal{G}_X(x)$$

is well-defined on $(0, R_{\mathcal{G}_X}]$ where $R_{\mathcal{G}_X} \geq 1$ is the radius of convergence of the series $\sum_{k=0}^{\infty} \mathbf{P}(X = k) z^k$. The function $\mathfrak{F}_X$, called the *factorial cumulant function*, is C^∞ on $(0, R_{\mathcal{G}_X})$. The n-th *factorial cumulant* of X is defined by

$$c_{(n)} = \lim_{x \uparrow 1} \mathfrak{F}_X^{(n)}(x),$$

whenever the limit exists. When $R_{\mathcal{G}_X} > 1$, the right-hand side of the above equation equals $\mathfrak{F}_X^{(n)}(1)$ which is finite.

Remark 13.A.27. *Assume that $m_n = \mathbf{E}[X^n] < \infty$. Then by Corollary 13.A.20, $m_{(k)} < \infty$, $\forall k \leq n$. It follows from Lemma 13.A.21 that*

$$\mathcal{G}_X(x) = 1 + \sum_{k=1}^{n} m_{(k)} \frac{(x-1)^k}{k!} + o(x-1)^n, \quad \forall x \in \mathbb{R}, |x| \leq 1.$$

Then, $\mathcal{G}_X$ is C^n on $(0,1]$, then so is $\mathfrak{F}_X$ which by Taylor-Young theorem admits an expansion

$$\mathfrak{F}_X(x) = \sum_{k=1}^{n} c_{(k)} \frac{(x-1)^k}{k!} + o(x-1)^n, \quad x \in (0,1].$$

Since $\mathfrak{F}_X(x) = \log \mathcal{G}_X(x)$, it follows that, $c_{(k)}$ is a universal polynomial of $m_{(1)}, \ldots, m_{(k)}$ (the same as the polynomial expressing the cumulant c_k as function of the moments $m_1, \ldots, m_k$; cf. Remark 13.A.26). Analogously $m_{(k)}$ is a universal polynomial of $c_{(1)}, \ldots, c_{(k)}$ (the same as the polynomial expressing m_k as function of $c_1, \ldots, c_k$).

Example 13.A.28. *Let X be a Poisson random variable with parameter λ. Its generating function is*

$$\mathcal{G}_X(z) = e^{\lambda(z-1)} = \sum_{k=0}^{\infty} \lambda^k \frac{(z-1)^k}{k!}.$$

Then $m_{(k)} = \lambda^k$. Moreover,

$$\mathfrak{F}_X(x) = \log \mathcal{G}_X(x) = \lambda(x-1).$$

Thus

$$c_{(1)} = \lambda, \quad \text{and } c_{(k)} = 0, \quad k \geq 2. \tag{13.A.28}$$

(Recall that the ordinary cumulant of the Gaussian random variable vanish for $k \geq 3$; cf. Example 13.A.24.)

Example 13.A.29. Negative Binomial distribution. *Let $X = \sum_{n=1}^{N} Z_n$ where N is a Poisson random variable of parameter $\lambda \in \mathbb{R}_+^*$ and $Z, Z_1, Z_2, \ldots$ are i.i.d.*

random variables independent from N having the logarithmic distribution with parameter $\rho \in (0,1)$; that is

$$\mathbf{P}(Z = n) = -\log(1-\rho)\frac{\rho^n}{n}, \quad n \in \mathbb{N}^*.$$

Then

$$\mathcal{G}_N(x) = e^{\lambda(x-1)}$$

and

$$\mathcal{G}_Z(x) = \frac{\log(1-\rho x)}{\log(1-\rho)}, \quad x < \frac{1}{\rho}.$$

Thus by Lemma 13.A.17

$$\begin{aligned}\mathcal{G}_X(x) &= \mathcal{G}_N(\mathcal{G}_Z(x)) \\ &= \exp\left[\lambda\left(\frac{\log(1-\rho x)}{\log(1-\rho)} - 1\right)\right] \\ &= \exp\left[\alpha\left(-\log(1-\rho x) + \log(1-\rho)\right)\right] \\ &= \left(\frac{1-\rho}{1-\rho x}\right)^{\alpha},\end{aligned}$$

where $\alpha = -\frac{\lambda}{\log(1-\rho)}$. Thus X has the negative Binomial distribution *with parameters α and ρ.*

Using the expansion

$$(1+u)^{\beta} = \sum_{k=0}^{\infty} \beta^{(k)}\frac{u^k}{k!},$$

we get

$$\begin{aligned}\mathcal{G}_X(x) &= \left[1 - \frac{\rho}{1-\rho}(x-1)\right]^{-\alpha} \qquad (13.A.29) \\ &= \sum_{k=0}^{\infty} (-\alpha)^{(k)}\left(\frac{-\rho}{1-\rho}\right)^k \frac{(x-1)^k}{k!} \\ &= \sum_{k=0}^{\infty} (\alpha+k-1)^{(k)}\left(\frac{\rho}{1-\rho}\right)^k \frac{(x-1)^k}{k!},\end{aligned}$$

which is valid for any real x such that $\left|\frac{\rho}{1-\rho}(x-1)\right| < 1$; which is equivalent to $\frac{2\rho-1}{\rho} < x < \frac{1}{\rho}$. It follows from Lemma 13.A.13 that the factorial moments of X are

$$\mathbf{E}\left[X^{(k)}\right] = (\alpha+k-1)^{(k)}\left(\frac{\rho}{1-\rho}\right)^k.$$

(In particular, $\mathbf{E}[X] = \alpha\frac{\rho}{1-\rho} = \lambda\left(-\frac{1}{\log(1-\rho)}\frac{\rho}{1-\rho}\right)$ which equals $\mathbf{E}[N]\,\mathbf{E}[Z]$ as expected from Wald's identity (13.A.19).)

On the other hand,

$$\begin{aligned}\log \mathcal{G}_X(x) &= \alpha \log\left(\frac{1-\rho}{1-\rho x}\right)\\ &= -\alpha \log\left[1-\frac{\rho}{1-\rho}(x-1)\right]\\ &= \alpha \sum_{k=0}^{\infty}\left(\frac{\rho}{1-\rho}\right)^k \frac{(x-1)^k}{k}.\end{aligned}$$

It follows that the factorial cumulant moments of X are

$$c_{(k)} = \alpha (k-1)!\left(\frac{\rho}{1-\rho}\right)^k. \tag{13.A.30}$$

Cumulants versus moments The relations between the (ordinary or factorial) cumulant and moments are given the following lemma. Let for all $q \in \mathbb{N}^*, \lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{N}^q$,

$$\begin{aligned}|\lambda| &:= \lambda_1 + \ldots + \lambda_q,\\ \lambda! &:= \lambda_1! \ldots \lambda_q!.\end{aligned}$$

Lemma 13.A.30. *Let $n \in \mathbb{N}^*$ and let $g : \mathbb{R} \to \mathbb{C}$ be n times differentiable at 0 such that $g(0) = 0$. Then $f(t) = e^{g(t)}$ is n times differentiable at 0. Let*

$$m_k = f^{(k)}(0), \quad c_k = g^{(k)}(0), \quad k \in \{1, \ldots, n\}.$$

Then the following results hold for all $k \in \{1, \ldots, n\}$.

(i)

$$\begin{aligned}m_k &= \sum_{q=1}^{k} \frac{1}{q!} \sum_{\lambda \in (\mathbb{N}^*)^q : |\lambda| = k} \frac{k!}{\lambda!} \prod_{p=1}^{q} c_{\lambda_p} && (13.A.31)\\ &= \sum_{q=1}^{k} \sum_{\nu = \{\nu_1, \ldots, \nu_q\}} \prod_{p=1}^{q} c_{|\nu_p|}, && (13.A.32)\end{aligned}$$

where the last summation is over all partitions $\nu = \{\nu_1, \ldots, \nu_q\}$ of $\{1, \ldots, k\}$, and $|\nu_p|$ denotes the cardinality of ν_p.

$$\begin{aligned}c_k &= \sum_{q=1}^{k} \frac{(-1)^{q-1}}{q} \sum_{\lambda \in (\mathbb{N}^*)^q : |\lambda| = k} \frac{k!}{\lambda!} \prod_{p=1}^{q} m_{\lambda_p} && (13.A.33)\\ &= \sum_{q=1}^{k} (-1)^{q-1}(q-1)! \sum_{\nu = \{\nu_1, \ldots, \nu_q\}} \prod_{p=1}^{q} m_{|\nu_p|}, && (13.A.34)\end{aligned}$$

where the last summation is again over all partitions $\nu = \{\nu_1, \ldots, \nu_q\}$ of $\{1, \ldots, k\}$, and $|\nu_p|$ denotes the cardinality of ν_p.

(ii)

$$m_k = \det \begin{pmatrix} b_1 & k-1 & 0 & \cdots & 0 \\ b_2 & b_1 & k-2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{k-1} & b_{k-2} & b_{k-3} & \cdots & 1 \\ b_k & b_{k-1} & b_{k-2} & \cdots & b_1 \end{pmatrix}, \tag{13.A.35}$$

where $b_j = \frac{(-1)^{j-1}}{(j-1)!} c_j$, $j \in \{1, \ldots, n\}$.

Proof. Since the functions f and g are n times differentiable at 0, then they admit the following Taylor-Young expansions

$$f(t) = 1 + \sum_{k=1}^{n} \frac{m_k}{k!} t^k + o(t^n), \quad t \in \mathbb{R},$$

$$g(t) = \sum_{k=1}^{n} \frac{c_k}{k!} t^k + o(t^n), \quad t \in \mathbb{R}.$$

(i) Combining the Taylor-Young expansion of g and

$$e^x = 1 + \sum_{q=1}^{n} \frac{x^q}{q!} + o(x^n), \quad x \in \mathbb{R},$$

we get

$$\begin{aligned} f(t) &= 1 + \sum_{q=1}^{n} \frac{1}{q!} \left(\sum_{k=1}^{n} \frac{c_k}{k!} t^k \right)^q + o(t^n) \\ &= 1 + \sum_{q=1}^{n} \frac{1}{q!} \sum_{\lambda_1, \ldots, \lambda_q = 1}^{n} \prod_{p=1}^{q} \left(\frac{c_{\lambda_p}}{\lambda_p!} t^{\lambda_p} \right) + o(t^n) \\ &= 1 + \sum_{k=1}^{n} \left[\sum_{q=1}^{k} \frac{1}{q!} \sum_{\lambda \in (\mathbb{N}^*)^q : |\lambda| = k} \left(\prod_{p=1}^{q} \frac{c_{\lambda_p}}{\lambda_p!} \right) \right] t^k + o(t^n), \end{aligned}$$

which proves (13.A.31). On the other hand, combining the Taylor-Young of f and

$$\log(1+x) = \sum_{q=1}^{n} \frac{(-1)^{q-1}}{q} x^q + o(x^n), \quad x \in \mathbb{R} : x > -1,$$

we obtain

$$\begin{aligned}
g(t) &= 1 + \sum_{q=1}^{n} \frac{(-1)^{q-1}}{q} \left(\sum_{k=1}^{n} m_k \frac{t^k}{k!} \right)^q + o(t^n) \\
&= 1 + \sum_{q=1}^{n} \frac{(-1)^{q-1}}{q} \sum_{\lambda_1, \ldots, \lambda_q = 1}^{n} \prod_{p=1}^{q} \left(m_{\lambda_p} \frac{t^{\lambda_p}}{\lambda_p!} \right) + o(t^n) \\
&= 1 + \sum_{k=1}^{n} \left[\sum_{q=1}^{k} \frac{(-1)^{q-1}}{q} \sum_{\lambda \in (\mathbb{N}^*)^q : |\lambda| = k} \prod_{p=1}^{q} \left(\frac{c_{\lambda_p}}{\lambda_p!} \right) \right] t^k + o(t^n),
\end{aligned}$$

which proves (13.A.33). Recall that, for any integers $k, \lambda_1, \ldots, \lambda_q$ such that $\lambda_1 + \cdots + \lambda_q = k$, the multinomial

$$\frac{k!}{\lambda_1! \ldots \lambda_q!}$$

is the number of ordered partitions of $\{1, \ldots, k\}$ into q sets of respective cardinals $\lambda_1, \ldots, \lambda_q$. Then the number of unordered partitions of $\{1, \ldots, k\}$ of cardinals $\{\lambda_1, \ldots, \lambda_q\}$ equals

$$\frac{k!}{q! \lambda_1! \ldots \lambda_q!}.$$

Equations (13.A.32) and (13.A.32) then follow. (ii) Since the functions f' and g' are $n-1$ times differentiable at 0, then they admit the following Taylor-Young expansions

$$f'(t) = \sum_{k=1}^{n} \frac{m_k}{(k-1)!} t^{k-1} + o\left(t^{n-1}\right), \quad t \in \mathbb{R},$$

$$g'(t) = \sum_{k=1}^{n} \frac{c_k}{(k-1)!} t^{k-1} + o(t^n), \quad t \in \mathbb{R}.$$

On the other hand, since $f(t) = e^{g(t)}$, then $f'(t) = g'(t) f(t)$. Thus for all $k \in \{1, \ldots, n\}$,

$$\frac{m_k}{(k-1)!} = \sum_{j=1}^{k} \frac{m_{k-j}}{(k-j)!} \frac{c_j}{(j-1)!},$$

which imples

$$\begin{aligned}
m_k &= \sum_{j=1}^{k} m_{k-j} \frac{c_j}{(j-1)!} \frac{(k-1)!}{(k-j)!} \\
&= \sum_{j=1}^{k} m_{k-j} b_j (-1)^{j-1} \frac{(k-1)!}{(k-j)!},
\end{aligned}$$

where b_j is as in the statement. We now prove (13.A.35) by induction. Since $g(0) = 0$, then $f(0) = m_0 = 1$, thus (13.A.35) holds for $k = 1$. Assume that (13.A.35) holds for $m_1, \dots, m_{k-1}$, then developping the determinant in the right-hand side of (13.A.35) in minors with respect to the first column, we get

$$\det \begin{pmatrix} b_1 & k-1 & 0 & \cdots & 0 \\ b_2 & b_1 & k-2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{k-1} & b_{k-2} & b_{k-3} & \cdots & 1 \\ b_k & b_{k-1} & b_{k-2} & \cdots & b_1 \end{pmatrix} = \sum_{j=1}^{k} b_j \left(-1\right)^{j-1} M_j,$$

where M_j is the jth minor; that is

$$\begin{aligned} M_j &= \det \begin{pmatrix} k-1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b_1 & k-2 & & 0 & 0 & 0 & & 0 \\ \vdots & & & & \vdots & \vdots & & \vdots \\ b_{j-2} & b_{j-3} & & k-j+1 & 0 & 0 & \cdots & 0 \\ b_j & b_{j-1} & & b_2 & b_1 & k-j-1 & \ddots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & \\ b_{k-2} & b_{k-3} & & b_{k-j} & b_{k-j-1} & b_{k-j-2} & \ddots & 1 \\ b_{k-1} & b_{k-2} & & b_{k-j+1} & b_{k-j} & b_{k-j-1} & \cdots & b_1 \end{pmatrix} \\ &= \left(k-1\right)\left(k-2\right)\dots\left(k-j+1\right) m_{k-j} \\ &= m_{k-j} \frac{(k-1)!}{(k-j)!}. \end{aligned}$$

Combining the above three displays shows that (13.A.35) holds for m_k. □

Remark 13.A.31. Bibliographic notes. *Lemma 13.A.30(ii) is from [83, Lemma 7 p.332].*

Example 13.A.32. *Let X be a random variable and denote by $m_n = \mathbf{E}\left[X^n\right]$ and c_n be its moments and cumulants respectively. Then*

$$m_1 = c_1,$$

$$\begin{aligned} m_2 &= 2!\left(\frac{c_2}{2} + \frac{1}{2!}c_1^2\right) \\ &= c_2 + c_1^2, \end{aligned}$$

$$\begin{aligned} m_3 &= 3!\left(\frac{c_3}{3!} + \frac{1}{2!}\left(\frac{c_2}{2}c_1 + c_1\frac{c_2}{2}\right) + \frac{1}{3!}c_1^3\right) \\ &= c_3 + 3c_2c_1 + c_1^3. \end{aligned}$$

Example 13.A.33. *Let X be a random variable and denote by $m_n = \mathbf{E}[X^n]$ and c_n be its moments and cumulants respectively. Then*

$$c_1 = m_1,$$

$$\begin{aligned} c_2 &= m_2 - m_1^2, \\ m_2 &= c_2 + c_1^2, \end{aligned}$$

$$\begin{aligned} c_3 &= m_3 - 3m_2 m_1 + 2m_1^3, \\ m_3 &= c_3 + 3c_2 c_1 + c_1^3, \end{aligned}$$

$$\begin{aligned} c_4 &= \overbrace{m_4}^{q=1} \overbrace{-3m_2^2 - 4m_1 m_3}^{q=2} + 2 \times \overbrace{6m_1^2 m_2}^{q=3} - 6\overbrace{m_1^4}^{q=4}, \\ m_4 &= c_4 + 3c_2^2 + 4c_1 c_3 + 6c_1^2 c_2 + c_1^4, \end{aligned}$$

which are consistent with results of Example 13.A.32. Denote by $m_n' = \mathbf{E}[(X - m_1)^n]$ the centered moments of X. Note that

$$c_2 = m_2', \quad c_3 = m_3', \quad c_4 = m_4' - 3m_2'^2.$$

Example 13.A.34. *Denote by $m_n' = \mathbf{E}[(X - m_1)^n]$ the centered moments of X. Combining the results of Examples 13.A.19 and 13.A.33, we get*

$$c_{(1)} = m_{(1)} = m_1 = c_1,$$

$$\begin{aligned} c_{(2)} &= m_{(2)} - m_{(1)}^2 \\ &= -m_1 + m_2 - m_1^2 = m_2' - m_1 \\ &= -c_1 + c_2 + c_1^2 - c_1^2 = c_2 - c_1, \end{aligned} \tag{13.A.36}$$

$$\begin{aligned} c_{(3)} &= m_{(3)} - 3m_{(2)} m_{(1)} + 2m_{(1)}^3 \\ &= 2m_1 - 3m_2 + m_3 - 3(-m_1 + m_2) m_1 + 2m_1^3 = m_3' - 3m_2' + 2m_1 \\ &= c_3 - 3c_2 + 2c_1. \end{aligned}$$

13.B Nonnegative random variables

13.B.1 Laplace transform

For a nonnegative random variable X, the *Laplace transform*, denoted by $\mathcal{L}_X$, is defined as

$$\mathcal{L}_X(t) = \mathbf{E}\left[e^{-tX}\right], \quad t \in \mathbb{R}_+$$

It is shown in [36, Theorem XIII.1.1 p.430] that the Laplace transform of a random variable characterizes its distribution; i.e., if two nonnegative random variables X and Y have the same Laplace transform, then $X \stackrel{\text{dist.}}{=} Y$.

The Laplace transform of a random variable may be used in place of the characteristic function to get its moments as illustrated in the following lemma.

Lemma 13.B.1. *Let X be a nonnegative random variable. Then the following results hold true.*

(i) For any $k \in \mathbb{N}^*$, the k-th order derivative of the Laplace transform equals

$$\mathcal{L}_X^{(k)}(t) = (-1)^k \mathbf{E}\left[X^k e^{-tX}\right], \quad t \in \mathbb{R}_+^*. \tag{13.B.1}$$

Moreover, the k-th moment of X equals

$$\mathbf{E}\left[X^k\right] = (-1)^k \lim_{t \downarrow 0} \mathcal{L}_X^{(k)}(t). \tag{13.B.2}$$

In particular, $\mathbf{E}\left[X^k\right] < \infty$ iff $\lim_{t\downarrow 0} \mathcal{L}_X^{(k)}(t) < \infty$.

(ii) If $\mathbf{E}[X^n] < \infty$ for some $n \in \mathbb{N}^*$, then

$$\mathcal{L}_X(t) = 1 + \sum_{k=1}^{n} (-1)^k \mathbf{E}\left[X^k\right] \frac{t^k}{k!} + \frac{t^n}{n!}\varepsilon_n(t), \quad t \in \mathbb{R}_+, \tag{13.B.3}$$

where $|\varepsilon_n(t)| \leq \mathbf{E}[X^n]$ and $\lim_{t\downarrow 0} \varepsilon_n(t) = 0$.

Proof. (i) We will prove (13.B.1) by induction on k. By definition of Laplace transform, Equation (13.B.1) holds for $k = 0$. Assume that Equation (13.B.1) holds true for some $k \in \mathbb{N}$, then for all $h \in \mathbb{R}^*$ such that $|h| < t$

$$\frac{\mathcal{L}_X^{(k)}(t+h) - \mathcal{L}_X^{(k)}(t)}{h} = (-1)^{k+1} \mathbf{E}\left[X^k e^{-tX} \frac{1 - e^{-hX}}{h}\right].$$

For fixed $x \in \mathbb{R}_+$, let $\varphi(h) = \frac{1-e^{-xh}}{h}$ for $h \in \mathbb{R}^*$ and $\varphi(0) = x$. Then,

$$\varphi'(h) = \frac{(1 + xh)\, e^{-xh} - 1}{h^2}.$$

If $h > -\frac{1}{x}$ then $xh + 1 > 0$, moreover, $\log(xh + 1) \leq xh \Rightarrow -\log(xh + 1) \geq -xh \Rightarrow \frac{1}{xh+1} \geq e^{-xh} \Rightarrow \varphi'(h) \leq 0$. On the other hand, if $h < -\frac{1}{x}$ then $xh + 1 < 0 \Rightarrow \varphi'(h) \leq 0$. Thus φ is nonincreasing on $\mathbb{R}$. Then it follows from the monotone convergence theorem that

$$\lim_{h \downarrow 0} \mathbf{E}\left[X^k e^{-tX} \frac{1 - e^{-hX}}{h}\right] = \mathbf{E}\left[X^{k+1} e^{-tX}\right].$$

Thus

$$\mathcal{L}_X^{(k+1)}(t) = (-1)^{k+1} \mathbf{E}\left[X^{k+1} e^{-tX}\right].$$

Thus Equation (13.B.1) holds true for $k+1$ which completes the proof of (13.B.1) by induction. Using (13.B.1) and the monotone convergence theorem we get (13.B.2).
(ii) We aim now to prove (13.B.3). Assume that $\mathbf{E}[X^n] < \infty$. Since $|x|^k \leq 1 + |x|^n$, $\forall x \in \mathbb{R}$ we deduce that $\mathbf{E}\left[X^k\right] < \infty$, $\forall k \leq n$. Let $f(t) = e^{-tX}$, then

$f^{(k)}(t) = (-X)^k e^{-tx}$. By Taylor-Lagrange theorem, for any , there exists some $\xi_t \in [0,t]$ such that

$$\begin{aligned} e^{-tX} &= 1 + \sum_{k=1}^{n-1} (-X)^k \frac{t^k}{k!} + f^{(n)}(\xi_t)\frac{t^n}{n!} \\ &= 1 + \sum_{k=1}^{n} (-X)^k \frac{t^k}{k!} + (-X)^n \left(e^{-\xi_t X} - 1\right) \frac{t^n}{n!}. \end{aligned}$$

Taking the expectation of both sides of the above equality, we get

$$\mathcal{L}_X(t) = 1 + \sum_{k=1}^{n} (-1)^k \mathbf{E}\left[X^k\right] \frac{t^k}{k!} + \varepsilon_n(t) \frac{t^n}{n!},$$

where $\varepsilon_n(t) = \mathbf{E}\left[(-X)^n \left(e^{-\xi_t X} - 1\right)\right]$. Observe that

$$|\varepsilon_n(t)| \leq \mathbf{E}\left[\left|(-X)^n \left(e^{-\xi_t X} - 1\right)\right|\right] = \mathbf{E}\left[X^n \left(1 - e^{-\xi_t X}\right)\right] \leq \mathbf{E}\left[X^n\right].$$

Further, since

$$\begin{cases} \lim_{t\downarrow 0}(-X)^n \left(e^{-\xi_t X} - 1\right) = 0, \\ \left|(-X)^n \left(\exp(-\xi_t X) - 1\right)\right| \leq X^n, \\ \mathbf{E}\left[X^n\right] < \infty, \end{cases}$$

then by the dominated convergence theorem $\lim_{t\downarrow 0} \varepsilon_n(t) = 0$. □

Corollary 13.B.2. *Let X be a nonnegative random variable.*

(i) $\mathcal{L}_X$ is C^∞ (i.e., $\mathcal{L}_X$ is infinitely differentiable) on $\mathbb{R}_+^*$.

(ii) For any $n \in \mathbb{N}^*$, $\mathbf{E}[X^n] < \infty$ iff $\mathcal{L}_X$ is C^n (i.e., $\mathcal{L}_X$ is n times differentiable with $\mathcal{L}_X^{(n)}$ continuous) on $\mathbb{R}_+$. If either of these conditions holds, then

$$\mathbf{E}\left[X^k\right] = (-1)^k \mathcal{L}_X^{(k)}(0), \quad k \leq n.$$

Proof. (i) For given $k \in \mathbb{N}$ and $t \in \mathbb{R}_+^*$ let $f(x) = x^k e^{-tx}$, $x \in \mathbb{R}_+$. Clearly, f is continuous and $\lim_{x\downarrow 0} f(x) = \mathbf{1}_{\{k=0\}}$, $\lim_{x\to\infty} f(x) = 0$. Then, f is bounded, that is there exists some $M \in \mathbb{R}_+^*$ such that $f(x) \leq M$, $\forall x \in \mathbb{R}_+$. Thus,

$$X^k e^{-tX} \leq M \Rightarrow \mathbf{E}\left[X^k e^{-tX}\right] \leq M \leq \infty.$$

We deduce from (13.B.1) that $\left|\mathcal{L}_X^{(k)}(t)\right| < \infty$. This being true for all $k \in \mathbb{N}$ and $t \in \mathbb{R}_+^*$, it follows that $\mathcal{L}_X$ is C^∞ on $\mathbb{R}_+^*$. (ii) *Necessity.* Assume that $\mathbf{E}[X^n] < \infty$. It follows from (13.B.3) that

$$\mathcal{L}_X^{(k)}(0) = (-1)^k \mathbf{E}\left[X^k\right], \quad k \leq n,$$

which combined with (13.B.2) shows that

$$\lim_{t\downarrow 0} \mathcal{L}_X^{(k)}(t) = \mathcal{L}_X^{(k)}(0), \quad k \leq n.$$

Then, $\mathcal{L}_X$ is C^n at 0. Therefore $\mathcal{L}_X$ is C^n on $\mathbb{R}_+$. *Sufficiency.* Assume that $\mathcal{L}_X$ is C^n on $\mathbb{R}_+$. Then by (13.B.2)

$$\mathbf{E}\left[X^n\right] = (-1)^n \lim_{t\downarrow 0} \mathcal{L}_X^{(n)}(t) = \mathcal{L}_X^{(n)}(0) < \infty.$$

□

Definition 13.B.3. *Let X be a nonnegative random variable such that $\mathbf{E}\left[X^n\right] < \infty$ for all $n \in \mathbb{N}^*$. The radius of convergence of the series $z \mapsto \sum_{n=1}^{\infty} \frac{\mathbf{E}[X^n]}{n!} z^n$ is denoted by $R_{\mathcal{L}_X}$ and called the* radius of convergence of the Laplace transform *of X.*

Note that for any nonnegative random variable X, $R_{\mathcal{L}_X} = R_{\Psi_X}$ the radius of convergence of the characteristic function.

Proposition 13.B.4. Infinite expansion of the Laplace transform. *Let X be a nonnegative random variable such that $\mathbf{E}\left[X^n\right] < \infty$ for all $n \in \mathbb{N}^*$ and let $R_{\mathcal{L}_X}$ be the radius of convergence of the series $z \mapsto \sum_{n=1}^{\infty} \frac{\mathbf{E}[X^n]}{n!} z^n$. Then the following results hold true.*

(i)

$$R_{\mathcal{L}_X} = \left[e \limsup_{n\to\infty} \frac{\left|\mathbf{E}\left[X^n\right]\right|^{1/n}}{n}\right]^{-1}.$$

(ii) If $R_{\mathcal{L}_X} > 0$, then

$$\mathcal{L}_X(t) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\mathbf{E}\left[X^n\right]}{n!} t^n < \infty, \quad t \in [0, R_{\mathcal{L}_X}).$$

(iii) Assume moreover that X is integer-valued. Then $R_{\mathcal{L}_X} > 0$ iff $R_{\mathcal{G}_X} > 1$. If either of these conditions holds true, then

$$R_{\mathcal{L}_X} = \log\left(R_{\mathcal{G}_X}\right).$$

Proof. (i) This follows from Proposition 13.A.7(i) and the fact that, in the case of a nonnegative random variable, $R_{\mathcal{L}_X}$ equals R_{Ψ_X}. (ii) The proof follows in the same lines as that of Proposition 13.A.7(ii). It relies on Lemma 13.B.1(ii). (It may also be deduced from [81, Theorem 3.5].) (iii) For any $t \in \mathbb{R}_+$,

$$e^{tX} = 1 + \sum_{n=1}^{\infty} \frac{X^n}{n!} t^n.$$

Taking the expectation and invoking the monotone convergence theorem, we get

$$\mathbf{E}\left[e^{tX}\right] = 1 + \sum_{n=1}^{\infty} \frac{\mathbf{E}\left[X^n\right]}{n!} t^n. \tag{13.B.4}$$

Step 1. Assume that $R_{\mathcal{G}_X} > 1$. If $t \in [0, \log(R_{\mathcal{G}_X}))$, then $\mathbf{E}\left[e^{tX}\right] = \mathcal{G}_X(e^t) < \infty$, thus the series in the right-hand side of (13.B.4) converges. This being true for any $t \in [0, \log(R_{\mathcal{G}_X}))$, it follows that

$$R_{\mathcal{L}_X} \geq \log(R_{\mathcal{G}_X}) > 0.$$

Step 2. Assume that $R_{\mathcal{L}_X} > 0$. If $x \in (1, \exp(R_{\mathcal{L}_X}))$, then $\sum_{n=1}^{\infty} \frac{\mathbf{E}[X^n]}{n!} (\log x)^n < \infty$. Thus, by (13.B.4), $\mathbf{E}\left[e^{X \log x}\right] < \infty$ and therefore, $\mathcal{G}_X(x) = \sum_{n=0}^{\infty} \mathbf{P}(X = n)x^n < \infty$. This being true for any $x \in (1, \exp(R_{\mathcal{L}_X}))$, it follows that

$$R_{\mathcal{G}_X} \geq \exp(R_{\mathcal{L}_X}) > 1.$$

Step 3. Therefore $R_{\mathcal{G}_X} > 1$ iff $R_{\mathcal{L}_X} > 0$. If either of these conditions hold, then the other one also holds true; and therefore $R_{\mathcal{L}_X} \geq \log(R_{\mathcal{G}_X})$ and $R_{\mathcal{G}_X} \geq \exp(R_{\mathcal{L}_X})$; thus $R_{\mathcal{L}_X} = \log(R_{\mathcal{G}_X})$. □

Note that $\mathcal{L}_X(t) = \mathbf{E}\left[e^{-tX}\right]$ may be defined for any $t \in \mathbb{R}$ such that $\mathbf{E}\left[\left|e^{-tX}\right|\right] < \infty$.

Corollary 13.B.5. *Let X be a nonnegative random variable such that $\mathbf{E}[X^n] < \infty$ for all $n \in \mathbb{N}^*$ and such that $R_{\mathcal{L}_X} > 0$. Let Y be a nonnegative random variable such that $\mathcal{L}_Y(t) = \mathcal{L}_X(t)$ over an interval $[0, \varepsilon)$ for some $\varepsilon \in \mathbb{R}_+^*$. Then $Y \stackrel{\text{dist.}}{=} X$.*

Proof. By (13.B.2), for any $k \in \mathbb{N}^*$,

$$\mathbf{E}\left[Y^k\right] = (-1)^k \lim_{t \downarrow 0} \mathcal{L}_Y^{(k)}(t) = (-1)^k \lim_{t \downarrow 0} \mathcal{L}_X^{(k)}(t) = \mathbf{E}\left[X^k\right].$$

That is, Y has the same moments as X. Since moreover $R_{\Psi_X} = R_{\mathcal{L}_X} > 0$, then Proposition 13.A.7(iv) implies $X \stackrel{\text{dist.}}{=} Y$. □

13.B.2 Cumulants

In the particular case when X is nonnegative, we may use the Laplace transform rather than the characteristic function to define the cumulants. Note that the Laplace transform $\mathcal{L}_X(t) > 0,\ \forall t \in \mathbb{R}_+$, then the following function

$$\zeta_X(t) = \log \mathcal{L}_X(t), \quad t \in \mathbb{R}_+$$

is well-defined. Moreover, since $\mathcal{L}(t)$ is C^∞ on $\mathbb{R}_+^*$, then so is $\zeta(t)$. The n-th cumulant of X is defined by

$$c_n = (-1)^n \lim_{t \downarrow 0} \zeta_X^{(n)}(t),$$

whenever the above limit exists.

If $\mathbf{E}[X^n] < \infty$ then $\mathcal{L}_X$ is C^n on $\mathbb{R}_+$, then so is ζ_X. Thus

$$c_k = (-1)^k \zeta_X^{(k)}(0), \quad \forall k \leq n$$

and consequently c_k is well-defined and finite.

13.C Random vectors

13.C.1 Moments from transforms

We will extend the previous results for random vectors. Similarly to the one dimensional case, different transforms are defined according to the type (real, nonnegative or integer) of the components of the random vector.

Characteristic function Let $X = (X_1, \ldots, X_l)$ be a l-dimensional real random vector. Its *characteristic function* is the function $\Psi_X : \mathbb{R}^l \to \mathbb{C}$ defined by

$$\Psi_X(t) = \mathbf{E}\left[e^{\boldsymbol{i} t^{\mathrm{T}} X}\right], \quad t \in \mathbb{R}^l,$$

where t^{T} denotes the transpose of t. We will denote

$$\begin{aligned} |t| &:= |t_1| + \ldots + |t_l|, \quad t = (t_1, \ldots, t_l) \in \mathbb{R}^l, \\ \nu! &:= \nu_1! \ldots \nu_l!, \quad \nu = (\nu_1, \ldots, \nu_l) \in \mathbb{N}^l, \\ t^\nu &:= t_1^{\nu_1} \ldots t_l^{\nu_l}, \quad t \in \mathbb{R}^l, \nu \in \mathbb{N}^l. \end{aligned}$$

Lemma 13.C.1. *Assume that for some $\nu \in \mathbb{N}^l$,*

$$\mathbf{E}[|X_1|^{\nu_1} \ldots |X_l|^{\nu_l}] < \infty,$$

then

$$\frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} \Psi_X(t) = \boldsymbol{i}^{|\nu|} \mathbf{E}\left[X^\nu e^{\boldsymbol{i} t^{\mathrm{T}} X}\right],$$

which is continuous on $\mathbb{R}^l$. In particular,

$$\mathbf{E}[X^\nu] = \frac{1}{\boldsymbol{i}^{|\nu|}} \frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} \Psi_X(0).$$

Proof. The proof may be carried by induction along the same lines to the proof of first part of Lemma 13.A.1. The continuity of the partial derivatives follows from the convergence dominated theorem. □

Corollary 13.C.2. *If* $\mathbf{E}\left[|X|^n\right] < \infty$ *for some* $n \in \mathbb{N}^*$*, then* Ψ_X *is* C^n *on* $\mathbb{R}^l$*. Moreover,*

$$\Psi_X(t) = 1 + \sum_{\nu \in \mathbb{N}^l : 1 \le |\nu| \le n} \frac{\boldsymbol{i}^{|\nu|}}{\nu!} m_\nu t^\nu + o\left(|t|^n\right), \quad t \in \mathbb{R}^l,$$

where

$$m_\nu := \mathbf{E}\left[X^\nu\right], \quad \nu \in \mathbb{N}^l$$

and call it the ν-moment *of* X*.*

Proof. First observe that for all $\nu \in \mathbb{N}^l$ such that $|\nu| \le n$ we have $\mathbf{E}[|X_1|^{\nu_1} \ldots |X_l|^{\nu_l}] < \infty$. It follows that the partial derivatives of order $l \le n$ are continuous by Lemma 13.C.1. Therefore Ψ_X is C^n is on $\mathbb{R}^l$. Then the announced expansion follows from Taylor-Young theorem. □

Generating function Let X be a l-dimensional integer-valued random vector. Its *generating function*, denoted by $\mathcal{G}_X$, is defined by

$$\mathcal{G}_X(z) = \mathbf{E}\left[z^X\right], \quad z \in \mathbb{C}^l \text{ such that } \mathbf{E}\left[|z_1|^{X_1} \ldots |z_l|^{X_l}\right] < \infty.$$

Lemma 13.C.3. *For any* $\nu \in \mathbb{N}^l$*,*

$$\mathbf{E}\left[X_1^{(\nu_1)} \ldots X_l^{(\nu_l)}\right] = \lim_{x_1,\ldots,x_l \uparrow 1} \frac{\partial^{|\nu|}}{\partial_{x_1}^{\nu_1} \ldots \partial_{x_l}^{\nu_l}} \mathcal{G}_X(x).$$

Proof. Analogous to the one dimensional case. □

We will denote

$$m_{(\nu)} := \mathbf{E}\left[X_1^{(\nu_1)} \ldots X_l^{(\nu_l)}\right]$$

and call it the ν*-factorial moment* of X.

13.C.2 Cumulants

Ordinary cumulants Let X be a l-dimensional real random vector and let $\Psi_X(t) = \mathbf{E}\left[e^{\boldsymbol{i}t^{\mathrm{T}}X}\right]$, $t \in \mathbb{R}^l$ be its characteristic function. Assume that for some $n \in \mathbb{N}^*$, $\mathbf{E}\left[|X|^n\right] < \infty$. Then Ψ_X is C^n on $\mathbb{R}^l$ by Corollary 13.C.2. Moreover, $\Psi_X(0) = 1 > 0$, then the function

$$\zeta_X(t) = \log \Psi_X(t)$$

called the *cumulant function*, is well-defined and C^n on a neighborhood of 0. For any $\nu \in \mathbb{N}^l$ such that $|\nu| \le n$, the ν*-cumulant* of X is defined by

$$c_\nu := \frac{1}{\boldsymbol{i}^{\nu_1 + \ldots \nu_l}} \frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} \zeta_X(0). \tag{13.C.1}$$

Factorial cumulants Let X be a l-dimensional integer-valued random vector and let $\mathcal{G}_X(x)$ be its generating function. Note that $\mathcal{G}_X(x) > 0$, $\forall x \in \left(\mathbb{R}_+^*\right)^l$, then the function

$$\mathfrak{F}_X(x) = \log \mathcal{G}_X(x),$$

called the *factorial cumulant function*, is well-defined on $(0,1]^l$ and is C^∞ on $(0,1)^l$. For any $\nu \in \mathbb{N}^l$ the *ν-factorial cumulant* of X is defined by

$$c_{(\nu)} := \lim_{x_1,\ldots,x_l \uparrow 1} \frac{\partial^{|\nu|}}{\partial_{x_1}^{\nu_1} \ldots \partial_{x_l}^{\nu_l}} \mathfrak{F}_X(0), \tag{13.C.2}$$

whenever the limit exists.

Cumulants versus moments

Lemma 13.C.4. *Let $n \in \mathbb{N}^*$ and let $g : \mathbb{R}^l \to \mathbb{C}$ be n times differentiable at 0 such that $g(0) = 0$. Then $f(t) = e^{g(t)}$ is n times differentiable at 0. For any $\nu = (\nu_1, \ldots, \nu_l) \in \mathbb{N}^l, |\nu| \le n$, let*

$$m_\nu := \frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} f(0), \quad c_\nu := \frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} g(0).$$

Then,

$$m_\nu = \sum_{q=1}^{|\nu|} \frac{1}{q!} \sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(q)} \in \left(\mathbb{N}^l\right)^* \\ \lambda^{(1)}+\ldots+\lambda^{(q)}=\nu}} \frac{\nu!}{\lambda^{(1)}! \ldots \lambda^{(q)}!} \prod_{p=1}^{q} c_{\lambda_p}, \tag{13.C.3}$$

$$c_\nu = \sum_{q=1}^{|\nu|} \frac{(-1)^{q-1}}{q} \sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(q)} \in \left(\mathbb{N}^l\right)^* \\ \lambda^{(1)}+\ldots+\lambda^{(q)}=\nu}} \frac{\nu!}{\lambda^{(1)}! \ldots \lambda^{(q)}!} \prod_{p=1}^{q} m_{\lambda_p}, \tag{13.C.4}$$

where $\left(\mathbb{N}^l\right)^* = \mathbb{N}^l \setminus \{0\}$.

Proof. Cf. [89, p. 290]. Combining the Taylor-Young expansions

$$g(t) = \sum_{\lambda \in \mathbb{N}^l : 1 < |\lambda| \le n} \frac{c_\lambda}{\lambda!} t^\lambda + o(|t|^n), \quad t \in \mathbb{R}^l$$

and

$$e^x = 1 + \sum_{q=1}^{n} \frac{x^q}{q!} + o(x^n), \quad x \in \mathbb{R},$$

we get

$$\begin{aligned}
f(t) &= 1 + \sum_{q=1}^{n} \frac{1}{q!} \left(\sum_{\lambda \in \mathbb{N}^l : 1 \leq |\lambda| \leq n} \frac{c_\lambda}{\lambda!} t^\lambda \right)^q + o\left(|t|^n\right) \\
&= 1 + \sum_{q=1}^{n} \frac{1}{q!} \sum_{\substack{\lambda^{(1)}, \ldots, \lambda^{(q)} \in \mathbb{N}^l \\ 1 \leq \left|\lambda^{(1)}\right|, \ldots, \left|\lambda^{(q)}\right| \leq n}} \prod_{p=1}^{q} \left(\frac{c_{\lambda^{(p)}}}{\lambda^{(p)}!} t^{\lambda^{(p)}} \right) + o\left(|t|^n\right) \\
&= 1 + \sum_{\nu \in \mathbb{N}^l : 1 \leq |\nu| \leq n} \left[\sum_{q=1}^{|\nu|} \frac{1}{q!} \sum_{\substack{\lambda^{(1)}, \ldots, \lambda^{(q)} \in \left(\mathbb{N}^l\right)^* \\ \lambda^{(1)} + \ldots + \lambda^{(q)} = \nu}} \left(\prod_{p=1}^{q} \frac{c_{\lambda_p}}{\lambda^{(p)}!} \right) \right] t^\nu + o\left(|t|^n\right),
\end{aligned}$$

which proves (13.C.3). On the other hand, combining the Taylor-Young expansions

$$f(t) = 1 + \sum_{\lambda \in \mathbb{N}^l : 1 < |\lambda| \leq n} \frac{m_\lambda}{\lambda!} t^\lambda + o\left(|t|^n\right), \quad t \in \mathbb{R}^l$$

and

$$\log(1+x) = \sum_{q=1}^{n} \frac{(-1)^{q-1}}{q} x^q + o\left(x^n\right), \quad x \in \mathbb{R} : x > -1,$$

we obtain

$$\begin{aligned}
g(t) &= 1 + \sum_{q=1}^{n} \frac{(-1)^{q-1}}{q} \left(\sum_{\lambda \in \mathbb{N}^l : 1 < |\lambda| \leq n} \frac{m_\lambda}{\lambda!} t^\lambda \right)^q + o\left(t^n\right) \\
&= 1 + \sum_{q=1}^{n} \frac{(-1)^{q-1}}{q} \sum_{\substack{\lambda^{(1)}, \ldots, \lambda^{(q)} \in \mathbb{N}^l \\ 1 \leq \left|\lambda^{(1)}\right|, \ldots, \left|\lambda^{(q)}\right| \leq n}} \prod_{p=1}^{q} \left(\frac{m_{\lambda^{(p)}}}{\lambda^{(p)}!} t^{\lambda^{(p)}} \right) + o\left(t^n\right) \\
&= 1 + \sum_{\nu \in \mathbb{N}^l : 1 \leq |\nu| \leq n} \left[\sum_{q=1}^{|\nu|} \frac{(-1)^{q-1}}{q} \sum_{\substack{\lambda^{(1)}, \ldots, \lambda^{(q)} \in \left(\mathbb{N}^l\right)^* \\ \lambda^{(1)} + \ldots + \lambda^{(q)} = \nu}} \left(\prod_{p=1}^{q} \frac{m_{\lambda_p}}{\lambda^{(p)}!} \right) \right] t^\nu + o\left(|t|^n\right),
\end{aligned}$$

which proves (13.C.4). □

When $\nu = (1, \ldots, 1)$ we call m_ν and c_ν *simple* moments and cumulants respectively. They are related to each other by the general relations (13.C.3) and (13.C.4). Nevertheless, we will express these relations in a form specific to this particular case. To do so, let, and for all $J \subset \{1, \ldots, l\}$,

$$m(J) = m_{\xi(J)}, \quad c(J) = c_{\xi(J)},$$

where $\xi(J)$ is a l-dimensional vector whose jth coordinate is $\mathbf{1}\{j \in J\}$ for all $j = 1, \ldots, l$. That is, for $J = \{j_1, \ldots, j_k\}$,

$$m(J) := \frac{\partial^k}{\partial_{t_{j_1}} \ldots \partial_{j_k}} f(0), \quad c(J) := \frac{\partial^k}{\partial_{t_{j_1}} \ldots \partial_{j_k}} g(0).$$

Corollary 13.C.5. *In the conditions of Lemma 13.C.4, for all $J \subset \{1, \ldots, l\}$ such that $|\xi(J)| \le n$,*

$$m(J) = \sum_{q=1}^{|\xi(J)|} \sum_{\{J_1,\ldots,J_q\}} \prod_{p=1}^{q} c(J_p),$$

$$c(J) = \sum_{q=1}^{|\xi(J)|} (-1)^{q-1} (q-1)! \sum_{\{J_1,\ldots,J_q\}} \prod_{p=1}^{q} m(J_p),$$

where the summation is over all partitions $\{J_1, \ldots, J_q\}$ of J.

Proof. Cf. [89, p. 292]. Let $\nu = \xi(J)$ and let $\lambda^{(1)}, \ldots, \lambda^{(q)} \in (\mathbb{N}^l)^*$ be such that $\lambda^{(1)} + \ldots + \lambda^{(q)} = \nu$. Then $\nu! = \lambda^{(1)}! = \ldots = \lambda^{(q)}! = 1$. Moreover, let $J_p \subset J$ be such that $\xi(J_p) = \lambda^{(p)}$ for $p = 1, \ldots, q$. Then the J_p are pairwise disjoint, and thus there are $q!$ permutations of them; all having the same contribution in (13.C.3) as well as in (13.C.4) which give the announced formulae. □

Let $X = (X_1, \ldots, X_l)$ be a random vector, and for all $J \subset \{1, \ldots, l\}$, let

$$m(J) = m_{\xi(J)} = \mathbf{E}\left[\prod_{j \in J} X_j\right], \quad c(J) = c_{\xi(J)}$$

called *simple moments and cumulants* respectively. They are related by the formulae in the above Corollary.

Example 13.C.6. *Let $X = (X_1, \ldots, X_l)$ be a random vector. Then*

$$\begin{aligned}
m(1) &= c(1), \\
m(1,2) &= c(1,2) + c(1)\, c(2), \\
m(1,2,3) &= c(1,2,3) \\
&\quad + c(1,2)\, c(3) + c(1,3)\, c(2) + c(2,3)\, c(1) \\
&\quad + c(1)\, c(2)\, c(3),
\end{aligned}$$

whereas

$$\begin{aligned}
c(1,2) &= m(1,2) - m(1)\, m(2), \\
c(1,2,3) &= m(1,2,3) \\
&\quad - m(1,2)\, m(3) - m(1,3)\, m(2) - m(2,3)\, m(1) \\
&\quad + 2m(1)\, m(2)\, m(3).
\end{aligned}$$

13.C.3 Nonnegative random vectors

Laplace transform Let X be a l-dimensional nonnegative random vector. Its *Laplace transform*, denoted by $\mathcal{L}_X$, is defined by

$$\mathcal{L}_X(t) = \mathbf{E}\left[e^{-t^{\mathrm{T}}X}\right], \quad t \in \mathbb{R}_+^l.$$

Lemma 13.C.7. *For any $\nu_1, \ldots, \nu_l \in \mathbb{N}$, the partial derivative of the Laplace transform equals*

$$\frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} \mathcal{L}_X(t) = (-1)^{|\nu|} \mathbf{E}\left[X^\nu e^{-t^{\mathrm{T}}X}\right], \quad t \in \left(\mathbb{R}_+^*\right)^l.$$

Moreover, the moments of X are related to its Laplace transform as follows

$$\mathbf{E}[X^\nu] = (-1)^{|\nu|} \lim_{t_1,\ldots,t_l \downarrow 0} \frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} \mathcal{L}_X(t).$$

In particular, one side of the above equality is finite iff the other one is so.

Proof. The proof may be carried by induction along the same lines to the proof of first part of Lemma 13.B.1. □

Corollary 13.C.8. *$\mathcal{L}_X$ is C^∞ on $\left(\mathbb{R}_+^*\right)^l$.*

Proof. Similar to Corollary 13.B.2. □

The following lemma is useful to get the probability that a random vector vanishes from its Laplace transform.

Lemma 13.C.9. *Let X be a l-dimensional nonnegative random vector. Then*

$$\lim_{t_1,\ldots,t_l \to \infty} \mathcal{L}_X(t) = \mathbf{P}(X_1 = \ldots = X_l = 0).$$

Proof. Note that

$$\begin{aligned}
\mathcal{L}_X(t) &= \mathbf{E}\left[e^{-t^{\mathrm{T}}X}\right] \\
&= \mathbf{E}[e^{-t^{\mathrm{T}}X}\mathbf{1}\{X_1 = \ldots = X_l = 0\}] + \mathbf{E}\left[e^{-t^{\mathrm{T}}X}\mathbf{1}\{X_1 > 0 \text{ or } \ldots \text{ or } X_l > 0\}\right] \\
&= \mathbf{P}(X_1 = \ldots = X_l = 0) + \mathbf{E}\left[e^{-t^{\mathrm{T}}X}\mathbf{1}\{X_1 > 0 \text{ or } \ldots \text{ or } X_l > 0\}\right].
\end{aligned}$$

Since for $t \in \mathbb{R}_+^l$, $e^{-t^{\mathrm{T}}X}\mathbf{1}\{X_1 > 0 \text{ or } \ldots \text{ or } X_l > 0\}$ is bounded by 1 and tends to 0 as $t_1, \ldots, t_l \to \infty \uparrow \infty$, then by dominated convergence theorem

$$\lim_{t_1,\ldots,t_l \to \infty} \mathbf{E}\left[e^{-t^{\mathrm{T}}X}\mathbf{1}\{X_1 > 0 \text{ or } \ldots \text{ or } X_l > 0\}\right] = 0.$$

□

Cumulants Let X be a l-dimensional real random vector. Note that $\mathcal{L}_X(t) > 0, \ \forall t \in \mathbb{R}_+^l$, then the following function

$$\zeta_X(t) = \log \mathcal{L}_X(t), \quad t \in \mathbb{R}_+^l$$

is well-defined. Moreover, since $\mathcal{L}(t)$ is C^∞ on $\left(\mathbb{R}_+^*\right)^l$, then so is $\zeta(t)$. For any $\nu \in \mathbb{N}^l$, the ν-cumulant of X equals

$$c_\nu := (-1)^{|\nu|} \lim_{t_1,\ldots,t_l \downarrow 0} \frac{\partial^{|\nu|}}{\partial_{t_1}^{\nu_1} \ldots \partial_{t_l}^{\nu_l}} \zeta_X(0),$$

whenever the above limit exists.

Chapter 14

Useful results in measure theory

14.A Basic results

The following lemma is useful in the main part of the book as well as in the subsequent part of the annex.

Lemma 14.A.1. Negligible product sets. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$.*

(i) If for some $\mathbb{G}_1 \in \mathcal{G}$, $\mu(\mathbb{G} \backslash \mathbb{G}_1) = 0$, then, for any $k \in \mathbb{N}^*$, $\mu^k\left(\mathbb{G}^k \backslash \mathbb{G}_1^k\right) = 0$.

(ii) If for some $\mathbb{G}_2 \in \mathcal{G} \otimes \mathcal{G}$, $\mu^2\left(\mathbb{G}^2 \backslash \mathbb{G}_2\right) = 0$, then, for any $k \geq 2$,

$$\mu^k\left(\left\{(x_1, \ldots, x_k) \in \mathbb{G}^k : (x_i, x_j) \in \mathbb{G}^2 \backslash \mathbb{G}_2 \text{ for some } i \neq j\right\}\right) = 0.$$

Proof. (i) For any $k \in \mathbb{N}^*$, observe that

$$\mathbb{G}_1^k = \bigcap_{i=1}^{k} \left\{(x_1, \ldots, x_k) \in \mathbb{G}^k : x_i \in \mathbb{G}_1\right\}.$$

Then, for any $B \in \mathcal{G}$,

$$B^k \backslash \mathbb{G}_1^k = \bigcup_{i=1}^{k} \left\{(x_1, \ldots, x_k) \in B^k : x_i \in B \backslash \mathbb{G}_1\right\}.$$

Thus

$$\begin{aligned}\mu^k\left(B^k\backslash\mathbb{G}_1^k\right) &\leq \sum_{i=1}^k \mu^k\left((x_1,\ldots,x_k)\in B^k : x_i\in B\backslash\mathbb{G}_1\right)\\ &= \sum_{i=1}^k \mu\left(B\right)^{k-1}\mu\left(B\backslash\mathbb{G}_1\right)\\ &\leq \sum_{i=1}^k \mu\left(B\right)^{k-1}\mu\left(\mathbb{G}\backslash\mathbb{G}_1\right),\end{aligned}$$

which vanishes if $\mu(B)<\infty$. Since μ is σ-finite , there exists a sequence of sets $B_1,B_2,\ldots$ with finite measures increasing to $\mathbb{G}$. By continuity from below of measures, $\mu^k\left(B_n^k\backslash\mathbb{G}_1^k\right)$ converges to $\mu^k\left(\mathbb{G}^k\backslash\mathbb{G}_1^k\right)$ as $n\to\infty$. Thus $\mu^k\left(\mathbb{G}^k\backslash\mathbb{G}_1^k\right)=0$. (ii) Let

$$A_k=\left\{(x_1,\ldots,x_k)\in\mathbb{G}^k : (x_i,x_j)\in\mathbb{G}^2\backslash\mathbb{G}_2 \text{ for some } i\neq j\right\}.$$

Observe that

$$A_k\subset\bigcup_{i\neq j=1}^k\left\{(x_1,\ldots,x_k)\in\mathbb{G}^k : (x_i,x_j)\in\mathbb{G}^2\backslash\mathbb{G}_2\right\}.$$

Then, for any $B\in\mathcal{G}$,

$$\begin{aligned}\mu^k\left(A_k\cap B^k\right) &\leq \sum_{i\neq j=1}^k \mu^k\left(\left\{(x_1,\ldots,x_k)\in B^k : (x_i,x_j)\in\mathbb{G}_2\right\}\right)\\ &= \sum_{i\neq j=1}^k \int_{B^k}\mathbf{1}\left\{(x_i,x_j)\in\mathbb{G}^2\backslash\mathbb{G}_2\right\}\mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right)\\ &\leq \sum_{i\neq j=1}^k \mu\left(B\right)^{k-2}\mu^2\left(\mathbb{G}^2\backslash\mathbb{G}_2\right),\end{aligned}$$

which vanishes if $\mu(B)<\infty$. By the same argument at the end of Item (i), it follows that $\mu^k(A_k)=0$. □

The following lemma gives a useful property of symmetric measures.

Lemma 14.A.2. Symmetric measures. *Let $(\mathbb{G},\mathcal{G})$ be a measurable space and S be a symmetric measure on $\mathbb{G}^n$ for some $n\in\mathbb{N}^*$. Then for all $A\in\mathcal{G}$ and all $A_1,\ldots,A_k\in\mathcal{G}$ forming a partition of A, we have*

$$S\left(A^n\right)=\sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=n}}\binom{n}{n_1,\ldots,n_k}S\left(A_1^{n_1}\times\cdots\times A_k^{n_k}\right).$$

Proof. Note that

$$S\left(A^n\right) = S\left((A_1 \cup \cdots \cup A_k)^n\right) = \sum_{j_1,\ldots,j_n=1}^{k} S\left(A_{j_1} \times \cdots \times A_{j_n}\right).$$

Since S is symmetric

$$S\left(A_{j_1} \times \cdots \times A_{j_n}\right) = S\left(A_1^{n_1} \times \cdots \times A_k^{n_k}\right),$$

where, for any $i \in \{1, \ldots, k\}$, n_i is the number of indexes among $j_1, \ldots, j_n$ which equal i. The number of ways to choose $j_1, \ldots, j_n$ such that n_i of them are equal to i is $\binom{n}{n_1,\ldots,n_k}$. Therefore continuing the equation at the beginning of the proof gives the announced result. □

The following result called the *inclusion-exclusion principle* will be useful in the main part of the book.

Lemma 14.A.3. Inclusion-exclusion principle. *For any events $B_1, B_2, \ldots B_m$, define their n-th* symmetric sum *as*

$$S_n := \sum_{\substack{J \subset \{1,\ldots,m\} \\ |J|=n}} \mathbf{P}\left(\bigcap_{j \in J} B_j\right), \quad n \in \mathbb{N}^* \tag{14.A.1}$$

and let $S_0 := 1$. Let N be the number of the aforementioned events that occur, that is $N = \sum_{i=1}^{m} \mathbf{1}_{B_i}$, then

$$\mathbf{P}\left(N \geq k\right) = \sum_{n=k}^{m} (-1)^{n-k} \binom{n-1}{k-1} S_n, \tag{14.A.2}$$

$$\mathbf{P}\left(N = k\right) = \sum_{n=k}^{m} (-1)^{n-k} \binom{n}{k} S_n, \tag{14.A.3}$$

$$\mathbf{E}[z^N] = \sum_{n=0}^{m} (z-1)^n S_n, \quad z \in [0,1], \tag{14.A.4}$$

$$\mathbf{E}[N] = S_1. \tag{14.A.5}$$

Proof. Cf. [35, IV.5 and IV.3] for (14.A.2) and (14.A.3) respectively. Cf. [40, §8.6] for (14.A.4). Applying (14.A.1) with $n = 1$ gives

$$S_1 = \sum_{j=1}^{m} \mathbf{P}\left(B_j\right) = \sum_{i=1}^{m} \mathbf{E}\left[\mathbf{1}_{B_i}\right] = \mathbf{E}[N],$$

which proves Equation (14.A.5). □

14.B Support of a measure

We begin by defining the support of a measure on a topological space.

Definition 14.B.1. Support of a measure. *Let* $\mathbb{G}$ *is a topological space and* $\mathcal{B}(\mathbb{G})$ *the Borel* σ*-algebra on* $\mathbb{G}$*. The* support of a measure μ *on* $(\mathbb{G}, \mathcal{B}(\mathbb{G}))$*, denoted* $\operatorname{supp}(\mu)$*, is defined by*

$$\operatorname{supp}(\mu) = \{x \in \mathbb{G} : \mu(A) > 0 \textit{ for any neighborhood } A \textit{ of } x\}. \tag{14.B.1}$$

Note that 'any neighborhood' may be replaced by 'any open neighborhood' in the above display.

Lemma 14.B.2. Alternative expression of the support. *Let* μ *be a measure on a topological space* $\mathbb{G}$ *endowed with its Borel* σ*-algebra* $\mathcal{B}(\mathbb{G})$*. Then* $\operatorname{supp}(\mu)$ *is the smallest closed set* $F \subset \mathbb{G}$ *with* $\mu(F^{\mathrm{c}}) = 0$*; that is*

$$\operatorname{supp}(\mu) = \bigcap_{F \subset \mathbb{G} \textit{ closed: } \mu(F^{\mathrm{c}})=0} F. \tag{14.B.2}$$

Equivalently, $\operatorname{supp}(\mu)^{\mathrm{c}}$ *is the the union of all open sets of measure zero. In particular,* $\operatorname{supp}(\mu)$ *is closed.*

Proof. Let S be the right-hand side of (14.B.2). (i) We first prove that $\operatorname{supp}(\mu) \subset S$. Let $F \subset \mathbb{G}$ be closed with $\mu(F^{\mathrm{c}}) = 0$. Any $x \in \operatorname{supp}(\mu)$ is within F (otherwise, F^{c} would be a neighborhood of x with measure zero). Then $\operatorname{supp}(\mu) \subset F$. Thus $\operatorname{supp}(\mu) \subset S$. (ii) We prove now that $\operatorname{supp}(\mu)^{\mathrm{c}} \subset S^{\mathrm{c}}$. Let $x \in \operatorname{supp}(\mu)^{\mathrm{c}}$. Then there exists an open neighborhood A of x with measure zero. Then $x \in A \subset S^{\mathrm{c}}$. □

Lemma 14.B.3. $\operatorname{supp}(\mu)^{\mathrm{c}}$ is μ-negligible. *Let* μ *be a measure on a topological space* $\mathbb{G}$ *endowed with its Borel* σ*-algebra* $\mathcal{B}(\mathbb{G})$*. If* $\mathbb{G}$ *is second countable (i.e., its topology has a countable basis), then*

$$\mu(\operatorname{supp}(\mu)^{\mathrm{c}}) = 0.$$

In particular, for any measurable function $f : \mathbb{G} \to \bar{\mathbb{C}}$ *with well defined integral* $\int_{\mathbb{G}} f \mathrm{d}\mu$*, we have* $\int_{\mathbb{G}} f \mathrm{d}\mu = \int_{\operatorname{supp}(\mu)} f \mathrm{d}\mu$*.*

Proof. For any $x \in \operatorname{supp}(\mu)^{\mathrm{c}}$, let A_x be an open neighborhood of x such that $\mu(A_x) = 0$. Then

$$\operatorname{supp}(\mu)^{\mathrm{c}} \subset \bigcup_{x \in \operatorname{supp}(\mu)^{\mathrm{c}}} A_x.$$

By Lindelöf theorem [57, Theorem 15 p.49], there is a countable subcover of the above cover, which permits to conclude. □

Example 14.B.4. *Here are some examples of the support of measures.*

(i) *The support of the Lebegue measure* ℓ^d *on* $\mathbb{R}^d$ *equals all* $\mathbb{R}^d$*; that is* $\mathrm{supp}\left(\ell^d\right) = \mathbb{R}^d$.

(ii) *The support of the Dirac measure* δ_x *is* $\{x\}$*; that is* $\mathrm{supp}\left(\delta_x\right) = \{x\}$.

(iii) *Let* $F : \mathbb{R} \to \mathbb{R}$ *be nondecreasing and right-continuous and let* μ *be the measure on* $\mathbb{R}$ *such that* $\mu(a, b] = F(b) - F(a)$ *for all* $a, b \in \mathbb{R}$*; cf. [11, Theorem 12.4 p.176] for existence and uniqueness of such measure. Then*

$$\mathrm{supp}\left(\mu\right) = \left\{x \in \mathbb{R} : F(x - \varepsilon) < F(x + \varepsilon), \forall \varepsilon \in \mathbb{R}_+^*\right\}.$$

which follows from the fact that A *is a neighborhood of* x *iff* A *contains an interval* $(x - \varepsilon, x + \varepsilon]$ *for some* $\varepsilon \in \mathbb{R}_+^*$.

Lemma 14.B.5. *Let* μ *be a measure on a topological space* $\mathbb{G}$ *endowed with its Borel* σ*-algebra* $\mathcal{B}(\mathbb{G})$ *and let* $f : \mathbb{G} \to \mathbb{R}_+$ *be continuous. If* $\int_{\mathbb{G}} f \mathrm{d}\mu = 0$*, then* $f(x) = 0$ *for any* $x \in \mathrm{supp}\left(\mu\right)$.

Proof. Assume that there exists some $x \in \mathrm{supp}\left(\mu\right)$ such that $f(x) > 0$. Since f is continous, then there exists is a neighborhood A of x such that $f(y) > 0$ for all $y \in A$. Since $x \in \mathrm{supp}\left(\mu\right)$, then $\mu(A) > 0$, thus $\int_A f \mathrm{d}\mu > 0$. □

Lemma 14.B.6. Support of the product of two measures. *Let* μ_i *be a* σ*-finite measure on a topological space* $\mathbb{G}_i$ *endowed with its Borel* σ*-algebra* $\mathcal{B}(\mathbb{G}_i)$ *(for each* $i \in \{1, 2\}$*) and let* $\mu_1 \times \mu_2$ *be the product of* μ_1 *and* μ_2*. Then*

$$\mathrm{supp}\left(\mu_1 \times \mu_2\right) = \mathrm{supp}\left(\mu_1\right) \times \mathrm{supp}\left(\mu_2\right).$$

Proof. Let $\mu := \mu_1 \times \mu_2$. (i) Let $x_1 \in \mathrm{supp}\left(\mu_1\right)^{\mathrm{c}}$. Then there exists a neighborhood A_1 of x_1 such that $\mu_1(A_1) = 0$. For any $x_2 \in \mathbb{G}_2$, the set $A = A_1 \times \mathbb{G}_2$ is a neighborhood of (x_1, x_2) and $\mu(A) = 0$. Then $(x_1, x_2) \in \mathrm{supp}\left(\mu\right)^{\mathrm{c}}$. Thus

$$\mathrm{supp}\left(\mu_1\right)^{\mathrm{c}} \times \mathbb{G}_2 \subset \mathrm{supp}\left(\mu\right)^{\mathrm{c}}.$$

By symmetry, we have also $\mathbb{G}_1 \times \mathrm{supp}\left(\mu_2\right)^{\mathrm{c}} \subset \mathrm{supp}\left(\mu\right)^{\mathrm{c}}$. Therefore,

$$\mathrm{supp}\left(\mu\right) \subset \mathrm{supp}\left(\mu_1\right) \times \mathrm{supp}\left(\mu_2\right).$$

(ii) Let $(x_1, x_2) \in \mathrm{supp}\left(\mu\right)^{\mathrm{c}}$, then there exists an open neighborhood A of (x_1, x_2) such that $\mu(A) = 0$. By definition of the product topology, $A = \bigcup_{i \in I} A_{1,i} \times A_{2,i}$ for some open sets $A_{1,i}$ in $\mathbb{G}_1$ and $A_{2,i}$ in $\mathbb{G}_2$ and some arbitrary set I. Since $(x_1, x_2) \in A$, then $(x_1, x_2) \in A_{1,i} \times A_{2,i}$ for some $i \in I$. Since $\mu(A_{1,i})\,\mu(A_{2,i}) = \mu(A_{1,i} \times A_{2,i}) \leq \mu(A) = 0$, then either $\mu(A_{1,i}) = 0$ or $\mu(A_{2,i}) = 0$. Thus $x_1 \in \mathrm{supp}\left(\mu_1\right)^{\mathrm{c}}$ or $x_2 \in \mathrm{supp}\left(\mu_2\right)^{\mathrm{c}}$. Therefore,

$$\mathrm{supp}\left(\mu_1\right) \times \mathrm{supp}\left(\mu_2\right) \subset \mathrm{supp}\left(\mu\right).$$

□

14.C Functional monotone class theorems

There are many versions of the *functional monotone class theorem*. We give the following version which will be often useful.

Theorem 14.C.1. Monotone class theorem for nonnegative functions. *Let S be a given set and let $\mathcal{H}$ be a collection of functions $f : S \to \bar{\mathbb{R}}_+$ such that:*

(i) The constant function 1 is in $\mathcal{H}$.

(ii) For any bounded $f, g \in \mathcal{H}$ and any $\alpha, \beta \in \mathbb{R}$ such that $\alpha f + \beta g$ is nonnegative,

$$\alpha f + \beta g \in \mathcal{H}.$$

(iii) $\mathcal{H}$ is closed under nondecreasing pointwise limits. (That is, if $\{f_n\}$ is a nondecreasing sequence of functions in $\mathcal{H}$, then their pointwise limit as $n \to \infty$ is in $\mathcal{H}$.)

(iv) $\mathcal{I}$ is a π-system (i.e., closed with respect to finite intersections) such that the indicator function $\mathbf{1}_C \in \mathcal{H}$ for all $C \in \mathcal{I}$.

Then $\mathcal{H}$ contains all functions $f : S \to \bar{\mathbb{R}}_+$ that are $\sigma(\mathcal{I})$-measurable.

Proof. Let

$$\mathcal{D} = \{A \subset S : \mathbf{1}_A \in \mathcal{H}\}.$$

Observe that $\mathcal{D}$ is a Dynkin system on S; i.e.

$$S \in \mathcal{D},$$

$$A \in \mathcal{D} \Rightarrow A^{\mathrm{c}} \in \mathcal{D},$$

and for every nondecreasing sequence $\{A_n\}_{n \in \mathbb{N}}$ of sets in $\mathcal{D}$, we have

$$\lim_{n \to \infty} A_n \in \mathcal{D},$$

which follow respectively from Assumptions (i), (ii) and (iii). Since $\mathcal{D}$ contains the π-system $\mathcal{I}$, then $\mathcal{D}$ contains $\sigma(\mathcal{I})$ by the Dynkin's theorem [11, Theorem 3.2 p.42]. Thus $\mathcal{H}$ contains the indicator functions of all the sets in $\sigma(\mathcal{I})$. Then, by Assumption (ii), $\mathcal{H}$ contains all the nonnegative simple $\sigma(\mathcal{I})$-measurable functions. Let $f : S \to \bar{\mathbb{R}}_+$ be $\sigma(\mathcal{I})$-measurable. The simple approximation theorem [11, Theorem 13.5 p.185] ensures that there exists a nondecreasing sequence of simple $\sigma(\mathcal{I})$-measurable functions $\{f_n\}_{n \in \mathbb{N}}$ converging pointwise to f. Then, by Assumption (iii), $f \in \mathcal{H}$. $\square$

14.D Mixture and disintegration of measures

14.D.1 Mixture of measures

We will show how to mix a family of measures. In this regard, we introduce the notions of *measure kernel* and *probability kernel*.

Definition 14.D.1. Measure kernel and probability Kernel. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces. A* measure kernel *from X to Y is a function $\kappa : X \times \mathcal{Y} \to \bar{\mathbb{R}}_+$ such that:*

(i) for each $B \in \mathcal{Y}$, $\kappa(\cdot, B)$ is a nonnegative $\mathcal{X}$-measurable function;

(ii) for each $x \in X$, $\kappa(x, \cdot)$ is a measure on $(Y, \mathcal{Y})$.

If morover, $\kappa(x, Y) = 1$ for any $x \in X$, then κ is called a probability kernel *from X to Y.*

Definition 14.D.2. Mixture and disintegration of measures. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces, μ a measure on $(X, \mathcal{X})$, κ a measure kernel from X to Y, and λ a measure on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$.*

(i) If

$$\lambda(C) = \int_{\mathcal{X}} \left(\int_{\mathcal{Y}} \mathbf{1}\{(x, y) \in C\} \kappa(x, \mathrm{d}y) \right) \mu(\mathrm{d}x), \quad C \in \mathcal{X} \otimes \mathcal{Y}, \tag{14.D.1}$$

then we say that λ is the mixture *of κ with respect to μ.*

(ii) If

$$\lambda(A \times B) = \int_A \kappa(x, B) \mu(\mathrm{d}x), \quad A \in \mathcal{X}, B \in \mathcal{Y}, \tag{14.D.2}$$

then we say that κ is a disintegration kernel *of λ with respect to μ.*

Obviously, (14.D.1) implies (14.D.2), but the converse is not true in general. We will see in Theorem 14.D.4(i) that the converse is true when μ is σ-finite and κ is a probability kernel.

Lemma 14.D.3. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces, μ a measure on $(X, \mathcal{X})$, κ a probability kernel from X to Y, and λ a measure on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ satisfying (14.D.2). Then the following results hold.*

(i) If μ is finite, then so is λ and $\lambda(X \times Y) = \mu(X)$. In particular, if μ is a probability measure, then so is λ.

(ii) If μ is σ-finite, then so is λ.

Proof. (i) By (14.D.2), $\lambda(X \times Y) = \int_X \kappa(x, Y) \mu(\mathrm{d}x) = \mu(X) < \infty$. (ii) Let $\{A_n\}_{n \in \mathbb{N}}$ be a partition of X into measurable sets such that $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Observe that, for any $n \in \mathbb{N}$, $\lambda(A_n \times Y) = \int_{A_n} \kappa(x, Y) \mu(\mathrm{d}x) = \mu(A_n)$. Since $\{A_n \times Y\}_{n \in \mathbb{N}}$ is a measurable partition of $X \times Y$, then λ is σ-finite. □

Theorem 14.D.4. Measure mixture theorem. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces, μ a σ-finite measure on $(X, \mathcal{X})$, and κ a probability kernel from X to Y. Then the following results hold.*

(i) There exists a unique measure λ on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ satisfying (14.D.2).

(ii) For any $\mathcal{X} \otimes \mathcal{Y}$-measurable function $\varphi : X \times Y \to \bar{\mathbb{R}}_+$, the function $\bar{\varphi}$ defined on X by

$$\bar{\varphi}(x) = \int_Y \varphi(x, y)\, \kappa(x, \mathrm{d}y), \quad x \in X \tag{14.D.3}$$

is $\mathcal{X}$-measurable. Moreover

$$\int_{X \times Y} \varphi(x, y)\, \lambda(\mathrm{d}x \times \mathrm{d}y) = \int_X \int_Y \varphi(x, y)\, \kappa(x, \mathrm{d}y)\, \mu(\mathrm{d}x). \tag{14.D.4}$$

(iii) Let $\varphi : X \times Y \to \bar{\mathbb{R}}$ be $\mathcal{X} \otimes \mathcal{Y}$-measurable such that either of the following conditions

$$\int_{X \times Y} |\varphi(x, y)|\, \lambda(\mathrm{d}x \times \mathrm{d}y) < \infty, \quad \text{or} \int_X \int_Y |\varphi(x, y)|\, \kappa(x, \mathrm{d}y)\, \mu(\mathrm{d}x) < \infty$$

holds true. Then the other one holds, the function $\bar{\varphi}(x)$ given by (14.D.3) is well defined[1] for μ-almost all $x \in X$ and μ-integrable, and equality (14.D.4) holds true.

Proof. (i) Let $\mathcal{I}$ be the family of all finite disjoint unions of sets of the form $A \times B$ where $A \in \mathcal{X}, B \in \mathcal{Y}$. Note first that $\mathcal{I}$ is an algebra of sets. The set function λ defined by (14.D.2) can be extended to a finitely additive set function on $\mathcal{I}$. Moreover, λ is obviously σ-finite on $\mathcal{I}$. Then it follows from the Carathéodory's extension theorem [44, §13 Theorem A] that λ admits a unique extension on $\sigma(\mathcal{I})$ which is precisely the product σ-algebra on $\mathcal{X} \otimes \mathcal{Y}$. (ii) **Case 1**: Assume that μ is finite. We will apply Theorem 14.C.1. Let $\mathcal{H}$ be the collection of functions $\varphi : X \times Y \to \bar{\mathbb{R}}_+$ such that the function $\bar{\varphi}$ defined by (14.D.3) is $\mathcal{X}$-measurable and such that (14.D.4) holds true. Clearly, the constant function 1 is in $\mathcal{H}$. For any bounded $\varphi \in \mathcal{H}$, $\bar{\varphi}$ is also bounded and both sides of (14.D.4) are finite. Consider two bounded functions $f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha f + \beta g$ is nonnegative. By the linearity of the integral, for any $x \in X$,

$$\overline{\alpha f + \beta g}(x) = \int_Y [\alpha f(x, y) + \beta g(x, y)]\, \kappa(x, \mathrm{d}y) = \alpha \bar{f}(x) + \beta \bar{g}(x).$$

Then $\overline{\alpha f + \beta g}$ is $\mathcal{X}$-measurable. Since μ is assumed finite and $\bar{f}$ and $\bar{g}$ are

[1] For the $x \in X$ such that the integral in (14.D.3) is not well defined, set $\bar{\varphi}(x) = 0$.

bounded, then $\bar{f}$ and $\bar{g}$ are μ-integrable. Integrating the above equation, we get

$$\begin{aligned}\int_X \overline{\alpha f + \beta g}\,(x)\,\mu(\mathrm{d}x) &= \alpha \int_X \bar{f}(x)\,\mu(\mathrm{d}x) + \beta \int_X \bar{g}(x)\,\mu(\mathrm{d}x) \\ &= \alpha \int_{X\times Y} f(x,y)\,\lambda(\mathrm{d}x\times\mathrm{d}y) + \beta \int_{X\times Y} g(x,y)\,\lambda(\mathrm{d}x\times\mathrm{d}y) \\ &= \int_{X\times Y} [\alpha f(x,y) + \beta g(x,y)]\,\lambda(\mathrm{d}x\times\mathrm{d}y),\end{aligned}$$

where the last equality is due to the fact that the integrals are finite. Therefore, $\alpha f+\beta g \in \mathcal{H}$. Moreover, $\mathcal{H}$ is closed under nondecreasing pointwise limits by the monotone convergence theorem [11, Theorem 16.2 p.208]. For any $A \in \mathcal{X}, B \in \mathcal{Y}$, the function $\varphi(x,y) = \mathbf{1}_A(x)\,\mathbf{1}_B(y)$ is in $\mathcal{H}$. (Indeed, $\bar{\varphi}(x) = \mathbf{1}_A(x)\,\kappa(x,B)$ which is $\mathcal{X}$-measurable and the equality in (14.D.4) follows from (14.D.2).) It follows that $\mathbf{1}_C \in \mathcal{H}$ for all $C \in \mathcal{I}$ defined in Point (i). Theorem 14.C.1 and the fact that $\sigma(\mathcal{I})$ is precisely $\mathcal{X}\otimes\mathcal{Y}$ allows one to conclude. **Case 2**: Assume now that μ is σ-finite. Let $\{A_n\}_{n\in\mathbb{N}}$ be a partition of X into measurable sets such that $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Let μ_n be the restriction of μ to A_n; that is

$$\mu_n(A) = \mu(A\cap A_n), \quad A \in \mathcal{X}.$$

Then obviously, μ_n is finite, $\mu = \sum_{n\in\mathbb{N}} \mu_n$ and by (14.D.2), for any $A \in \mathcal{X}, B \in \mathcal{Y}$,

$$\lambda(A\times B) = \sum_{n\in\mathbb{N}} \int_A \kappa(x,B)\,\mu_n(\mathrm{d}x) = \sum_{n\in\mathbb{N}} \lambda_n(A\times B),$$

where λ_n is a mixture of $\{\kappa(x,\cdot)\}_{x\in X}$ with respect to μ_n for any $n \in \mathbb{N}$. It is then enough to apply the result of Case 1 to each μ_n and corresponding mixture λ_n, and then add the integrals over all $n \in \mathbb{N}$. (iii) Applying Point (ii) to $|\varphi|$ gives

$$\int_{X\times Y} |\varphi(x,y)|\,\lambda(\mathrm{d}x\times\mathrm{d}y) = \int_X \int_Y |\varphi(x,y)|\,\kappa(x,\mathrm{d}y)\,\mu(\mathrm{d}x) < \infty.$$

Then for μ-almost all $x \in X$, $\int_Y |\varphi(x,y)|\,\kappa(x,\mathrm{d}y) < \infty$ and therefore $\bar{\varphi}(x)$ is well defined. We write $\varphi = \varphi^+ - \varphi^-$ where φ^+ and φ^- are nonnegative $\mathcal{X}\otimes\mathcal{Y}$-measurable functions. Observe that, $\bar{\varphi}$ is $\mathcal{X}$-measurable since

$$\bar{\varphi}(x) = \begin{cases} \overline{\varphi^+}(x) - \overline{\varphi^-}(x), & \text{if } \overline{\varphi^+}(x) < \infty \text{ or } \overline{\varphi^-}(x) < \infty, \\ 0, & \text{if } \overline{\varphi^+}(x) = \overline{\varphi^-}(x) = \infty. \end{cases}$$

Moreover, $\bar{\varphi}$ is μ-integrable since

$$\int_X |\bar{\varphi}(x)|\,\mu(\mathrm{d}x) \le \int_X \int_Y |\varphi(x,y)|\,\kappa(x,\mathrm{d}y)\,\mu(\mathrm{d}x) < \infty.$$

Applying (14.D.4) for φ^+ and φ^- and then subtracting the two equations proves (14.D.4) for φ. □

Remark 14.D.5. *Theorem 14.D.4 is stated in [77, Proposition III.2.1] in the particular case when μ is a probability measure.*

14.D.2 Disintegration of measures

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces. Let λ be a σ-finite measure on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$. We define the *marginal measure* λ_X to be the projection of λ onto $(X, \mathcal{X})$, that is, the measure defined by

$$\lambda_X (A) = \lambda (A \times Y), \quad \forall A \in \mathcal{X}. \tag{14.D.5}$$

For any fixed $B \in \mathcal{Y}$, $\lambda (\cdot \times B)$ may be regarded as a measure on $(X, \mathcal{X})$, which is clearly absolutely-continuous with respect to the marginal λ_X. If λ_X is σ-finite, it follows from the Radon-Nikodym theorem [11, Theorem 32.2 p.422] that there exists a nonnegative $\mathcal{X}$-measurable function $\kappa (\cdot, B)$ on X such that

$$\lambda (A \times B) = \int_A \kappa (x, B) \lambda_X (\mathrm{d}x), \quad \forall A \in \mathcal{X}. \tag{14.D.6}$$

Moreover, the function $x \mapsto \kappa (x, B)$ is unique λ_X-almost everywhere. The following lemma gives some properties of the function κ.

Lemma 14.D.6. *[44, p.208] Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces, λ a σ-finite measure on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ such that its projection λ_X onto $(X, \mathcal{X})$ is also σ-finite, and for each $B \in \mathcal{Y}$, let $\kappa (\cdot, B)$ be the Radon-Nikodym derivative*

$$\kappa (\cdot, B) = \frac{\mathrm{d}\lambda (\cdot \times B)}{\mathrm{d}\lambda_X}. \tag{14.D.7}$$

Then the following results hold.

(i) For any $B \in \mathcal{Y}$,

$$0 \leq \kappa (x, B) \leq 1, \quad \textit{for } \lambda_X\textit{-almost all } x \in X.$$

(ii) For any disjoint sets $B, C \in \mathcal{Y}$,

$$\kappa (x, B \cup C) = \kappa (x, B) + \kappa (x, C), \quad \textit{for } \lambda_X\textit{-almost all } x \in X. \tag{14.D.8}$$

For any sequence of disjoint sets $B_1, B_2, \ldots \in \mathcal{Y}$,

$$\kappa \left(x, \bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \kappa (x, B_n), \quad \textit{for } \lambda_X\textit{-almost all } x \in X. \tag{14.D.9}$$

(iii)

$$\kappa (x, Y) = 1, \quad \textit{for } \lambda_X\textit{-almost all } x \in X. \tag{14.D.10}$$

$$\kappa (x, \emptyset) = 0, \quad \textit{for } \lambda_X\textit{-almost all } x \in X. \tag{14.D.11}$$

If $B_1 \subset B_2$, then

$$\kappa (x, B_1) \leq \kappa (x, B_2), \quad \textit{for } \lambda_X\textit{-almost all } x \in X. \tag{14.D.12}$$

(iv) For any increasing sequence of sets $B_1, B_2, \ldots \in \mathcal{Y}$,

$$\kappa\left(x, \bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \kappa(x, B_n), \quad \text{for } \lambda_X\text{-almost all } x \in X. \tag{14.D.13}$$

(v) For any decreasing sequence of sets $B_1, B_2, \ldots \in \mathcal{Y}$,

$$\kappa\left(x, \bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \kappa(x, B_n), \quad \text{for } \lambda_X\text{-almost all } x \in X. \tag{14.D.14}$$

Proof. (i) Let $E = \{x \in X : \kappa(x, B) > 1\}$. Since $\lambda(E \times B) \le \lambda_X(E)$, then

$$0 \le \lambda(E \times B) - \lambda_X(E) = \int_E (\kappa(x, B) - 1)\, \lambda_X(\mathrm{d}x) \le 0,$$

then

$$\int_E (\kappa(x, B) - 1)\, \lambda_X(\mathrm{d}x) = 0.$$

Thus $\lambda_X(E) = 0$. (ii) We first prove (14.D.9). For any $A \in \mathcal{X}$,

$$\begin{aligned}
\int_A \kappa\left(x, \bigcup_{n=1}^{\infty} B_n\right) \lambda_X(\mathrm{d}x) &= \lambda\left(A \times \bigcup_{n=1}^{\infty} B_n\right) \\
&= \lambda\left(\bigcup_{n=1}^{\infty} (A \times B_n)\right) \\
&= \sum_{n=1}^{\infty} \lambda(A \times B_n) \\
&= \sum_{n=1}^{\infty} \int_A \kappa(x, B_n)\, \lambda_X(\mathrm{d}x) = \int_A \sum_{n=1}^{\infty} \kappa(x, B_n)\, \lambda_X(\mathrm{d}x),
\end{aligned}$$

where the last equality is due to the monotone convergence theorem. The uniqueness assertion of the Radon-Nikodym theorem allows one to conclude the proof of (14.D.9). Consider now two disjoint sets $B, C \in \mathcal{Y}$. Applying (14.D.9) with $B_1 = B$, $B_2 = C$ and $B_k = \emptyset$ for $k = 2, 3, \ldots$ gives (14.D.8). (iii) Equations (14.D.10) and (14.D.11) are immediate from (14.D.7). If $B_1 \subset B_2$, then $B_2 = B_1 \cup (B_2 \backslash B_1)$. Applying (14.D.8) we get for λ_X-almost all $x \in X$,

$$\kappa(x, B_2) = \kappa(x, B_1) + \kappa(x, B_2 \backslash B_1),$$

from which inequality (14.D.12) follows. (iv) Let

$$C_1 = B_1, \quad C_k = B_k \backslash C_{k-1}, \quad k = 2, 3, \ldots$$

Observe that, for λ_X-almost all $x \in X$,

$$\kappa\left(x, \bigcup_{n=1}^{\infty} B_n\right) = \kappa\left(x, \bigcup_{n=1}^{\infty} C_n\right)$$
$$= \sum_{n=1}^{\infty} \kappa(x, C_n) = \lim_{n\to\infty} \kappa(x, B_n),$$

where the second equality is due to (14.D.9). (v) Let

$$C_k = Y \backslash B_k, \quad k = 1, 2, \ldots$$

Applying (14.D.8) and then (14.D.10), we get for λ_X-almost all $x \in X$,

$$\kappa(x, B_k) + \kappa(x, C_k) = \kappa(x, Y) = 1$$

and

$$\kappa\left(x, \bigcap_{n=1}^{\infty} B_n\right) + \kappa\left(x, \bigcup_{n=1}^{\infty} C_n\right) = \kappa(x, Y) = 1.$$

On the other hand, since $C_1, C_2, \ldots$ is increasing, then by (14.D.13), for λ_X-almost all $x \in X$,

$$\kappa\left(x, \bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \kappa(x, C_n)$$

Combining the above three equalities gives the announced result. □

Observe that the properties stated in the previous lemma for $\kappa(x, B)$ hold true, except for x in some set of λ_X-measure zero. Unfortunately, this set depends on the particular set B under consideration, so one may not conclude that $\kappa(x, \cdot)$ is a measure for λ_X-almost all x.

We will show that this holds true under some conditions to be specified later. In this regard, we define the notion of *isomorphism* of measurable spaces.

Definition 14.D.7. Isomorphism of measurable spaces *[56, §10.B p.66]. We say that two measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are* isomorphic *if there exists a measurable bijection $\varphi : X \to Y$ such that φ^{-1} is also measurable.*

Definition 14.D.8. Polish measurable space. *We will say that a measurable space $(X, \mathcal{X})$ is* Polish *if (i) $\mathcal{X}$ is the σ-algebra generated by a topology $\mathcal{T}$ and (ii) $(X, \mathcal{T})$ is Polish; i.e., there exists some metric d on X such that the topology induced by d is equal to $\mathcal{T}$ and such that (X, d) is a complete separable metric space.*

Lemma 14.D.9. *Any Polish measurable space is isomorphic to a subset of* $[0, 1]$ *with the corresponding Borel σ-algebra.*

Proof. Cf. [54, Theorem 1.1(ii) p.18]. □

Theorem 14.D.10. Measure disintegration theorem. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces, and assume that $(Y, \mathcal{Y})$ is Polish. Let λ be a σ-finite measure on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ such that the projection λ_X of λ onto $(X, \mathcal{X})$ is also σ-finite. Then:*

(i) There exists a disintegration probability kernel κ of λ with respect to λ_X.

(ii) Moreover, κ is unique λ_X-almost everywhere; i.e., if $\hat{\kappa}$ is another disintegration probability kernel of λ with respect to λ_X, then $\hat{\kappa}(x, \cdot) = \kappa(x, \cdot)$ for λ_X-almost all $x \in X$.

Proof. We will proceed in three steps taking Y to be the entire real line, then a Borel subset of $\mathbb{R}$, and finally any Polish space. **Case 1**. Suppose first that $Y = \mathbb{R}$ and $\mathcal{Y} = \mathcal{B}(\mathbb{R})$. (i) The following arguments follow those in [11, Theorem 33.3 p.439]. For each $B \in \mathcal{Y}$, let $\kappa_0(\cdot, B)$ be the Radon-Nikodym derivative [11, Theorem 32.2 p.422]

$$\kappa_0(\cdot, B) = \frac{\mathrm{d}\lambda(\cdot \times B)}{\mathrm{d}\lambda_X}.$$

Consider the function φ defined on $X \times \mathbb{Q}$ by

$$\varphi(x, r) := \kappa_0(x, (-\infty, r]), \quad x \in X, r \in \mathbb{Q}.$$

If $r, s \in \mathbb{Q}$ such that $r \leq s$, then by (14.D.12),

$$\varphi(x, r) \leq \varphi(x, s),$$

for x outside a $\mathcal{X}$-measurable set $A_{r,s}$ of λ_X-measure 0. Applying (14.D.14) with $B_n = \left(-\infty, r + n^{-1}\right]$, we get

$$\varphi(x, r) = \lim_{n \to \infty} \varphi\left(x, r + n^{-1}\right),$$

for x outside a $\mathcal{X}$-measurable set C_r of λ_X-measure 0. Applying (14.D.11) and (14.D.14) and then (14.D.10) and (14.D.13),

$$\lim_{r \to -\infty} \varphi(x, r) = 0, \quad \lim_{r \to +\infty} \varphi(x, r) = 1$$

for x outside a $\mathcal{X}$-measurable set D of λ_X-measure 0. Observe that the set

$$A_0 = \left(\bigcup_{r \leq s \in \mathbb{Q}} A_{r,s}\right) \cup \left(\bigcup_{r \in \mathbb{Q}} C_r\right) \cup D$$

lies in $\mathcal{X}$ and has λ_X-measure 0. Now define for any $x \in X, y \in \mathbb{R}$,

$$F(x, y) = \mathbf{1}_{A_0^c}(x) \inf_{r \geq y} \{\varphi(x, r)\} + \mathbf{1}_{A_0}(x) \mathbf{1}\{y \geq 0\}.$$

Observe that if $x \in A_0^c, r \in \mathbb{Q}$, then $F(x,r) = \varphi(x,r)$. It is easy to see that for every fixed $x \in X$, the function $y \mapsto F(x,y)$ is nondecreasing and right-continuous with limits 1 and 0 at $+\infty$ and $-\infty$ respectively. Then, by [11, Theorem 14.1 p.188], there exists a probability measure $\kappa(x,\cdot)$ on $(Y, \mathcal{Y})$ such that

$$\kappa(x,(-\infty,y]) = F(x,y), \quad x \in X, y \in \mathbb{R}.$$

For each $r \in \mathbb{Q}$ let $B = (-\infty, r]$ and observe that

$$\begin{aligned}
\kappa(x,B) &= \kappa(x,(-\infty,r]) \\
&= F(x,r) \\
&= \mathbf{1}_{A_0^c}(x)\,\varphi(x,r) + \mathbf{1}_{A_0}(x)\,\mathbf{1}\{r \geq 0\} \\
&= \mathbf{1}_{A_0^c}(x)\,\kappa_0(x,(-\infty,r]) + \mathbf{1}_{A_0}(x)\,\mathbf{1}\{r \geq 0\},
\end{aligned}$$

from which we deduce that

$$\lambda(A \times B) = \int_A \kappa(x,B)\,\lambda_X(\mathrm{d}x), \quad \text{for any } A \in \mathcal{X} \tag{14.D.15}$$

and

$$\kappa(x,B) \text{ is an } \mathcal{X}\text{-measurable function of } x \in X. \tag{14.D.16}$$

Let $\mathcal{D}$ be the set of all $B \in \mathcal{Y}$ such that (14.D.16) and (14.D.15) are true. Clearly $\mathcal{D}$ is a Dynkin system and contains $\mathcal{I} = \{B = (-\infty, r] : r \in \mathbb{Q}\}$ which is closed under finite intersections. Then $\mathcal{D}$ contains $\sigma(\mathcal{I})$ by the Dynkin's theorem [11, Theorem 3.2 p.42]. Observing that $\sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R}) = \mathcal{Y}$ concludes the proof of the existence part (i) in the case $Y = \mathbb{R}$. (ii) Observe that for each $B \in \mathcal{Y}$, $\kappa(\cdot, B)$ is the Radon-Nikodym derivative

$$\kappa(\cdot,B) = \frac{\mathrm{d}\lambda(\cdot \times B)}{\mathrm{d}\lambda_X}.$$

Then by the Radon-Nikodym theorem [11, Theorem 32.2 p.422], $\kappa(x,B)$ is unique λ_X-almost everywhere. Then $\{\kappa(\cdot,(-\infty,r])\}_{r \in \mathbb{Q}}$ is unique λ_X-almost everywhere. By the continuity from above of measures [44, Theorem E p.38], for any $x \in X$ and any $y \in Y$,

$$\kappa(x,(-\infty,y]) = \lim_{r \downarrow y, r \in \mathbb{Q}} \kappa(x,(-\infty,r]).$$

The announced uniqueness of κ follows. **Case 2**. Suppose now that Y is a Borel subset of $\mathbb{R}$. Let $\tilde{\lambda}$ be a probability measure on $(X \times \mathbb{R}, \mathcal{X} \otimes \mathcal{B}(\mathbb{R}))$ such that

$$\tilde{\lambda}(A \times B) = \lambda(A \times (B \cap Y)), \quad A \in \mathcal{X}, B \in \mathcal{B}(\mathbb{R}).$$

Observe that the projections of $\tilde{\lambda}$ and λ onto $(X, \mathcal{X})$ are equal. Applying the result of Case 1 to $\tilde{\lambda}$ shows that there exists a probability kernel $\tilde{\kappa}$ from X to $\mathbb{R}$, such that $\tilde{\lambda}$ is the mixture of $\{\tilde{\kappa}(x,\cdot)\}_{x \in X}$ with respect to λ_X. In particular,

$$\lambda(A \times B) = \tilde{\lambda}(A \times B) = \int_A \tilde{\kappa}(x,B)\,\lambda_X(\mathrm{d}x), \quad A \in \mathcal{X}, B \in \mathcal{Y}.$$

Letting

$$\kappa(x,B)=\tilde{\kappa}(x,B), \quad A\in\mathcal{X}, B\in\mathcal{Y},$$

allows one to concludes the proof of the theorem in the case $Y\in\mathcal{B}(\mathbb{R})$. **Case 3**. Using Lemma 14.D.9, the case of a Polish space Y, is reduced to the case where Y is a Borel subset Y_0 of $\mathbb{R}$ treated in Case 2. Indeed, let $\varphi: Y\to Y_0$ a measurable bijection such that φ^{-1} is also measurable. Let $\tilde{\lambda}$ be a probability measure on $(X\times Y_0, \mathcal{X}\otimes\mathcal{Y}_0)$ such that

$$\tilde{\lambda}(A\times B_0)=\lambda(A\times\varphi(B_0)), \quad A\in\mathcal{X}, B_0\in\mathcal{Y}_0.$$

Observe that the projections of $\tilde{\lambda}$ and λ onto $(X,\mathcal{X})$ are equal. Applying the result of Point (i) to $\tilde{\lambda}$ shows that there exists a probability kernel $\tilde{\kappa}$ from X to Y_0, such that $\tilde{\lambda}$ is the mixture of $\{\tilde{\kappa}(x,\cdot)\}_{x\in X}$ with respect to λ_X. Then

$$\lambda(A\times B)=\tilde{\lambda}\left(A\times\varphi^{-1}(B)\right)=\int_A\tilde{\kappa}\left(x,\varphi^{-1}(B)\right)\lambda_X(\mathrm{d}x), \quad A\in\mathcal{X}, B\in\mathcal{Y}.$$

Letting

$$\kappa(x,B)=\tilde{\kappa}\left(x,\varphi^{-1}(B)\right), \quad A\in\mathcal{X}, B\in\mathcal{Y},$$

allows one to conclude the proof the theorem. $\square$

Remark 14.D.11. Bibliographic notes. *Theorem 14.D.10 is stated in [29, Proposition A.1.5.III p.605] for probability measures without proof. Cf. also related statements in [53, Theorem 6.3] and for the particular case $Y=\mathbb{R}$ in [44, §48(5) p.210] and [11, Theorem 33.3 p.439].*

We will extend Theorem 14.D.10 to the case when the measure $\lambda_{\mathcal{X}}$ is not necessarily σ-finite. In this regard we need the following lemmas.

Lemma 14.D.12. *Let $(X,\mathcal{X})$ be a measurable space and μ a σ-finite measure on $(X,\mathcal{X})$. Then there exists a measurable function $h: X\to\,]0,1[$ such that the measure $h\mu$ is finite.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}^*}$ be a partition of X into measurable sets such that $\mu(A_n)<\infty$ for all $n\in\mathbb{N}$. Let h be a function defined on X by

$$h(x)=\sum_{n\in\mathbb{N}^*}2^{-n}\frac{\mathbf{1}\{x\in A_n\}}{1+\mu(A_n)}, \quad x\in X.$$

Observe that $h(x)\in\,]0,1[$ for any $x\in X$. Consider the measure $\tilde{\mu}=h\mu$ on $(X,\mathcal{X})$; that is

$$\tilde{\mu}(A)=\int_A h(x)\mu(\mathrm{d}x), \quad A\in\mathcal{X}.$$

The measure $\tilde{\mu}$ is finite since

$$\tilde{\mu}(X)=\sum_{n\in\mathbb{N}^*}2^{-n}\frac{\mu(A_n)}{1+\mu(A_n)}\le\sum_{n\in\mathbb{N}^*}2^{-n}=1.$$

$\square$

Recall that two measures μ and ν are said *equivalent* if they are absolutely-continuous with respect to each other. Equivalence of measures is denoted by $\mu \sim \nu$.

Lemma 14.D.13. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces, μ a measure on $(X, \mathcal{X})$, κ a measure kernel from X to Y, λ be a measure on $X \times Y$, and λ_X be the projection of λ onto X. Assume that κ is a disintegration kernel of λ with respect to μ. Then $\lambda_X \ll \mu$ and λ_X admits density $x \mapsto \kappa(x, Y)$ with respect to μ. In particular, $\mu \sim \lambda_X$ iff $\kappa(x, Y) \neq 0$ for μ-almost all $x \in X$.*

Proof. Observe from (14.D.2) that for any $A \in X$,

$$\lambda_X(A) = \lambda(A \times Y) = \int_A \kappa(x, Y)\,\mu(\mathrm{d}x).$$

Thus $\lambda_X \ll \mu$ and λ_X admits density $x \mapsto \kappa(x, Y)$ with respect to μ. The last statement is then obvious. □

Theorem 14.D.14. Measure disintegration theorem; extension. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces such that $(Y, \mathcal{Y})$ is Polish and let λ be a σ-finite measure on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$. Then:*

(i) There exists a σ-finite measure μ on $(X, \mathcal{X})$ and a measure kernel κ from X to Y such that $\mu \sim \lambda_X$, $\kappa(x, Y) \neq 0$ for any $x \in X$, and κ is a disintegration kernel of λ with respect to μ.

(ii) Assume that there exists two σ-finite measures μ and $\hat{\mu}$ on $\mathcal{X}$ and two measure kernels κ and $\hat{\kappa}$ from $\mathcal{X}$ to $\mathcal{Y}$ which are disintegration kernels of λ with respect to μ and $\hat{\mu}$ respectively. Assume moreover that $\hat{\mu} \ll \mu$. Then

$$\hat{\kappa}(x, \cdot)\,\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}\mu}(x) = \kappa(x, \cdot), \quad \text{for } \mu\text{-almost all } x \in \mathcal{X},$$

where $\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}\mu}$ is a Radon-Nikodym derivative of $\hat{\mu}$ with respect to μ.

Proof. (i) By Lemma 14.D.12, there exists a measurable function $h : X \times Y \to]0, 1[$ such that the measure $\tilde{\lambda} = h\lambda$ is finite. Then the projection $\tilde{\lambda}_X$ of $\tilde{\lambda}$ onto $(X, \mathcal{X})$ is also finite. Applying Theorem 14.D.10 to $\tilde{\lambda}$ shows that there exists a disintegration probability kernel $\tilde{\kappa}$ of $\tilde{\lambda}$ with respect to $\tilde{\lambda}_X$. Then for any $A \in \mathcal{X}, B \in \mathcal{Y}$,

$$\begin{aligned}
\lambda(A \times B) &= \int_{A \times B} \frac{1}{h(x, y)}\tilde{\lambda}(\mathrm{d}x \times \mathrm{d}y) \\
&= \int_{X \times Y} \frac{\mathbf{1}\{x \in A, y \in B\}}{h(x, y)}\tilde{\lambda}(\mathrm{d}x \times \mathrm{d}y) \\
&= \int_X \int_Y \frac{\mathbf{1}\{x \in A, y \in B\}}{h(x, y)}\tilde{\kappa}(x, \mathrm{d}y)\,\tilde{\lambda}_X(\mathrm{d}x) \\
&= \int_A \left[\int_B \frac{1}{h(x, y)}\tilde{\kappa}(x, \mathrm{d}y)\right]\tilde{\lambda}_X(\mathrm{d}x)
\end{aligned}$$

where the third equality is due to (14.D.4). Let

$$\kappa\left(x,B\right)=\int_{B}\frac{1}{h\left(x,y\right)}\tilde{\kappa}\left(x,\mathrm{d}y\right),\quad x\in X,B\in\mathcal{Y}.$$

Then λ is a mixture of $\{\kappa\left(x,\cdot\right)\}_{x\in X}$ with respect to $\mu:=\tilde{\lambda}_{X}$. It follows from the above equation that, for any $x\in X$,

$$\kappa\left(x,Y\right)=\int_{Y}\frac{1}{h\left(x,y\right)}\tilde{\kappa}\left(x,\mathrm{d}y\right),$$

which is positive since $h>0$ and $\tilde{\kappa}\left(x,\cdot\right)$ is a probability measure. Using the fact that for any $A\in\mathcal{X}$, $\tilde{\lambda}_{X}\left(A\right)=\int_{A\times Y}h\left(x,y\right)\lambda\left(\mathrm{d}x\times\mathrm{d}y\right)$, it follows that $\tilde{\lambda}_{X}\sim\lambda_{X}$. (ii) For any $A\in\mathcal{X},B\in\mathcal{Y}$,

$$\lambda\left(A\times B\right)=\int_{A}\hat{\kappa}\left(x,B\right)\hat{\mu}\left(\mathrm{d}x\right)=\int_{A}\hat{\kappa}\left(x,B\right)\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}\mu}\left(x\right)\mu\left(\mathrm{d}x\right).$$

Thus λ is a mixture of $\left\{\hat{\kappa}\left(x,\cdot\right)\frac{\mathrm{d}\hat{\mu}}{\mathrm{d}\mu}\left(x\right)\right\}_{x\in X}$ with respect to μ. Thus it is enough to prove the announced result in the case $\hat{\mu}=\mu$. Observe that, for any $B\in\mathcal{Y}$,

$$\hat{\kappa}\left(\cdot,B\right)=\frac{\mathrm{d}\lambda\left(\cdot\times B\right)}{\mathrm{d}\mu}=\kappa\left(\cdot,B\right).$$

Then

$$\hat{\kappa}\left(x,B\right)=\kappa\left(x,B\right),\quad\text{for }\mu\text{-almost all }x\in X.$$

Unfortunately, the exceptional sets of μ-measure zero depend on the particular set B under consideration. Following the same lines as the proof of Theorem 14.D.10(ii), we see that

$$\hat{\kappa}\left(x,\cdot\right)=\kappa\left(x,\cdot\right),\quad\text{for }\mu\text{-almost all }x\in X.$$

□

Remark 14.D.15. Bibliographic notes. *Theorem 14.D.14 is stated in [52, p.164] and proved in [54, Theorem 1.23 p.37]. Our Item (ii) extends uniqueness result therein.*

14.E Power and factorial powers of measures

Definition 14.E.1. *Let* $(\mathbb{G},\mathcal{G})$ *be a measurable space. Given a* σ*-finite measure* μ *on* $\mathbb{G}$ *and* $n\in\mathbb{N}^{*}$, *we denote by* μ^{n} *its* n-th power *in the sense of products of measures. In particular, for a* counting measure $\mu=\sum_{j\in\mathbb{Z}}\delta_{x_{j}}$ *(where atoms need not to be distinct) and* $n\in\mathbb{N}^{*}$,

$$\mu^{n}=\sum_{(j_{1},\ldots,j_{n})\in\mathbb{Z}^{n}}\delta_{(x_{j_{1}},\ldots,x_{j_{n}})}.$$

For such counting measure, we define its n-th factorial power *as the following counting measure on* $\mathbb{G}^n$

$$\mu^{(n)} = \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} \delta_{(x_{j_1},\ldots,x_{j_n})}, \tag{14.E.1}$$

where $\mathbb{Z}^{(n)} = \{(j_1,\ldots,j_n) \in \mathbb{Z}^n : j_k \neq j_l,\ \textit{for any } l \neq k\}$.

Observe that for all $B_1,\ldots,B_n \in \mathcal{G}$,

$$\begin{aligned}
&\mu^{(n)}(B_1 \times \ldots \times B_n) \\
&= \int_{B_1\times\ldots\times B_n} \left(\mu - \sum_{k=1}^{n-1} \delta_{x_k}\right)(\mathrm{d}x_n) \left(\mu - \sum_{k=1}^{n-2} \delta_{x_k}\right)(\mathrm{d}x_{n-1}) \ldots \mu(\mathrm{d}x_1).
\end{aligned}$$

Example 14.E.2. *For a σ-finite measure μ on a measurable space $(\mathbb{G},\mathcal{G})$,*

$$\mu^2(A \times B) = \mu(A)\,\mu(B), \quad A,B \in \mathcal{G} \tag{14.E.2}$$

and, more generally,

$$\mu^n(B_1 \times \ldots \times B_n) = \mu(B_1)\ldots\mu(B_n), \quad B_1,\ldots,B_n \in \mathcal{G}. \tag{14.E.3}$$

It is clear that $\mu^{(1)} = \mu$. If $\mu = \delta_x$, then $\mu^{(2)} = 0$. If $\mu = \delta_x + \delta_y$, then $\mu^{(2)} = \delta_{(x,y)} + \delta_{(y,x)}$; in the particular case $x = y$, $\mu^{(2)} = 2\delta_{(x,x)}$.

Lemma 14.E.3. *Let $(\mathbb{G},\mathcal{G})$ be a measurable space, μ be a counting measure on $\mathbb{G}$ and $n \in \mathbb{N}^*$. For any pairwise disjoint $B_1,\ldots,B_n \in \mathcal{G}$,*

$$\mu^n(B_1 \times \ldots \times B_n) = \mu^{(n)}(B_1 \times \ldots \times B_n). \tag{14.E.4}$$

For any $A,B \in \mathcal{G}$,

$$\mu^2(A \times B) = \mu^{(2)}(A \times B) + \mu(A \cap B). \tag{14.E.5}$$

For any $B \in \mathcal{G}$,

$$\mu^{(n)}(B^n) = \mu(B)(\mu(B) - 1)\ldots(\mu(B) - n + 1)^+. \tag{14.E.6}$$

Let $\mathcal{G}^{\otimes n}$ be the product σ-algebra. For any $A \in \mathcal{G}^{\otimes n}, B \in \mathcal{G}^{\otimes m}$,

$$\mu^{(n+m)}(A \times B) = \int_A \left(\mu - \sum_{i=1}^{n} \delta_{x_i}\right)^{(m)}(B)\,\mu^{(n)}(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n). \tag{14.E.7}$$

Proof. The two first identities are obvious. For the third one, it is enough to observe that $\mu^{(n)}(B^n)$ is the number of arrangements of ordered sequences of n points among the $\mu(B)$ atoms of μ in B where atoms are counted with their multiplicity. (Indeed, note first that

$$\mu^{(1)}(B) = \sum_j \mathbf{1}\{x_j \in B\} = \mu(B).$$

Moreover, for all $n \geq 2$,

$$
\begin{aligned}
\mu^{(n)}\left(B^{n}\right) &= \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} \mathbf{1}\left\{x_{j_1}\in B,\ldots,x_{j_n}\in B\right\} \\
&= \sum_{(j_1,\ldots,j_{n-1})\in\mathbb{Z}^{(n-1)}} \mathbf{1}\left\{x_{j_1}\in B,\ldots,x_{j_n}\in B\right\} \sum_{j_n\notin\{j_1,\ldots,j_{n-1}\}} \mathbf{1}\left\{x_{j_n}\in B\right\} \\
&= \sum_{(j_1,\ldots,j_{n-1})\in\mathbb{Z}^{(n-1)}} \mathbf{1}\left\{x_{j_1}\in B,\ldots,x_{j_n}\in B\right\}\left(\mu(B)-n+1\right)^{+} \\
&= \mu^{(n-1)}\left(B^{n-1}\right)\left(\mu(B)-n+1\right)^{+}.
\end{aligned}
$$

Then the announced result (14.E.6) follows by induction on n.) It remains to prove the fourth identity. By definition, writing $\mu=\sum_j \delta_{a_j}$, we have

$$
\mu^{(n+m)} = \sum_{(j_1,\ldots,j_{n+m})\in\mathbb{Z}^{(n+m)}} \delta_{(a_{j_1},\ldots,a_{j_{n+m}})}.
$$

Then for all $A\in\mathcal{G}^{\otimes n}, B\in\mathcal{G}^{\otimes m}$,

$$
\begin{aligned}
&\mu^{(n+m)}(A\times B) \\
&= \sum_{(j_1,\ldots,j_{n+m})\in\mathbb{Z}^{(n+m)}} \mathbf{1}\left\{(a_{j_1},\ldots,a_{j_{n+m}})\in A\times B\right\} \\
&= \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} \mathbf{1}\left\{(a_{j_1},\ldots,a_{j_n})\in A\right\} \sum_{\substack{(j_{n+1},\ldots,j_{n+m})\in\mathbb{Z}^{(m)}\\ j_{n+k}\notin\{j_1,\ldots,j_n\}}} \mathbf{1}\left\{(a_{j_{n+1}},\ldots,a_{j_{n+m}})\in B\right\} \\
&= \sum_{(j_1,\ldots,j_n)\in\mathbb{Z}^{(n)}} \mathbf{1}\left\{(a_{j_1},\ldots,a_{j_n})\in A\right\}\left(\mu-\sum_{i=1}^{n}\delta_{a_{j_i}}\right)^{(m)}(B) \\
&= \sum_{x\in\mu^{(n)}\cap A}\left(\mu-\sum_{i=1}^{n}\delta_{x_i}\right)^{(m)}(B) \\
&= \int_A \left(\mu-\sum_{i=1}^{n}\delta_{x_i}\right)^{(m)}(B)\,\mu^{(n)}\left(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_n\right).
\end{aligned}
$$

□

The following lemma generalizes (14.E.5).

Lemma 14.E.4. *Let $(\mathbb{G},\mathcal{G})$ be a measurable space, μ be a counting measure on $\mathbb{G}$ and $n\in\mathbb{N}^*$. Then for any $B_1,\ldots,B_n\in\mathcal{G}$,*

$$
\mu^{n}\left(B_1\times\ldots\times B_n\right)=\sum_{q=1}^{n}\sum_{\{J_1,\ldots,J_q\}}\mu^{(q)}\left(\prod_{p=1}^{q}\left(\bigcap_{j\in J_p}B_j\right)\right), \tag{14.E.8}
$$

where the summation is over all partitions $\{J_1,\ldots,J_q\}$ of $\{1,\ldots,n\}$.

Proof. The above identities hold true for all $B_1, \ldots, B_n \in \mathcal{G}$ which are pairwise either identical or disjoint. Two measures which coincide on such $B_1 \times \ldots \times B_n$ are identical. Indeed, for all $A_1, \ldots, A_n \in \mathcal{G}$, we may show inductively that the product $A_1 \times \ldots \times A_n$ may be written as a finite union of sets of the form $B_1 \times \ldots \times B_n$ where $B_1, \ldots, B_n \in \mathcal{G}$ are pairwise either identical or disjoint. Indeed, this is trivial for $n = 1$, and induction heredity follows from the observation that for all $n \geq 1$ and all $A_1, \ldots, A_{n+1} \in \mathcal{G}$,

$$\begin{aligned}
&A_1 \times \ldots \times A_n \times A_{n+1} \\
&= \left(\bigcup B_1 \times \ldots \times B_n\right) \times A_{n+1} \\
&= \bigcup \left((B_1 \cap A_{n+1}) \cup \left(B_1 \cap A_{n+1}^{\mathrm{c}}\right)\right) \times \ldots \times (B_n \cap A_{n+1}) \cup \left(B_n \cap A_{n+1}^{\mathrm{c}}\right) \times B_{n+1}.
\end{aligned}$$

□

We give now the factorial power of the superposition of counting measures.

Lemma 14.E.5. Factorial power of superposition of counting measures. *Let $(\mathbb{G}, \mathcal{G})$ be a measurable space, $\mu_1, \ldots, \mu_m$ be counting measures on $\mathbb{G}$ and $\mu = \mu_1 + \ldots + \mu_m$. Then for any $k \in \mathbb{N}^*$ and any $B_1, \ldots, B_k \in \mathcal{G}$,*

$$\mu^{(k)}(B_1 \times \ldots \times B_k) = \sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^{m} \mu_l^{(|I_l|)} \left(\prod_{u \in I_l} B_u\right),$$

where the summation is over all partitions $\{I_1, \ldots, I_m\}$ of $\{1, \ldots, n\}$ and $|I_l|$ is the cardinal of I_l.

Proof. Let $\mu_i = \sum_{j \in \mathbb{Z}} \delta_{x_{l,j}}$ $(i = 1, \ldots, m)$ and $\mu = \sum_{j \in \mathbb{Z}} \delta_{x_j}$. Then

$$\begin{aligned}
\mu^{(k)}(B_1 \times \ldots \times B_k) &= \sum_{(j_1,\ldots,j_k) \in \mathbb{Z}^{(k)}} \mathbf{1}\{x_{j_1} \in B_1\} \ldots \mathbf{1}\{x_{j_k} \in B_k\} \\
&= \sum_{\substack{i_1,\ldots,i_k \in \{1,\ldots,m\} \\ j_1,\ldots,j_k \in \mathbb{Z} \\ ((i_1,j_1),\ldots,(i_k,j_k)) \in (\mathbb{Z}^2)^{(k)}}} \mathbf{1}\{x_{i_1,j_1} \in B_1\} \ldots \mathbf{1}\{x_{i_k,j_k} \in B_k\} \\
&= \sum_{\{I_1,\ldots,I_m\}} \sum_{\substack{(j_u^1)_{u \in I_1} \in \mathbb{Z}^{|I_1| \neq} \\ \vdots \\ (j_u^k)_{u \in I_1} \in \mathbb{Z}^{|I_k| \neq}}} \prod_{l=1}^{m} \prod_{u \in I_l} \mathbf{1}\{x_{l,j_u^l} \in B_u\} \\
&= \sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^{m} \sum_{(j_u^l)_{u \in I_l} \in \mathbb{Z}^{|I_l| \neq}} \prod_{u \in I_l} \mathbf{1}\{x_{l,j_u^l} \in B_u\} \\
&= \sum_{\{I_1,\ldots,I_m\}} \prod_{l=1}^{m} \mu_l^{(|I_l|)} \left(\prod_{u \in I_l} B_u\right),
\end{aligned}$$

where the summation $\sum_{\{I_1,\ldots,I_m\}}$ is over all partitions $\{I_1,\ldots,I_m\}$ of $\{1,\ldots,n\}$ and $\mathbb{Z}^{n\neq}$ is the set of n-tuples $(j_1,\ldots,j_n)\in\mathbb{Z}^n$ where $j_1,\ldots,j_n$ are pairwise distinct. □

14.F Equality of measures

The following lemma will be useful.

Lemma 14.F.1. *Let $(\mathbb{G},\mathcal{G})$ be a measurable space, $k\in\mathbb{N}^*$ and μ and η be two σ-finite measures on $\mathbb{G}^k$ such that for any bounded measurable function $\varphi:\mathbb{G}\to\mathbb{R}_+$,*

$$\int_{\mathbb{G}^k}\left(\prod_{i=1}^k\varphi(x_i)\right)\,\mu(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k)=\int_{\mathbb{G}^k}\left(\prod_{i=1}^k\varphi(x_i)\right)\,\eta(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k)<\infty.$$

Then $\mu=\eta$.

Proof. Let $t_1,\ldots,t_k\in\mathbb{R}_+$, let $h_1,\ldots,h_k:\mathbb{G}\to\mathbb{R}_+$ be bounded measurable functions and let

$$\varphi(x):=\sum_{j=1}^k t_jh_j(x),\quad x\in\mathbb{G}.$$

Observe that

$$\prod_{i=1}^k\varphi(x_i)=\sum_{j_1,\ldots,j_k=1}^k t_{j_1}h_{j_1}(x_1)\ldots t_{j_k}h_{j_k}(x_k).$$

Then

$$\begin{aligned}&\int_{\mathbb{G}^k}\left(\prod_{i=1}^k\varphi(x_i)\right)\,\mu(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k)\\&=\sum_{j_1,\ldots,j_k=1}^k t_{j_1}\ldots t_{j_k}\int_{\mathbb{G}^k}h_{j_1}(x_1)\ldots h_{j_k}(x_k)\,\mu(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k),\end{aligned}$$

with an analogous equality for η. The right-hand side of the above equality is a multinomial on $t_1,\ldots,t_k$. The equality of this multinomial with that corresponding to η implies the equality of their coefficients. In particular,

$$\begin{aligned}&\int_{\mathbb{G}^k}h_1(x_1)\ldots h_k(x_k)\,\mu(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k)\\&\qquad=\int_{\mathbb{G}^k}h_1(x_1)\ldots h_k(x_k)\,\eta(\mathrm{d}x_1\times\ldots\times\mathrm{d}x_k).\end{aligned}$$

In particular, for any $B_1,\ldots,B_k\in\mathcal{G}$, applying the above equality for $h_j(x)=\mathbf{1}_{B_j}(x)$ $(j\in\{1,\ldots,k\},x\in\mathbb{G})$ implies

$$\mu(B_1\times\ldots B_k)=\eta(B_1\times\ldots B_k).$$

Since the measurable rectangles form a π-system generating $\mathcal{G}\otimes\ldots\otimes\mathcal{G}$, then $\mu=\eta$ by [11, Theorem 10.3 p.163]. □

14.G Symmetric complex Gaussian random variables

Definition 14.G.1. Symmetric complex Gaussian.

(i) *A complex random variable Z is said to be* Gaussian *if its real and imaginary part are jointly Gaussian. If moreover, $Z \stackrel{\text{dist.}}{=} e^{i\theta} Z$ for any $\theta \in \mathbb{R}$, then Z is said to be* symmetric complex Gaussian *[50, §1.4 p.13] (also known as* circularly symmetric Gaussian *[38, Definition 3.7.1]). If moreover its variance equals 1, it is called a* standard complex Gaussian.

(ii) *A complex random vector $Z = (Z_1, \ldots, Z_m)$ is said to be* Gaussian *if the real and imaginary parts of $Z_1, \ldots, Z_m$ are jointly Gaussian. If moreover, $Z \stackrel{\text{dist.}}{=} e^{i\theta} Z$ for any $\theta \in \mathbb{R}$, then Z is said to be* symmetric complex Gaussian random vector.

(iii) *A family of complex random variables is called a (*symmetric*)* complex Gaussian *if any finite subfamily is (symmetric) complex Gaussian vector. In particular, a complex stochastic process $\{Z(i)\}_{i \in I}$, where I is an arbitrary index set, is called a (*symmetric*)* complex Gaussian *iff it is a (symmetric) complex Gaussian family.*

If Z is a symmetric complex Gaussian random vector, then it is centered. Moreover, the distribution of Z is characterized by its covariance matrix $\Gamma_Z = \mathbf{E}[ZZ^*]$ (cf. [38, Theorem 3.7.13]), and we denote $Z \stackrel{\text{dist.}}{\sim} \mathcal{CN}(0, \Gamma_Z)$. The moments of the products of the components of such random vector are given by the following Wick's formula.

Lemma 14.G.2. Wick's formula. *Let $n \in \mathbb{N}^*$ and let $(Z_1, \ldots, Z_n, W_1, \ldots, W_n)$ be a symmetric complex Gaussian random vector. Then*

$$\mathbf{E}[Z_1 \ldots Z_n W_1^* \ldots W_n^*] = \operatorname{per}\left(\mathbf{E}\left[Z_i W_j^*\right]\right)_{1 \le i,j \le n}. \tag{14.G.1}$$

Proof. Cf. [49, Lemma 2.1.7]. □

Recall that the space $L^2_{\mathbb{C}}(\mathbf{P}, \Omega)$ of complex square-integrable random variables is a Hilbert space by the Riesz-Fischer theorem [22, Theorem 4.8 p.93].

Definition 14.G.3. Generated Hilbert subspace *[21, Definition 3.2.2]. Let $\{Z(i)\}_{i \in I}$ be a collection of random variables in $L^2_{\mathbb{C}}(\mathbf{P}, \Omega)$, where I is an arbitrary index set. The closure of the vector space of finite linear complex combinations of elements of $\{Z(i)\}_{i \in I}$ is a Hilbert subspace of $L^2_{\mathbb{C}}(\mathbf{P}, \Omega)$ called the* Hilbert subspace generated by $\{Z(i)\}_{i \in I}$. *More explicitly, this is the space of all limits in the quadratic mean (that is, limits in $L^2_{\mathbb{C}}(\mathbf{P}, \Omega)$) of some sequence of finite complex linear combinations of elements of $\{Z(i)\}_{i \in I}$.*

We will also need the following extension of [50, §1.4 p.13], [21, Theorem 3.2.3] to the complex case.

Lemma 14.G.4. *Let $\{Z(i)\}_{i\in I}$ be a (symmetric) complex Gaussian stochastic process in $L^2_{\mathbb{C}}(\mathbf{P},\Omega)$. Then the Hilbert subspace generated by $\{Z(i)\}_{i\in I}$ is a (symmetric) complex Gaussian family.*

Proof. Let $\mathcal{H}$ be the Hilbert subspace generated by $\{Z(i)\}_{i\in I}$ and let $Y = (Y_1,\ldots,Y_k) \in \mathcal{H}^k$. (i) **Complex Gaussian**. We have to show that for all $a \in \mathbb{C}^k$, $X := \sum_{j=1}^k a_j Y_j$ is complex Gaussian. By definition of $\mathcal{H}$, for each $j \in \{1,\ldots,k\}$, there exists a sequence $\{Y_j^{(n)}\}_{n\geq 1}$ such that: for all $n \geq 1$, $Y_j^{(n)}$ is a finite linear combination of elements of $\{Z(i)\}_{i\in I}$; and $\lim_{n\to\infty} Y_j^{(n)} = Y_j$ in $L^2_{\mathbb{C}}(\mathbf{P},\Omega)$. For a given $a \in \mathbb{C}^k$, the variables $X^{(n)} := \sum_{j=1}^k a_j Y_j^{(n)}$ are complex Gaussian and converge as $n \to \infty$ in quadratic mean to X. Hence $X = \sum_{j=1}^k a_j Y_j$ is complex Gaussian by [21, Example 2.2.1]. This being true for any $a \in \mathbb{C}^k$, it follows that $Y = (Y_1,\ldots,Y_k)$ is a complex Gaussian random vector. (ii) **Symmetric complex Gaussian**. We have to prove that Y is symmetric complex Gaussian. To do so, it is enough to prove that $\mathbf{E}\left[YY^{\mathrm{T}}\right] = 0$; cf. [38, Theorem 3.7.6]. Let $Y^{(n)} = \left(Y_1^{(n)},\ldots,Y_k^{(n)}\right)$. Observe that

$$\begin{aligned}
\left|\mathbf{E}\left[Y_i^{(n)}Y_j^{(n)}\right] - \mathbf{E}\left[Y_iY_j\right]\right| &= \Big|\mathbf{E}\left[\left(Y_i^{(n)} - Y_i\right)\left(Y_j^{(n)} - Y_j\right)\right] \\
&\quad + \mathbf{E}\left[\left(Y_i^{(n)} - Y_i\right)Y_j\right] + \mathbf{E}\left[Y_i\left(Y_j^{(n)} - Y_j\right)\right]\Big| \\
&\leq \left|\mathbf{E}\left[\left(Y_i^{(n)} - Y_i\right)\left(Y_j^{(n)} - Y_j\right)\right]\right| \\
&\quad + \left|\mathbf{E}\left[\left(Y_i^{(n)} - Y_i\right)Y_j\right]\right| + \left|\mathbf{E}\left[Y_i\left(Y_j^{(n)} - Y_j\right)\right]\right|.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we see that the right-hand side of the above inequality goes to 0 as $n \to \infty$. Then $\mathbf{E}\left[Y^{(n)}Y^{(n)\mathrm{T}}\right]$ converges componentwise to $\mathbf{E}\left[YY^{\mathrm{T}}\right]$. Since $\mathbf{E}\left[Y^{(n)}Y^{(n)\mathrm{T}}\right] = 0$, then $\mathbf{E}\left[YY^{\mathrm{T}}\right] = 0$. $\square$

Chapter 15

Useful results in algebra

15.A Matrices

Notation 15.A.1. Submatrices. *Let* $A = (A_{ij})_{1\leq i,j\leq n}$ *be a complex matrix. For any nonempty* $\gamma, \beta \subset \{1, \ldots, n\}$, *let* $A_{\gamma\beta} = (A_{ij})_{i\in\gamma, j\in\beta}$ *and* $A_\gamma = (A_{ij})_{i,j\in\gamma}$ *where indexes are considered in the increasing order. By convension* $\det(A_\emptyset) = 1$.

Note that $\det(A_\gamma)$ does not depend on the order in which the index set γ is considered.

15.A.1 Inequalities

The following lemma gives inequalities for the determinants of Hermitian nonnegative-definite matrices.

Lemma 15.A.2. Hadamard, Fischer and Koteljanskii inequalities. *Let* $A = (A_{ij})_{1\leq i,j\leq n}$ *be a Hermitian nonnegative-definite matrix, where* $n \in \mathbb{N}^*$. *Then the following results hold true.*

(i) Hadamard's inequality*:*

$$0 \leq \det(A) \leq \prod_{i=1}^{n} A_{ii}. \tag{15.A.1}$$

(ii) Fischer's inequality*:*

$$\det(A) \leq \det(A_\gamma)\det(A_{\gamma^c}), \tag{15.A.2}$$

where $\gamma \subset \{1, \ldots, n\}$ *and* $A_\gamma = (A_{ij})_{i,j\in\gamma}$ *with the convension* $\det(A_\emptyset) = 1$.

(iii) Koteljanskii's inequality*:*

$$\det\left(A_{\gamma\cup\beta}\right)\det\left(A_{\gamma\cap\beta}\right) \le \det\left(A_\gamma\right)\det\left(A_\beta\right), \tag{15.A.3}$$

where $\gamma,\beta \subset \{1,\ldots,n\}$*. (The above inequality is called the* Hadamard-Fischer inequality *by some authors.)*

(iv)

$$|A_{ij}|^2 \le A_{ii}A_{jj}, \quad \textit{for any } i,j \in \{1,\ldots,n\}. \tag{15.A.4}$$

Proof. (i) By [48, Theorem 7.8.1], $\det(A) \le \prod_{i=1}^{n} A_{ii}$. On the other hand, by [48, p.398], when A is positive definite $\det(A) > 0$. If A is only nonnegative-definite, then for any $\epsilon > 0$, $A+\epsilon I$ is positive definite where I is the identity matrix. Then $\det(A+\epsilon I) > 0$. Letting $\epsilon \to 0$ and invoking the continuity of the determinant implies $\det(A) \ge 0$. (ii) Inequality (15.A.2) is proved in [48, Theorem 7.8.3] when A is positive definite. Extend the result for a nonnegative-definite matrix A as in (i). (iii) Cf [100, Theorem 2.2] when A is positive definite and extend the result for a nonnegative-definite matrix A as in (i). (iv) Every principal sub matrix of A is Hermitian nonnegative-definite by [48, Observation 7.1.2 p.397]. In particular, the 2-by-2 principal sub matrix

$$\begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{pmatrix}$$

is Hermitian nonnegative-definite. Thus its determinant is nonnegative which give the announced inequality. □

15.A.2 Schur complement

The following lemma relates the determinant of a matrix (partitioned into blocks) to the so-called Schur complement.

Lemma 15.A.3. Schur complement formula*. Let* $A = (A_{ij})_{1\le i,j\le n}$ *be a complex matrix. For any nonempty* $\gamma,\beta \subset \{1,\ldots,n\}$*, let* $A_{\gamma\beta} = (A_{ij})_{i\in\gamma,j\in\beta}$ *and* $A_\gamma = (A_{ij})_{i,j\in\gamma}$*.*

(i) Consider a nonempty $\gamma \subset \{1,\ldots,n\}$ *(strictly) such that* A_γ *is invertible. Then*

$$\det(A) = \det(A_\gamma)\det\left(A_{\gamma^c} - A_{\gamma^c\gamma}(A_\gamma)^{-1}A_{\gamma\gamma^c}\right). \tag{15.A.5}$$

where $(A_\gamma)^{-1}$ *is the inverse of the matrix* A_γ*. The above formula is called* Schur complement formula*. The matrix*

$$B = A_{\gamma^c} - A_{\gamma^c\gamma}A_\gamma^{-1}A_{\gamma\gamma^c} \tag{15.A.6}$$

is called the Schur complement *of* A_γ *in* A*.*

If A is invertible, then so is B with inverse

$$B^{-1} = \left(A^{-1}\right)_{\gamma^c}, \tag{15.A.7}$$

and

$$\det\left(\left(A^{-1}\right)_{\gamma^c}\right) = \frac{\det(A_\gamma)}{\det(A)}. \tag{15.A.8}$$

(ii) Assume that $A_{11} \neq 0$ and let $B = (B_{ij})_{2\leq i,j\leq n}$ be the matrix defined by

$$B_{ij} = \frac{1}{A_{11}} \det\begin{pmatrix} A_{ij} & A_{i1} \\ A_{1j} & A_{11} \end{pmatrix} = A_{ij} - \frac{A_{i1}A_{1j}}{A_{11}}, \quad 2 \leq i,j \leq n.$$

Then

$$\det(A) = A_{11} \times \det(B).$$

Proof. (i) Cf. [48, §0.8.5 p.22] for the Schur complement formula (15.A.5). Assume now that A is invertible. The fact that B is invertible with inverse given by (15.A.7) follows from the inverse of a partitioned matrix [48, §0.7.3 p.18]. Combining (15.A.5) and (15.A.7) implies (15.A.8). (ii) This follows from (15.A.5) with $\gamma = \{1\}$. (Note that B is indeed the Schur complement of A_{11} in A.) □

15.A.3 Diagonal expansion of the determinant

Lemma 15.A.4. Diagonal expansion of the determinant. *Let $A = (A_{ij})_{1\leq i,j\leq n}$ be a complex matrix. Then the following results hold.*

(i) Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then

$$\sum_{\beta\subset\{1,\ldots,n\}} \det(A_\beta)\det(\Lambda_{\beta^c}) = \det(A + \Lambda), \tag{15.A.9}$$

where $A_\beta = (A_{ij})_{i,j\in\beta}$.

(ii) For any $\gamma \subset \{1, \ldots, n\}$,

$$\sum_{\beta\subset\gamma} \det(A_{\beta\cup\gamma^c}) = \det(A + \Gamma), \tag{15.A.10}$$

where the matrix Γ is defined by $\Gamma_{ij} = \mathbf{1}\{i = j \in \gamma\}$ for $1 \leq i,j \leq n$.

(iii)

$$\sum_{\beta\subset\{1,\ldots,n\}} \det(A_\beta) = \det(A + I). \tag{15.A.11}$$

Proof. (i) Cf. [27, Equation (2.1)]. (ii) Note that for any $\beta \subset \{1, \ldots, n\}$, $\det(\Gamma_{\beta^c}) = \mathbf{1}\{\beta^c \subset \gamma\} = \mathbf{1}\{\gamma^c \subset \beta\} = \mathbf{1}\{\beta = \gamma^c \cup \alpha : \alpha \subset \gamma\}$. Then applying (15.A.9) with $\Lambda := \Gamma$, we get

$$\begin{aligned}\det(A + \Gamma) &= \sum_{\beta \subset \{1,\ldots,n\}} \det(A_\beta) \det(\Gamma_{\beta^c}) \\ &= \sum_{\alpha \subset \gamma} \det(A_{\alpha \cup \gamma^c}).\end{aligned}$$

(iii) This follows from (15.A.10) with $\gamma = \{1, \ldots, n\}$. □

Remark 15.A.5. Bibliographic notes. *A direct proof of (15.A.10) is given in [61, Theorem 2.1].*

15.A.4 α-determinant of a matrix

Definition 15.A.6. α-determinant *[97]. For all matrices $A = (A_{ij})_{1 \le i,j \le k} \in \mathbb{C}^{k \times k}$ and all $\alpha \in \mathbb{R}$, we define the* α-determinant *of A by*

$$\det_\alpha(A) = \sum_{\pi \in S_k} \alpha^{k - \mathrm{cyc}(\pi)} \prod_{i=1}^{k} A_{i\pi(i)}, \tag{15.A.12}$$

where S_k is the set of permutations of $\{1, \ldots, k\}$ and $\mathrm{cyc}(\pi)$ denotes the number of cycles in π.

For $\alpha = -1$, we retrieve the usual determinant of A and for $\alpha = 1$, we get the permanent *of A; that is*

$$\det(A) = \det_{-1}(A), \quad \mathrm{per}(A) = \det_1(A).$$

Moreover, by convension $0^0 = 1$,

$$\det_0(A) = \prod_{i=1}^{k} A_{ii}.$$

In the particular case of a one dimensional matrix $A \in \mathbb{C}$, we have $\det_\alpha(A) = A$ for any $\alpha \in \mathbb{R}$. For the identity matrix I, $\det_\alpha(I) = 1$ for any $\alpha \in \mathbb{R}$.

The following lemma compares the permanent and the determinant of a nonnegative-definite matrix.

Lemma 15.A.7. *Let $A = (A_{ij})_{1 \le i,j \le k} \in \mathbb{C}^{k \times k}$ be a Hermitian nonnegative-definite matrix. Then*

$$\mathrm{per}(A) \ge \prod_{i=1}^{k} A_{ii} \ge \det(A) \ge 0.$$

Proof. The first inequality is proved in [66]. Hadamard's inequality (15.A.1) concludes the proof. □

15.B Power series composition

The following lemma concerns the composition of two power series; i.e., the substitution of a power series in another power series.

Lemma 15.B.1. Power series composition (substitution). *Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and assume that the power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ have radii of convergence R_A and R_B respectively. Assume moreover that $b_0 = 0$. Then $A(B(z))$ admits a power series expansion*

$$A(B(z)) = \sum_{n=0}^{\infty} c_n z^n,$$

which converges for any $z \in \mathbb{C}$ such that $|z| < R_B$ and $\sum_{n=0}^{\infty} |b_n z^n| < R_A$, with

$$c_n = \sum_{k=0}^{\infty} a_k b_{nk},, \quad n \in \mathbb{N}.$$

where b_{nk} is the coefficient of z^n in the expansion of $B(z)^k$. In particular, if $R_A = \infty$, then the radius of convergence of the power series $A(B(z)) = \sum_{n=0}^{\infty} c_n z^n$ is at least R_B.

Proof. Recall that a power series is absolutely convergent within the interior of its disc of convergence [59, Proposition 1.1.1 p.1]. Observe that $B(z)^k$ has a radius of convergence at least R_B by [59, Proposition 1.1.4 p.4], thus

$$B(z)^k = \sum_{n=0}^{\infty} b_{nk} z^n, \quad z \in \mathbb{C} : |z| < R_B.$$

Similarly, let b'_{nk} be the coefficient of z^n in the expansion of $\left(\sum_{n=0}^{\infty} |b_n| z^n\right)^k$; that is

$$\left(\sum_{n=0}^{\infty} |b_n| z^n\right)^k = \sum_{n=0}^{\infty} b'_{nk} z^n, \quad z \in \mathbb{C} : |z| < R_B.$$

Obviously,

$$|b_{nk}| \leq b'_{nk}, \quad n, k \in \mathbb{N}.$$

Let $z \in \mathbb{C}$ be such that $|z| < R_B$ and $Z := \sum_{n=0}^{\infty} |b_n z^n| < R_A$. Since Z is in the interior of the disc of convergence of $A(z)$, then $\sum_{k=0}^{\infty} \left|a_k Z^k\right| < \infty$. Moreover,

$$\begin{aligned}
\sum_{k=0}^{\infty} \left|a_k Z^k\right| &= \sum_{k=0}^{\infty} |a_k| \left(\sum_{n=0}^{\infty} |b_n z^n|\right)^k \\
&= \sum_{k=0}^{\infty} |a_k| \left(\sum_{n=0}^{\infty} b'_{nk} |z|^n\right) \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} b'_{nk} |a_k z^n|.
\end{aligned}$$

On the other hand, since $|B(z)| \leq Z < R_A$, then

$$\begin{aligned} A(B(z)) &= \sum_{k=0}^{\infty} a_k B(z)^k \\ &= \sum_{k=0}^{\infty} a_k \left(\sum_{n=0}^{\infty} b_{nk} z^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k b_{nk} \right) z^n, \end{aligned}$$

where the last equality is due to Fubini's theorem and the fact that

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |b_{nk} a_k z^n| \leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} b'_{nk} |a_k z^n| < \infty.$$

□

Chapter 16

Useful results in functional analysis

16.A Integral operators

We shall introduce the notion of integral operators which is a special class of linear operators defined on the space L^2 of square-integrable functions. We firstly recall the definition and basic properties of the space L^2.

16.A.1 L^2 space properties

Definition 16.A.1. Space L^2 of square-integrable functions. *Let $(\mathbb{G}, \mathcal{G}, \mu)$ be a measure space. We denote by $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ the set of measurable functions $f : \mathbb{G} \to \mathbb{C}$ which are square-integrable with respect to μ. Two functions $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$ such that $f(x) = g(x)$, for μ-almost all $x \in \mathbb{G}$ are said to be* equal in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$.

The space $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ equipped with the inner product

$$\langle f, g\rangle = \int_{\mathbb{G}} f(x)\, g(x)^*\, \mu(\mathrm{d}x), \quad f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G}),$$

is a Hilbert space by the Riesz-Fischer theorem [22, Theorem 4.8 p.93]. The norm *in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is denoted by*

$$\|f\| = \left(\int_{\mathbb{G}} |f(x)|^2\, \mu(\mathrm{d}x)\right)^{1/2}, \quad f \in L^2_{\mathbb{C}}(\mu, \mathbb{G}).$$

Recall that a Hilbert space is called *separable* if it contains a countable dense subset. We give now sufficient conditions for the space $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ to be separable.

Lemma 16.A.2. Separability of L^2. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$.*

(i) If $\mathcal{G}$ is countably-generated *(i.e., admits a countable basis), then the space $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable.*

(ii) If $\mathbb{G}$ is a separable metric space, then the Borel σ-algebra $\mathcal{B}(\mathbb{G})$ generated by the metric is countably-generated.

(iii) If $\mathbb{G}$ is l.c.s.h *(i.e., locally compact, second countable, and Hausdorff), then $\mathbb{G}$ is Polish (i.e., complete separable metric space) and $\mathcal{B}(\mathbb{G})$ is countably-generated.*

Proof. (i) Cf. [26, Proposition 3.4.5 p.110]. (ii) Cf. [56, Proposition 12.1 p.73]. (iii) The fact that $\mathbb{G}$ is Polish follows from [56, Theorem 5.3 p.29]. Then $\mathcal{B}(\mathbb{G})$ is countably-generated by (ii). □

Recall that a separable Hilbert space admits at least one denumerable orthonormal basis [20, Theorem C2.3 p.150].

Lemma 16.A.3. L^2 for product measure. *For $i \in \{i, 2\}$, let μ_i be a σ-finite measure on measurable spaces $(\mathbb{G}_i, \mathcal{G}_i)$ and assume that $L^2_{\mathbb{C}}(\mu_i, \mathbb{G}_i)$ is separable with orthonormal basis $\left\{\varphi_n^{(i)}\right\}_{n\in\mathbb{N}^*}$. Then $L^2_{\mathbb{C}}(\mu_1 \times \mu_2, \mathbb{G}_1 \times \mathbb{G}_2)$ is separable with orthonormal basis $\left\{\varphi_n^{(1)}\varphi_m^{(2)}\right\}_{n,m\in\mathbb{N}^*}$ (where $\varphi_n^{(1)}\varphi_m^{(2)}$ is the mapping defined on $\mathbb{G}_1 \times \mathbb{G}_2$ by $(x_1, x_2) \mapsto \varphi_n^{(1)}(x_1)\varphi_m^{(2)}(x_2)$).*

Proof. It is easy to check that $\left\{\varphi_n^{(1)}\varphi_m^{(2)}\right\}_{n,m\in\mathbb{N}^*}$ is an orthonormal family in $L^2_{\mathbb{C}}(\mu_1 \times \mu_2, \mathbb{G}_1 \times \mathbb{G}_2)$. Let $\mathcal{H}$ be the subspace of $L^2_{\mathbb{C}}(\mu_1 \times \mu_2, \mathbb{G}_1 \times \mathbb{G}_2)$ generated by $\left\{\varphi_n^{(1)}\varphi_m^{(2)}\right\}_{n,m\in\mathbb{N}^*}$ and let $f \in L^2_{\mathbb{C}}(\mu_1 \times \mu_2, \mathbb{G}_1 \times \mathbb{G}_2)$. It follows from [20, Theorem C2.1(c) p.146] that the projection of f on $\mathcal{H}$ is

$$\tilde{f} = \sum_{n,m\in\mathbb{N}^*} \left\langle f, \varphi_n^{(1)}\varphi_m^{(2)}\right\rangle \varphi_n^{(1)}\varphi_m^{(2)}.$$

Let $g = f - \tilde{f}$, then, for any $n, m \in \mathbb{N}^*$, $\left\langle g, \varphi_n^{(1)}\varphi_m^{(2)}\right\rangle = 0$, i.e.

$$\int_{\mathbb{G}_1\times\mathbb{G}_2} g(x, y)\,\varphi_n^{(1)}(x)^*\,\varphi_m^{(2)}(y)^*\,\mu_1(\mathrm{d}x)\,\mu_2(\mathrm{d}y) = 0.$$

This implies by Fubini's theorem [44, §36 Theorem C] (which is justified by the Cauchy-Schwarz inequality),

$$\int_{\mathbb{G}_2}\left(\int_{\mathbb{G}_1} g(x, y)\,\varphi_n^{(1)}(x)^*\,\mu(\mathrm{d}x)\right)\varphi_m^{(2)}(y)^*\,\mu(\mathrm{d}y) = 0.$$

Since $\left\{\varphi_m^{(2)}\right\}_{m\in\mathbb{N}^*}$ is a basis of $L^2_{\mathbb{C}}(\mu_2, \mathbb{G}_2)$, it follows that

$$\int_{\mathbb{G}_1} g(x, y)\,\varphi_n^{(1)}(x)^*\,\mu(\mathrm{d}x) = 0,$$

for all $y \in \mathbb{G}_2 \backslash B_n$ for some $B_n \in \mathcal{G}_2$ with $\mu(B_n) = 0$. Then for all $y \in \mathbb{G}_2 \backslash \bigcup_{k \in \mathbb{N}^*} B_k$, the above equality holds for any $n \in \mathbb{N}^*$, which implies $g(x, y) = 0$ for μ-almost all $x \in \mathbb{G}_1$. Thus $g(x, y) = 0$ for $\mu_1 \times \mu_2$-almost all $(x, y) \in \mathbb{G}_1 \times \mathbb{G}_2$. Therefore, $f = \tilde{f}$ in $L^2_{\mathbb{C}}(\mu_1 \times \mu_2, \mathbb{G}_1 \times \mathbb{G}_2)$. □

Remark 16.A.4. Bibliographic notes. *The proof of Lemma 16.A.3 is from [82, p.51].*

The following lemma gives a sufficient condition for equality in $L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$.

Lemma 16.A.5. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$ and let $K, \tilde{K} \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$. Assume that, for any $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,*

$$\int_{\mathbb{G}^2} K(x, y) f(y) g(x)^* \mu(\mathrm{d}x) \mu(\mathrm{d}y) = \int_{\mathbb{G}^2} \tilde{K}(x, y) f(y) g(x)^* \mu(\mathrm{d}x) \mu(\mathrm{d}y).$$

Then $\tilde{K}(x, y) = K(x, y)$ for μ^2-almost all $(x, y) \in \mathbb{G}^2$.

Proof. (i) Assume first that the measure μ is finite. Observe first that $K, \tilde{K} \in L^1_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$ by the Cauchy-Schwarz inequality and that $\mathbf{1}_{\mathbb{G}} \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Let

$$\mathcal{D} = \{C \in \mathcal{G} \otimes \mathcal{G} : \int_C \left(K(x, y) - \tilde{K}(x, y)\right) \mu(\mathrm{d}x) \mu(\mathrm{d}y) = 0\}.$$

Observe that $\mathcal{D}$ is a Dynkin system on $\mathbb{G}^2$; i.e.

$$\mathbb{G}^2 \in \mathcal{D},$$

$$C \in \mathcal{D} \Rightarrow C^{\mathrm{c}} \in \mathcal{D},$$

and for every nondecreasing sequence $\{C_n\}_{n \in \mathbb{N}}$ of sets in $\mathcal{D}$, we have

$$\lim_{n \to \infty} C_n \in \mathcal{D},$$

by the dominated convergence theorem. Let $\mathcal{I}$ be the family of finite union of rectangles $A \times B$ where $A, B \in \mathcal{G}$. Since $\mathcal{D}$ contains the π-system $\mathcal{I}$, then $\mathcal{D}$ contains $\sigma(\mathcal{I}) = \mathcal{G} \otimes \mathcal{G}$ by the Dynkin's theorem [11, Theorem 3.2 p.42].
(ii) Assume now that the measure μ is σ-finite. Let $\{B_n\}_{n \in \mathbb{N}}$ a nondecreasing sequence of sets in $\mathcal{G}$, such that $\bigcup_{n \in \mathbb{N}} B_n = \mathbb{G}$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$. Since μ restricted to B_n is finite, then $\tilde{K}(x, y) = K(x, y)$ for μ^2-almost all $(x, y) \in B_n^2$. Since $\bigcup_{n \in \mathbb{N}} B_n = \mathbb{G}$, then $\tilde{K}(x, y) = K(x, y)$ for μ^2-almost all $(x, y) \in \mathbb{G}^2$. □

16.A.2 Linear operators

We remind here basic definitions for linear operators on normed and Hilbert spaces with scalar field $\mathbb{R}$ or $\mathbb{C}$.

Definition 16.A.6. Bounded linear operator. *A linear operator $\mathcal{K}$ on a normed vector space $\mathcal{H}$ with norm $\|\cdot\|$ is called* bounded *if*

$$\sup\{\|\mathcal{K}f\| : f \in \mathcal{H}, \|f\| = 1\} < \infty.$$

In this case, the above supremum is called norm *of $\mathcal{K}$ and denoted by $\|\mathcal{K}\|$.*

Definition 16.A.7. *Let $\mathcal{K}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.*

(i) The adjoint operator *of $\mathcal{K}$, denoted by $\mathcal{K}^*$, is the unique bounded linear operator on $\mathcal{H}$ satisfying*

$$\langle \mathcal{K}f, g\rangle = \langle f, \mathcal{K}^* g\rangle, \quad f, g \in \mathcal{H}.$$

(For existence and uniqueness of the adjoint see [86, §12.9 p.311].)

(ii) $\mathcal{K}$ is called Hermitian *if $\mathcal{K}^* = \mathcal{K}$.*

(iii) Assume that $\mathcal{K}$ is Hermitian. $\mathcal{K}$ is called nonnegative-definite *if $\langle \mathcal{K}f, f\rangle \geq 0$ for any $f \in \mathcal{H}$.*

(iv) $\mathcal{K}$ is called compact *if the image by $\mathcal{K}$ of the unit ball is relatively compact; see. [82, Definition p.199].*

16.A.3 Integral operator basics

We now define integral operators.

Definition 16.A.8. Integral operator. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$ and let $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. For any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$, let $\mathcal{K}_{\mathbb{G}} f$ be the function defined on $\mathbb{G}$ by*

$$\mathcal{K}_{\mathbb{G}} f(x) = \int_{\mathbb{G}} K(x, y) f(y) \mu(\mathrm{d}y), \tag{16.A.1}$$

for all $x \in \mathbb{G}$ such that the integral in the right-hand side is well defined and finite; and $\mathcal{K}_{\mathbb{G}} f(x) = 0$ otherwise. The operator $\mathcal{K}_{\mathbb{G}}$ defined above is the integral operator *associated to the* kernel *K on $\mathbb{G}$.*

Here are the basic properties of the integral operator.

Lemma 16.A.9. Integral operator basic properties. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$ and let $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. Then the following results hold.*

(i) For any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$, the function $\mathcal{K}_{\mathbb{G}} f$ defined by (16.A.1) is in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Moreover, the equality (16.A.1) holds for μ-almost all $x \in \mathbb{G}$.

(ii) The linear operator $\mathcal{K}_{\mathbb{G}}$ on $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is bounded (and therefore continuous) with norm satisfying

$$\|\mathcal{K}_{\mathbb{G}}\| \leq \|K\|, \tag{16.A.2}$$

where $\|K\| = \left(\int_{\mathbb{G}^2} |K(x,y)|^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y)\right)^{1/2}$.

(iii) For any $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\langle \mathcal{K}_{\mathbb{G}} f, g\rangle = \int_{\mathbb{G}^2} K(x,y) f(y) g(x)^* \mu(\mathrm{d}x)\mu(\mathrm{d}y). \tag{16.A.3}$$

(iv) Let $\tilde{K} \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$ and $\tilde{\mathcal{K}}_{\mathbb{G}}$ be the corresponding integral operator. Then $\tilde{\mathcal{K}}_{\mathbb{G}} = \mathcal{K}_{\mathbb{G}}$ iff $\tilde{K}(x,y) = K(x,y)$ for μ^2-almost all $(x,y) \in \mathbb{G}^2$.

(v) Let $\mathcal{K}^*_{\mathbb{G}}$ be the adjoint operator of $\mathcal{K}_{\mathbb{G}}$. Then $\mathcal{K}^*_{\mathbb{G}}$ is an integral operator associated to the kernel $\tilde{K}$ defined by

$$\tilde{K}(x,y) := K(y,x)^*, \quad (x,y) \in \mathbb{G}^2.$$

(vi) $\mathcal{K}_{\mathbb{G}}$ is Hermitian iff $K(x,y) = K(y,x)^*$ for μ^2-almost all $(x,y) \in \mathbb{G}^2$.

Proof. (i) Note that, for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\begin{aligned}\int_{\mathbb{G}} |\mathcal{K}_{\mathbb{G}} f(x)|^2 \mu(\mathrm{d}x) &\leq \int_{\mathbb{G}} \left(\int_{\mathbb{G}} |K(x,y) f(y)| \mu(\mathrm{d}y)\right)^2 \mu(\mathrm{d}x) \\ &\leq \int_{\mathbb{G}} \left(\int_{\mathbb{G}} |K(x,y)|^2 \mu(\mathrm{d}y)\right) \left(\int_{\mathbb{G}} |f(y)|^2 \mu(\mathrm{d}y)\right) \mu(\mathrm{d}x) \\ &= \|K\|^2 \|f\|^2 < \infty,\end{aligned}$$

where the second line follows from the Cauchy-Schwarz inequality and the third line is due to Fubini-Tonelli theorem [44, §36 Theorem B]. It follows that $\mathcal{K}_{\mathbb{G}} f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Moreover, the above inequality shows that, for μ-almost all $x \in \mathbb{G}$, $\int_{\mathbb{G}} |K(x,y) f(y)| \mu(\mathrm{d}y) < \infty$ and therefore, equality (16.A.1) holds true. (ii) The above inequality shows that, for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$, $\|\mathcal{K}_{\mathbb{G}} f\| \leq c\|f\|$, where $c = \left(\int_{\mathbb{G}^2} |K(x,y)|^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y)\right)^{1/2}$. Then $\mathcal{K}_{\mathbb{G}}$ is bounded and $\|\mathcal{K}_{\mathbb{G}}\| \leq c$. (iii) For any $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\begin{aligned}\langle \mathcal{K}_{\mathbb{G}} f, g\rangle &= \int_{\mathbb{G}} \mathcal{K}_{\mathbb{G}} f(x) g(x)^* \mu(\mathrm{d}x) \\ &= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} K(x,y) f(y) \mu(\mathrm{d}y)\right) g(x)^* \mu(\mathrm{d}x) \\ &= \int_{\mathbb{G}^2} K(x,y) f(y) g(x)^* \mu(\mathrm{d}x)\mu(\mathrm{d}y),\end{aligned}$$

where the third equality is due to Fubini's theorem which is justified since by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_{\mathbb{G}^2} |K(x,y) f(y) g(x)| \mu(\mathrm{d}x) \mu(\mathrm{d}y) \\
&\leq \left(\int_{\mathbb{G}^2} |K(x,y)|^2 \mu(\mathrm{d}x) \mu(\mathrm{d}y) \right)^{1/2} \left(\int_{\mathbb{G}^2} |f(y) g(x)|^2 \mu(\mathrm{d}x) \mu(\mathrm{d}y) \right)^{1/2} \\
&= \|K\| \|f\| \|g\| < \infty. \qquad (16.A.4)
\end{aligned}$$

(iv) *Sufficiency.* Assume that $\tilde{K}(x,y) = K(x,y)$ for μ^2-almost all $(x,y) \in \mathbb{G}^2$. Then, for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\begin{aligned}
\int_{\mathbb{G}} \left| \left(\mathcal{K}_{\mathbb{G}} - \tilde{\mathcal{K}}_{\mathbb{G}} \right) f(x) \right|^2 \mu(\mathrm{d}x) &= \int_{\mathbb{G}} \left| \int_{\mathbb{G}} \left(K(x,y) - \tilde{K}(x,y) \right) f(y) \mu(\mathrm{d}y) \right|^2 \mu(\mathrm{d}x) \\
&\leq \int_{\mathbb{G}} \left(\int_{\mathbb{G}} \left| K(x,y) - \tilde{K}(x,y) \right|^2 \mu(\mathrm{d}y) \right) \left(\int_{\mathbb{G}} |f(y)|^2 \mu(\mathrm{d}y) \right) \mu(\mathrm{d}x) \\
&= \|f\|^2 \left(\int_{\mathbb{G}^2} \left| K(x,y) - \tilde{K}(x,y) \right|^2 \mu^2(\mathrm{d}x \times \mathrm{d}y) \right) = 0,
\end{aligned}$$

where the second line follows from the Cauchy-Schwarz inequality and the third line is due to Fubini-Tonelli theorem. Thus $\mathcal{K}_{\mathbb{G}} f = \tilde{\mathcal{K}}_{\mathbb{G}} f$ in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Therefore, $\tilde{\mathcal{K}}_{\mathbb{G}} = \mathcal{K}_{\mathbb{G}}$. *Necessity.* Assume that $\tilde{\mathcal{K}}_{\mathbb{G}} = \mathcal{K}_{\mathbb{G}}$. Then for any $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\langle \mathcal{K}_{\mathbb{G}} f, g \rangle = \left\langle \tilde{\mathcal{K}}_{\mathbb{G}} f, g \right\rangle,$$

thus by (16.A.3),

$$\int_{\mathbb{G}^2} K(x,y) f(y) g(x)^* \mu(\mathrm{d}x) \mu(\mathrm{d}y) = \int_{\mathbb{G}^2} \tilde{K}(x,y) f(y) g(x)^* \mu(\mathrm{d}x) \mu(\mathrm{d}y).$$

Therefore, $\tilde{K}(x,y) = K(x,y)$ for μ^2-almost all $(x,y) \in \mathbb{G}^2$ by Lemma 16.A.5.
(v) Let $\tilde{\mathcal{K}}_{\mathbb{G}}$ be the integral operator associated to the kernel $\tilde{K}(x,y) := K(y,x)^*$. It follows from (16.A.3) that for any $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\begin{aligned}
\left\langle \tilde{\mathcal{K}}_{\mathbb{G}} f, g \right\rangle &= \int_{\mathbb{G}^2} \tilde{K}(x,y) f(y) g(x)^* \mu(\mathrm{d}x) \mu(\mathrm{d}y) \\
&= \int_{\mathbb{G}^2} K(y,x)^* f(y) g(x)^* \mu(\mathrm{d}x) \mu(\mathrm{d}y) \\
&= \left(\int_{\mathbb{G}^2} K(y,x) f(y)^* g(x) \mu(\mathrm{d}x) \mu(\mathrm{d}y) \right)^* \\
&= \left(\int_{\mathbb{G}^2} K(x,y) g(y) f(x)^* \mu(\mathrm{d}x) \mu(\mathrm{d}y) \right)^* \\
&= \left(\langle \mathcal{K}_{\mathbb{G}} g, f \rangle \right)^* = \langle f, \mathcal{K}_{\mathbb{G}} g \rangle,
\end{aligned}$$

where the fourth equality is due to the change of variable $(y,x) \to (x,y)$ and Fubini's theorem which is justified by (16.A.4). It follows from the above equality that $\tilde{\mathcal{K}}_{\mathbb{G}}$ is the adjoint operator of $\mathcal{K}_{\mathbb{G}}$. (vi) This is immediate from (iv) and (v). $\square$

The following lemma shows that the composition of integral operators is also an integral operator and gives its kernel.

Lemma 16.A.10. Integral operators composition. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$, $K_1, \ldots, K_N \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$ for some integer $N \geq 2$, and $\mathcal{K}_1, \ldots, \mathcal{K}_N$ the integral operators associated respectively to the kernels $K_1, \ldots, K_N$. Then the following results hold.*

(i) The composition $\mathcal{K}_1 \ldots \mathcal{K}_N$ is an integral operator associated to the kernel K defined by

$$K(x,y) = \int_{\mathbb{G}^{N-1}} K_1(x,z_1)K_2(z_1,z_2)\ldots K_N(z_{N-1},y)\mu(\mathrm{d}z_1)\ldots\mu(\mathrm{d}z_{N-1}), \tag{16.A.5}$$

for all $(x,y) \in \mathbb{G}^2$ such that the integral in the right-hand side is well defined and finite (which holds for μ^2-almost all $(x,y) \in \mathbb{G}^2$). Moreover,

$$\|K\| \leq \|K_1\| \ldots \|K_N\|, \tag{16.A.6}$$

where $\|\cdot\|$ is the norm in $L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$.

(ii)

$$\int_{\mathbb{G}^N} |K_1(x,z_1)K_2(z_1,z_2)\ldots K_N(z_{N-1},x)|\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}z_1)\ldots\mu(\mathrm{d}z_{N-1}) < \infty. \tag{16.A.7}$$

In particular, the integral in the right-hand side of (16.A.5) is well defined and finite for $x = y$ for μ-almost all $x \in \mathbb{G}$.

(iii) Assume that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable, let K be defined by (16.A.5), and $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Let

$$\alpha_{n,m} = \left\langle K, \varphi_n(x)\,\varphi_m(y)^*\right\rangle = \int_{\mathbb{G}^2} K(x,y)\,\varphi_n(x)^*\,\varphi_m(y)\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}y), \quad n,m \in \mathbb{N}^*,$$

$$\alpha^{(i)}_{n,m} = \left\langle K_i, \varphi_n(x)\,\varphi_m(y)^*\right\rangle, \quad i \in \{1,\ldots,n\}, n,m \in \mathbb{N}^*.$$

Then

$$\alpha_{n,m} = \sum_{j_1,\ldots,j_{N-1}\in\mathbb{N}^*} \alpha^{(1)}_{n,j_1}\alpha^{(2)}_{j_1,j_2}\ldots\alpha^{(N)}_{j_{N-1},m}, \quad n,m \in \mathbb{N}^*,$$

where the series in the right-hand side converges absolutely.

Proof. (i) The proof is by induction on $N \geq 2$. Besides the announced result in the lemma, we shall prove that, for any $N \geq 2$,

$$\int_{\mathbb{G}^2} \left(\int_{\mathbb{G}^{N-1}} |K_1(x, z_1) \dots K_N(z_{N-1}, z)| \, \mu(\mathrm{d}z_1) \dots \mu(\mathrm{d}z_{N-1}) \right)^2 \mu(\mathrm{d}x) \, \mu(\mathrm{d}z) \\ \leq \|K_1\|^2 \dots \|K_N\|^2 . \tag{16.A.8}$$

Step 1: Proof for $N = 2$. We first prove (16.A.8); indeed,

$$\begin{aligned} &\int_{\mathbb{G}^2} \left(\int_{\mathbb{G}} |K_1(x, z) \, K_2(z, y)| \, \mu(\mathrm{d}z) \right)^2 \mu(\mathrm{d}x) \, \mu(\mathrm{d}y) \\ &\leq \int_{\mathbb{G}^2} \left(\int_{\mathbb{G}} |K_1(x, z)|^2 \, \mu(\mathrm{d}z) \int_{\mathbb{G}} |K_2(z, y)|^2 \, \mu(\mathrm{d}z) \right) \mu(\mathrm{d}x) \, \mu(\mathrm{d}y) \\ &= \|K_1\|^2 \, \|K_2\|^2 < \infty, \end{aligned}$$

where the first inequality is due to Cauchy-Schwarz inequality and the equality is due to Fubini-Tonelli theorem. This proves that $\int_{\mathbb{G}} |K_1(x, z) \, K_2(z, y)| \, \mu(\mathrm{d}z) < \infty$ for μ^2-almost all $(x, y) \in \mathbb{G}^2$. Moreover, the above inequality shows that $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$ and $\|K\| \leq \|K_1\| \times \|K_2\|$. Let $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$. By Lemma 16.A.9(i), $K_2 f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$ and for μ-almost all $x \in \mathbb{G}$,

$$\begin{aligned} (\mathcal{K}_1 \mathcal{K}_2) f(x) &= \int_{\mathbb{G}} K_1(x, z) \, K_2 f(z) \, \mu(\mathrm{d}z) \\ &= \int_{\mathbb{G}} K_1(x, z) \left(\int_{\mathbb{G}} K_2(z, y) \, f(y) \, \mu(\mathrm{d}y) \right) \mu(\mathrm{d}z) \\ &= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} K_1(x, z) \, K_2(z, y) \, \mu(\mathrm{d}z) \right) f(y) \, \mu(\mathrm{d}y), \end{aligned}$$

where the third equality is due to Fubini's theorem. (Indeed, by the Cauchy-Schwarz inequality

$$\begin{aligned} &\left[\int_{\mathbb{G}^2} |K_1(x, z) \, K_2(z, y) \, f(y)| \, \mu(\mathrm{d}z) \, \mu(\mathrm{d}y) \right]^2 \\ &= \left[\int_{\mathbb{G}} |K_1(x, z)| \left(\int_{\mathbb{G}} |K_2(z, y) \, f(y)| \, \mu(\mathrm{d}y) \right) \mu(\mathrm{d}z) \right]^2 \\ &\leq \left[\int_{\mathbb{G}} |K_1(x, z)|^2 \, \mu(\mathrm{d}z) \right] \left[\int_{\mathbb{G}} \left(\int_{\mathbb{G}} |K_2(z, y) \, f(y)| \, \mu(\mathrm{d}y) \right)^2 \mu(\mathrm{d}z) \right] \\ &\leq \left[\int_{\mathbb{G}} |K_1(x, z)|^2 \, \mu(\mathrm{d}z) \right] \left[\int_{\mathbb{G}} \left(\|f\|^2 \int_{\mathbb{G}} |K_2(z, y)|^2 \, \mu(\mathrm{d}y) \right) \mu(\mathrm{d}z) \right] \\ &= \int_{\mathbb{G}} |K_1(x, z)|^2 \, \mu(\mathrm{d}z) \, \|f\|^2 \, \|K_2\|^2 . \end{aligned}$$

Taking the integral with respect to $\mu(\mathrm{d}x)$ shows that the left-hand side of the above inequality is finite for μ-almost all $x \in \mathbb{G}$.) *Step 2:* Assume that the desired result holds for $N-1$ (where $N \geq 3$). We first prove (16.A.8); indeed,

$$\begin{aligned}
&\int_{\mathbb{G}^2}\left(\int_{\mathbb{G}^{N-1}} |K_1(x,z_1)K_2(z_1,z_2)\dots K_{N-1}(z_{N-2},z)K_N(z,y)|\right.\\
&\qquad\qquad\left.\times \mu(\mathrm{d}z_1)\dots\mu(\mathrm{d}z_{N-2})\,\mu(\mathrm{d}z)\right)^2 \mu(\mathrm{d}x)\,\mu(\mathrm{d}y)\\
&\leq \int_{\mathbb{G}^2}\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}^{N-2}} |K_1(x,z_1)\dots K_{N-1}(z_{N-2},z)|\,\mu(\mathrm{d}z_1)\dots\mu(\mathrm{d}z_{N-2})\right)^2 \mu(\mathrm{d}z)\right)\\
&\qquad\qquad\times\left(\int_{\mathbb{G}} |K_N(z,y)|^2\,\mu(\mathrm{d}z)\right)\mu(\mathrm{d}x)\,\mu(\mathrm{d}y)\\
&= \int_{\mathbb{G}^2}\left(\int_{\mathbb{G}^{N-2}} |K_1(x,z_1)\dots K_{N-1}(z_{N-2},z)|\,\mu(\mathrm{d}z_1)\dots\mu(\mathrm{d}z_{N-2})\right)^2\\
&\qquad\qquad\times\mu(\mathrm{d}x)\,\mu(\mathrm{d}z)\times\|K_N\|^2\\
&\leq \|K_1\|^2\dots\|K_{N-1}\|^{2\cdot}\times\|K_N\|^2,
\end{aligned}$$

where the last inequality follows from (16.A.8) for $N-1$. The composition $\mathcal{K}_1\dots\mathcal{K}_N = (\mathcal{K}_1\dots\mathcal{K}_{N-1})\,\mathcal{K}_N$ is an integral operator by Step 1 with kernel

$$K(x,y) = \int_{\mathbb{G}} K'(x,z)\,K_N(z,y)\,\mu(\mathrm{d}z), \tag{16.A.9}$$

where

$$K'(x,z) = \int_{\mathbb{G}^{N-2}} K_1(x,z_1)K_2(z_1,z_2)\dots K_{N-1}(z_{N-2},z)\mu(\mathrm{d}z_1)\dots\mu(\mathrm{d}z_{N-2}).$$

It follows from (16.A.8) that

$$\int_{\mathbb{G}^{N-1}} |K_1(x,z_1)K_2(z_1,z_2)\dots K_{N-1}(z_{N-2},z)K_N(z,y)|\,\mu(\mathrm{d}z_1)\dots\mu(\mathrm{d}z_{N-2})\,\mu(\mathrm{d}z) < \infty$$

for μ^2-almost all $(x,y)\in\mathbb{G}^2$. For any $(x,y)\in\mathbb{G}^2$ satisfying the above inequality, Fubini's theorem applied to (16.A.9) shows that (16.A.5) holds. (ii) *Step 1: Proof for* $N=2$. By Cauchy-Schwarz inequality,

$$\int_{\mathbb{G}^2} |K_1(x,z)\,K_2(z,x)|\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}z) \leq \|K_1\|\,\|K_2\|.$$

Since the above quantity is finite, then $\int_{\mathbb{G}} |K_1(x,z)\,K_2(z,x)|\,\mu(\mathrm{d}z) < \infty$ for μ-almost all $x\in\mathbb{G}$. *Step 2: Proof for* $N\geq 3$. Let

$$K'(z,x) = \int_{\mathbb{G}^{N-2}} |K_2(z,z_2)\dots K_N(z_{N-1},x)|\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}z_2)\dots\mu(\mathrm{d}z_{N-1}).$$

By Fubini-Tonelli theorem, the left-hand side of (16.A.7) equals

$$\int_{\mathbb{G}^2} |K_1(x,z)\,K'(z,x)|\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}z) \leq \|K_1\|\,\|K'\|,$$

by Cauchy-Schwarz inequality. Applying (16.A.6) to the kernels, $|K_1|, \ldots, |K_N|$, we get $\|K'\| \leq \|K_2\| \ldots \|K_N\|$, which combined with the above inequality permits to conclude. (iii) Lemma 16.A.3 shows that $\left\{\varphi_n(x)\varphi_m(y)^*\right\}_{n,m\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$. Then $\alpha_{n,m}$ (resp. $\alpha^{(i)}_{n,m}$) are the coordinates of K (resp. K_i) in this basis. Moreover, by (16.A.3),

$$\langle \mathcal{K}\varphi_m, \varphi_n\rangle = \int_{\mathbb{G}^2} K(x,y)\varphi_m(y)\varphi_n(x)^* \mu(\mathrm{d}x)\mu(\mathrm{d}y) = \alpha_{n,m}. \tag{16.A.10}$$

Similarly,

$$\langle \mathcal{K}_i\varphi_m, \varphi_n\rangle = \alpha^{(i)}_{n,m}.$$

Therefore,

$$\langle \mathcal{K}_i^*\varphi_m, \varphi_n\rangle = \alpha^{(i)*}_{m,n}.$$

Thus

$$\mathcal{K}_i\varphi_m = \sum_{n\in\mathbb{N}^*} \alpha^{(i)}_{n,m}\varphi_n, \quad \mathcal{K}_i^*\varphi_m = \sum_{n\in\mathbb{N}^*} \alpha^{(i)*}_{m,n}\varphi_n, \tag{16.A.11}$$

We now prove the announced result by induction on N. Besides the announced result in the lemma, we shall prove that, for any $N \geq 2$,

$$\sum_{j_1,\ldots,j_{N-1}\in\mathbb{N}^*} \left|\alpha^{(1)}_{n,j_1}\alpha^{(2)}_{j_1,j_2}\ldots\alpha^{(N)}_{j_{N-1},m}\right| \leq \prod_{l=1}^{N}\|K_l\|. \tag{16.A.12}$$

Step 1: Proof for $N = 2$. By (16.A.10)

$$\begin{aligned}\alpha_{n,m} &= \langle \mathcal{K}_1\mathcal{K}_2\varphi_m, \varphi_n\rangle \\ &= \langle \mathcal{K}_2\varphi_m, \mathcal{K}_1^*\varphi_n\rangle = \sum_{j_1\in\mathbb{N}^*}\alpha^{(1)}_{n,j_1}\alpha^{(2)}_{j_1,m}.\end{aligned}$$

where the last equality is due to (16.A.11). Moreover, by Cauchy-Schwarz inequality,

$$\sum_{j_1\in\mathbb{N}^*}\left|\alpha^{(1)}_{n,j_1}\alpha^{(2)}_{j_1,m}\right| \leq \left(\sum_{j_1\in\mathbb{N}^*}\left|\alpha^{(1)}_{n,j_1}\right|^2\right)^{1/2}\left(\sum_{j_1\in\mathbb{N}^*}\left|\alpha^{(2)}_{j_1,m}\right|^2\right)^{1/2} \leq \|K_1\|\,\|K_2\|,$$

which proves (16.A.12) for $N = 2$. *Step 2:* Assume that the desired result holds for $N-1$ (where $N \geq 3$). We begin by proving (16.A.12). Let K' be the kernel associated to $\mathcal{K}_2\ldots\mathcal{K}_N$. Again by Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{j_1,\ldots,j_{N-1}\in\mathbb{N}^*}\left|\alpha^{(1)}_{n,j_1}\alpha^{(2)}_{j_1,j_2}\ldots\alpha^{(N)}_{j_{N-1},m}\right| &\leq \left(\sum_{j_1\in\mathbb{N}^*}\left|\alpha^{(1)}_{n,j_1}\right|^2\right)^{1/2}\left(\sum_{j_1,\ldots,j_{N-1}\in\mathbb{N}^*}\left|\alpha^{(2)}_{j_1,j_2}\ldots\alpha^{(N)}_{j_{N-1},m}\right|^2\right)^{1/2} \\ &\leq \|K_1\|\,\|K'\| \\ &\leq \|K_1\|\,\|K_2\|\ldots\|K_N\|,\end{aligned}$$

where the last inequality is due to (16.A.6). Now, by (16.A.10),

$$\begin{aligned}\alpha_{n,m} &= \langle \mathcal{K}_1 \dots \mathcal{K}_N \varphi_m, \varphi_n \rangle \\ &= \langle \mathcal{K}_2 \dots \mathcal{K}_N \varphi_m, \mathcal{K}_1^* \varphi_n \rangle \\ &= \sum_{j_1,\dots,j_{N-1} \in \mathbb{N}^*} \alpha_{n,j_1}^{(1)} \sum_{j_2,\dots,j_{N-1} \in \mathbb{N}^*} \alpha_{j_2,j_3}^{(2)} \dots \alpha_{j_{N-1},m}^{(N)} \\ &= \sum_{j_1,\dots,j_{N-1} \in \mathbb{N}^*} \alpha_{n,j_1}^{(1)} \alpha_{j_1,j_2}^{(2)} \dots \alpha_{j_{N-1},m}^{(N)},\end{aligned}$$

where the last equality is justified by Fubini's theorem and (16.A.12). □

The following proposition gives a sufficient condition for the integral operator to be compact.

Proposition 16.A.11. Integral operator compactness. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$, $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$, and $\mathcal{K}_{\mathbb{G}}$ be the associated integral operator. Assume that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable. Then the following results hold.*

(i) The operator $\mathcal{K}_{\mathbb{G}}$ is compact.

(ii) The set of eigenvalues of $\mathcal{K}_{\mathbb{G}}$ is at most countable, and has at most one accumulation point, namely, 0. Moreover, each non-zero eigenvalue has finite multiplicity.

Proof. (i) Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Lemma 16.A.3 shows that $\{\varphi_n(x)\varphi_m(y)^*\}_{n,m\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. Letting $\alpha_{n,m} = \langle K, \varphi_n(x)\varphi_m(y)^* \rangle$, we get

$$K(x, y) = \sum_{n,m\in\mathbb{N}^*} \alpha_{n,m}\varphi_n(x)\varphi_m(y)^*,$$

where equality is in $L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. For each $N \in \mathbb{N}^*$, let

$$K^{(N)}(x, y) = \sum_{n,m=1}^{N} \alpha_{n,m}\varphi_n(x)\varphi_m(y)^*,$$

and let $\mathcal{K}^{(N)}_{\mathbb{G}}$ be the integral operator associated to the kernel $K^{(N)}$ which is bounded by (16.A.2). Observe that, for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\begin{aligned}\mathcal{K}^{(N)}_{\mathbb{G}} f(x) &= \sum_{n,m=1}^{N} \alpha_{n,m}\varphi_n(x) \int_{\mathbb{G}} f(y)\varphi_m(y)^* \mu(\mathrm{d}y) \\ &= \sum_{n,m=1}^{N} \alpha_{n,m} \langle f, \varphi_m \rangle \varphi_n(x),\end{aligned}$$

then $\mathcal{K}_{\mathbb{G}}^{(N)} f = \sum_{n,m=1}^{N} \alpha_{n,m} \langle f, \varphi_m \rangle \varphi_n$ which shows that the operator $\mathcal{K}_{\mathbb{G}}^{(N)}$ has finite rank. Then $\mathcal{K}_{\mathbb{G}}^{(N)}$ is a compact operator by [82, Example p.199]. Since $\left\| K - K^{(N)} \right\| \to 0$ as $N \to \infty$, it follows from (16.A.2) that $\left\| \mathcal{K}_{\mathbb{G}} - \mathcal{K}_{\mathbb{G}}^{(N)} \right\| \to 0$ as $N \to \infty$. Therefore, $\mathcal{K}_{\mathbb{G}}$ is compact by [82, Theorem VI.12 p.200]. (ii) This follows from the fact that the space $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is a separable Hilbert space and the operator $\mathcal{K}_{\mathbb{G}}$ is compact and invoking Riesz-Schauder theorem [82, Theorem VI.15 p.203]. □

Remark 16.A.12. Bibliographic notes. *Proposition 16.A.11(i) may be deduced from [82, Theorem VI.22(e) p.210] and [82, Theorem VI.23 p.210] (the assumptions that the measure μ is σ-finite and the space $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable are implicit there). Our proof follows theirs.*

Proposition 16.A.11(i) is stated in [23, Theorem 2.3.2 p.168] under the additional condition that $\mathbb{G}$ is locally compact which is not required as the above proof shows.

We now focus our attention on Hermitian integral operators.

Proposition 16.A.13. Hermitian integral operators. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$, let $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$, and let $\mathcal{K}_{\mathbb{G}}$ be the associated integral operator. Assume that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable, and $K(x, y) = K(y, x)^*$ for μ^2-almost all $(x, y) \in \mathbb{G}^2$ (or, equivalently, the operator $\mathcal{K}_{\mathbb{G}}$ is Hermitian). Then the following results hold.*

(i) There exists an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors $\{\varphi_n\}_{n \in \mathbb{N}^*}$ of $\mathcal{K}_{\mathbb{G}}$ with respective real eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}^*}$ such that $\lambda_n \to 0$ as $n \to \infty$.

(ii) For any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\mathcal{K}_{\mathbb{G}} f = \sum_{n \in \mathbb{N}^*} \langle f, \varphi_n \rangle \lambda_n \varphi_n, \tag{16.A.13}$$

where equality is in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$.

(iii)

$$K(x, y) = \sum_{n \in \mathbb{N}^*} \lambda_n \varphi_n(x) \varphi_n(y)^*, \tag{16.A.14}$$

where the equality is in $L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$; that is the the series in the right-hand side of (16.A.14) converges in $L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$ to K (in particular, the above equality holds for μ^2-almost all $(x, y) \in \mathbb{G}^2$).

(iv)

$$\|\mathcal{K}_{\mathbb{G}}\| = \sup_{n \in \mathbb{N}^*} |\lambda_n| .$$

(v) $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite iff $\lambda_n \geq 0$ for all $n \in \mathbb{N}^*$.

(vi) If $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite, then for any integer $k \geq 2$, the matrix $(K(x_i, x_j))_{1 \leq i,j \leq k}$ is Hermitian nonnegative-definite for μ^k-almost all $(x_1, \ldots, x_k) \in \mathbb{G}^k$.

(vii) Assume that there exists some constant $c \in \mathbb{R}_+^*$ such that

$$\int_{\mathbb{G}} |K(x,y)|^2 \mu(\mathrm{d}y) \leq c, \quad \text{for all } x \in \mathbb{G}. \tag{16.A.15}$$

Then for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\mathcal{K}_{\mathbb{G}} f(x) = \int_{\mathbb{G}} K(x,y) f(y) \mu(\mathrm{d}y) = \sum_{n \in \mathbb{N}^*} \langle f, \varphi_n \rangle \lambda_n \varphi_n(x), \quad \text{for any } x \in \mathbb{G}, \tag{16.A.16}$$

where the series in the right-hand side of the above equality converges uniformly in $x \in \mathbb{G}$.

Proof. (i) The fact that $\mathcal{K}_{\mathbb{G}}$ is Hermitian follows from Lemma 16.A.9(vi). Since $\mathcal{K}_{\mathbb{G}}$ is Hermitian, then its eigenvalues are real. Since $\mathcal{K}_{\mathbb{G}}$ is moreover compact by Proposition 16.A.11(i), the spectral theorem of Hermitian compact operators [82, Theorem VI.16p.203], shows the existence of a denumerable orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}^*}$ of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ with respective eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}^*}$ such that $\lambda_n \to 0$ as $n \to \infty$. (ii) Since $\{\varphi_n\}_{n \in \mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, then any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$ may be written as follows

$$f = \sum_{n \in \mathbb{N}^*} \langle f, \varphi_n \rangle \varphi_n.$$

Since the linear operator $\mathcal{K}_{\mathbb{G}}$ is bounded by Lemma 16.A.9(ii), then its is continuous. Thus

$$\mathcal{K}_{\mathbb{G}} f = \sum_{n \in \mathbb{N}^*} \langle f, \varphi_n \rangle \mathcal{K}_{\mathbb{G}} \varphi_n = \sum_{n \in \mathbb{N}^*} \langle f, \varphi_n \rangle \lambda_n \varphi_n.$$

(iii) By Lemma 16.A.3, $\{\varphi_n \varphi_m^*\}_{n,m \in \mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. Letting $\alpha_{n,m} = \langle K, \varphi_n \varphi_m^* \rangle$, we get

$$K(x,y) = \sum_{n,m \in \mathbb{N}^*} \alpha_{n,m} \varphi_n(x) \varphi_m(y)^*,$$

where equality is in $L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. Observe that

$$\begin{aligned}
\alpha_{n,m} &= \langle K, \varphi_n \varphi_m^* \rangle \\
&= \int_{\mathbb{G}^2} K(x,y) \varphi_n(x)^* \varphi_m(y) \mu(\mathrm{d}x) \mu(\mathrm{d}y) \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} K(x,y) \varphi_m(y) \mu(\mathrm{d}y) \right) \varphi_n(x)^* \mu(\mathrm{d}x) \\
&= \int_{\mathbb{G}} (\mathcal{K}_{\mathbb{G}} \varphi_m(x)) \varphi_n(x)^* \mu(\mathrm{d}x) \\
&= \lambda_m \mathbf{1}\{n = m\},
\end{aligned}$$

where the third equality is due to Fubini's theorem (which is justified by the Cauchy-Schwarz inequality). (iv) Let $\gamma = \sup_{n\in\mathbb{N}^*} |\lambda_n|$. Using (16.A.13) and invoking [21, Theorem 1.3.15(a)], it follows that that, for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\begin{aligned}\|\mathcal{K}_{\mathbb{G}} f\|^2 &= \sum_{n\in\mathbb{N}^*} \lambda_n^2 |\langle f, \varphi_n\rangle|^2 \\ &\le \gamma^2 \sum_{n\in\mathbb{N}^*} |\langle f, \varphi_n\rangle|^2 = \gamma^2 \|f\|^2 .\end{aligned}$$

Thus $\|\mathcal{K}_{\mathbb{G}}\| \le \gamma$. On the other hand, since $\gamma = \sup_{n\in\mathbb{N}^*} |\lambda_n|$, for any $\varepsilon \in \mathbb{R}^*_+$, there exists some $n \in \mathbb{N}^*$ such that $|\lambda_n| > \gamma - \varepsilon$, and therefore

$$\|\mathcal{K}_{\mathbb{G}}\| \ge \frac{\|\mathcal{K}_{\mathbb{G}}\varphi_n\|}{\|\varphi_n\|} = |\lambda_n| > \gamma - \varepsilon.$$

This being true for $\varepsilon \in \mathbb{R}^*_+$, it follows that $\|\mathcal{K}_{\mathbb{G}}\| \ge \gamma$. (v) Since $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, then, by [20, Equation (17) p.146] for any $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\langle g, f\rangle = \sum_{n\in\mathbb{N}^*} \langle g, \varphi_n\rangle \langle f, \varphi_n\rangle^* .$$

Let $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$, applying the above formula with $g = \mathcal{K}_{\mathbb{G}} f$ and invoking (16.A.13), we get

$$\langle \mathcal{K}_{\mathbb{G}} f, f\rangle = \sum_{n\in\mathbb{N}^*} \lambda_n |\langle f, \varphi_n\rangle|^2 .$$

The above quantity is nonnegative for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$ iff $\lambda_n \ge 0$ for all $n \in \mathbb{N}^*$. (vi) By (iii), there exists $\mathbb{G}_2 \subset \mathbb{G}^2$ with $\mu^2\left(\mathbb{G}^2 \backslash \mathbb{G}_2\right) = 0$ such that

$$K(x, y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x) \varphi_n(y)^*, \quad \text{for all } (x, y) \in \mathbb{G}_2. \tag{16.A.17}$$

For any $k \ge 2$, let

$$A_k = \left\{(x_1, \ldots, x_k) \in \mathbb{G}^k : (x_i, x_j) \in \mathbb{G}_2 \text{ for all } i \ne j\right\}.$$

Observe that $\mu^k\left(\mathbb{G}^k \backslash A_k\right) = 0$ by Lemma 14.A.1(ii). On the other hand, it follows from (16.A.17) that for any $(x_1, \ldots, x_k) \in A_k$,

$$(K(x_i, x_j))_{1\le i,j\le k} = \sum_{n\in\mathbb{N}^*} \lambda_n \left(\varphi_n(x_i) \varphi_n(x_j)^*\right)_{1\le i,j\le k} .$$

Since each matrix $\left(\varphi_n(x_i) \varphi_n(x_j)^*\right)_{1\le i,j\le k}$ is Hermitian nonnegative-definite and the eigenvalues $\{\lambda_n\}$ are nonnegative, then $(K(x_i, x_j))_{1\le i,j\le k}$ is Hermitian nonnegative-definite. (vii) Let $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$. For any $x \in \mathbb{G}$,

$$\begin{aligned}\left(\int_{\mathbb{G}} |K(x, y) f(y)| \mu(\mathrm{d}y)\right)^2 &\le \int_{\mathbb{G}} |K(x, y)|^2 \mu(\mathrm{d}y) \int_{\mathbb{G}} |f(y)|^2 \mu(\mathrm{d}y) \\ &\le c \|f\|^2 < \infty,\end{aligned}$$

where the second inequality follows from the Cauchy-Schwarz inequality. Then the integral in the right-hand side of (16.A.1) is well defined and finite for any $x \in \mathbb{G}$. This shows that the first equality in (16.A.16) holds for any $x \in \mathbb{G}$. Let

$$f_N(x) = \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n(x), \quad N \in \mathbb{N}^*, x \in \mathbb{G}.$$

Then, for any $x \in \mathbb{G}$,

$$\begin{aligned}
|\mathcal{K}_{\mathbb{G}} f(x) - \mathcal{K}_{\mathbb{G}} f_N(x)|^2 &= \left| \int_{\mathbb{G}} K(x,y) (f(y) - f_N(x)) \mu(\mathrm{d}y) \right|^2 \\
&\leq \int_{\mathbb{G}} |K(x,y)|^2 \mu(\mathrm{d}y) \int_{\mathbb{G}} |f(y) - f_N(x)|^2 \mu(\mathrm{d}y) \\
&\leq c \|f - f_N\|^2,
\end{aligned}$$

where the second inequality follows from the Cauchy-Schwarz inequality. This shows the uniform convergence of the series in the right-hand side of (16.A.16) to $\mathcal{K}_{\mathbb{G}} f(x)$. □

Remark 16.A.14. Bibliographic notes. *Proposition 16.A.11(i) is called* Hilbert-Schmidt theorem *in [23, Theorem 2.3.2 p.168]; whereas this name is used for Proposition 16.A.13(i) in [82, Theorem VI.16p.203]. Proposition 16.A.13(iii) is called* Schmidt's theorem *in [84, p.243].*

Proposition 16.A.13(vii) is an extension of [84, Theorem p.244] where $\mathbb{G}$ is the real line and μ is the Lebesgue measure.

16.A.4 Trace class operators

We now study the so-called *trace class* property of operators.

Definition 16.A.15. Trace of bounded, nonnegative-definite operators. *For any bounded, Hermitian, nonnegative-definite operator $\mathcal{K}$ on a separable Hilbert space $\mathcal{H}$, its* trace *is defined by*

$$\operatorname{tr}(\mathcal{K}) = \sum_{n \in \mathbb{N}^*} \langle \mathcal{K} \psi_n, \psi_n \rangle, \tag{16.A.18}$$

where $\{\psi_n\}_{n \in \mathbb{N}^*}$ is any orthonormal basis of $\mathcal{H}$. The series in the right-hand side of the above equality is independent of the choice of the basis; cf. [82, Theorem VI.18 p.206].

Definition 16.A.16. Trace class operator. *A bounded linear operator $\mathcal{K}$ on a separable Hilbert space $\mathcal{H}$ is called* trace class *when* $\operatorname{tr}\left(\sqrt{\mathcal{K}^* \mathcal{K}}\right) < \infty$*; or, equivalently, the series of its singular values (i.e., eigenvalues of $\sqrt{\mathcal{K}^* \mathcal{K}}$) is convergent; cf. [82, Definition p 207]. (Observe that the square root of the bounded nonnegative-definite operator $\mathcal{K}^* \mathcal{K}$ is well defined by [82, Theorem VI.9 p 196].)*

For any trace class operator $\mathcal{K}$, its trace *is defined by*

$$\operatorname{tr}(\mathcal{K}) = \sum_{n\in\mathbb{N}^*} \langle \mathcal{K}\psi_n, \psi_n \rangle, \tag{16.A.19}$$

where $\{\psi_n\}_{n\in\mathbb{N}^*}$ is any orthonormal basis of $\mathcal{H}$; cf. [82, Definition p.211]. The series in the right-hand side of the above equality is absolutely convergent and is independent of the choice of the basis; cf. [82, Theorem VI.24 p.211].

Lemma 16.A.17. *Let $\mathcal{K}$ and $\mathcal{L}$ be linear operators on a separable Hilbert space $\mathcal{H}$. Assume that $\mathcal{K}$ is trace class and $\mathcal{L}$ is bounded. Then $\mathcal{KL}$ and $\mathcal{LK}$ are trace class and $\operatorname{tr}(\mathcal{KL}) = \operatorname{tr}(\mathcal{LK})$.*

Proof. $\mathcal{KL}$ and $\mathcal{LK}$ are trace class by [82, Theorem VI.19(b) p.207] and they have equal traces by [82, Theorem VI.25(c) p.212]. □

We consider first the particular case of a Hermitian integral operator and show that its trace may easily be expressed in terms of its eigenvalues.

Corollary 16.A.18. Trace of Hermitian integral operator. *In the conditions of Proposition 16.A.13, the following results hold true.*

(i) The integral operator $\mathcal{K}_{\mathbb{G}}$ is trace class iff the series of its eigenvalues accounting for their multiplicities $\{\lambda_n\}_{n\in\mathbb{N}^*}$ is absolutely convergent; that is

$$\sum_{n\in\mathbb{N}^*} |\lambda_n| < \infty. \tag{16.A.20}$$

(ii) If $\mathcal{K}_{\mathbb{G}}$ is trace class, then

$$\operatorname{tr}(\mathcal{K}_{\mathbb{G}}) = \sum_{n\in\mathbb{N}^*} \lambda_n. \tag{16.A.21}$$

Proof. (i) By Proposition 16.A.13(i), the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$ of $\mathcal{K}_{\mathbb{G}}$ are real and there exists an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ composed of eigenvectors $\{\varphi_n\}_{n\in\mathbb{N}^*}$ of $\mathcal{K}_{\mathbb{G}}$. Then the singular values of $\mathcal{K}_{\mathbb{G}}$ are $\{|\lambda_n|\}_{n\in\mathbb{N}^*}$, thus

$$\operatorname{tr}\left(\sqrt{\mathcal{K}^*_{\mathbb{G}}\mathcal{K}_{\mathbb{G}}}\right) = \sum_{n\in\mathbb{N}^*} \langle \mathcal{K}_{\mathbb{G}}\varphi_n, \varphi_n\rangle = \sum_{n\in\mathbb{N}^*} |\lambda_n|,$$

where the first equality is due to (16.A.18). (ii) This follows from (16.A.19) by taking the $\{\psi_n\}_{n\in\mathbb{N}^*}$ there equal to the eigenvectors $\{\varphi_n\}_{n\in\mathbb{N}^*}$ of $\mathcal{K}_{\mathbb{G}}$. □

We express now the trace of an integral operator in terms of its kernel and give the trace of the composition of integral operators.

Lemma 16.A.19. Trace of composition of integral operators. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$. Assume that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable and let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$.*

(i) *Let $K \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$ and $\mathcal{K}$ be the associated integral operator assumed trace class. Then,*

$$\operatorname{tr}(\mathcal{K}) = \sum_{n \in \mathbb{N}^*} \int_{\mathbb{G}^2} K(x, y) \varphi_n(x)^* \varphi_n(y) \mu(\mathrm{d}x) \mu(\mathrm{d}y).$$

where the series is absolutely convergent.

(ii) *Let $K_1, \ldots, K_N \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$ for some integer $N \geq 2$, and $\mathcal{K}_1, \ldots, \mathcal{K}_N$ the integral operators associated respectively to the kernels $K_1, \ldots, K_N$. Assume that $\mathcal{K}_1, \ldots, \mathcal{K}_N$ are trace class. Then,*

$$\operatorname{tr}(\mathcal{K}_1 \ldots \mathcal{K}_N) = \int_{\mathbb{G}^N} K_1(x, z_1) K_2(z_1, z_2) \ldots K_N(z_{N-1}, x) \mu(\mathrm{d}x) \mu(\mathrm{d}z_1) \ldots \mu(\mathrm{d}z_{N-1}). \tag{16.A.22}$$

Proof. (i) Let

$$\alpha_{n,m} = \left\langle K, \varphi_n(x) \varphi_m(y)^* \right\rangle = \int_{\mathbb{G}^2} K(x, y) \varphi_n(x)^* \varphi_m(y) \mu(\mathrm{d}x) \mu(\mathrm{d}y), \quad n, m \in \mathbb{N}^*.$$

Observe from (16.A.3) that

$$\alpha_{n,m} = \langle \mathcal{K} \varphi_n, \varphi_m \rangle.$$

Since $\mathcal{K}$ is assumed trace class, then by (16.A.19),

$$\operatorname{tr}(\mathcal{K}) = \sum_{n \in \mathbb{N}^*} \langle \mathcal{K}_{\mathbb{G}} \varphi_n, \varphi_n \rangle = \sum_{n \in \mathbb{N}^*} \alpha_{n,n},$$

where the series is absolutely convergent by [82, Theorem VI.24 p.211]. (ii) By Lemma 16.A.10(i), the composition $\mathcal{K}_1 \ldots \mathcal{K}_N$ is an integral operator associated to the kernel K defined by (16.A.5). Moreover, by Lemma 16.A.17, $\mathcal{K}_1 \ldots \mathcal{K}_N$ is trace class. Then by Item (i),

$$\operatorname{tr}(\mathcal{K}) = \sum_{n \in \mathbb{N}^*} \alpha_{n,n},$$

where $\alpha_{n,m} = \left\langle K, \varphi_n(x) \varphi_m(y)^* \right\rangle$. By Lemma 16.A.10(iii),

$$\operatorname{tr}(\mathcal{K}) = \sum_{n \in \mathbb{N}^*} \sum_{j_1, \ldots, j_{N-1} \in \mathbb{N}^*} \alpha^{(1)}_{n,j_1} \alpha^{(2)}_{j_1,j_2} \ldots \alpha^{(N)}_{j_{N-1},n},$$

where $\alpha^{(i)}_{n,m} = \left\langle K_i, \varphi_n(x) \varphi_m(y)^* \right\rangle$ ($i \in \{1, \ldots, n\}, n, m \in \mathbb{N}^*$). *Step 1: Proof for $N = 2$.* The above display gives

$$\operatorname{tr}(\mathcal{K}) = \sum_{n \in \mathbb{N}^*} \sum_{j_1 \in \mathbb{N}^*} \alpha^{(1)}_{n,j_1} \alpha^{(2)}_{j_1,n}. \tag{16.A.23}$$

On the other hand,

$$K_i(x,y) = \sum_{n,m\in\mathbb{N}^*} \alpha^{(i)}_{n,m}\varphi_n(x)\varphi_m(y)^*,$$

where equality is in $L^2_{\mathbb{C}}(\mu^2,\mathbb{G}^2)$. In particular,

$$K_1(x,y) = \sum_{n,m\in\mathbb{N}^*} \alpha^{(1)}_{n,m}\varphi_n(x)\varphi_m(y)^*,$$

$$K_2(y,x) = \sum_{n,m\in\mathbb{N}^*} \alpha^{(2)}_{n,m}\varphi_n(y)\varphi_m(x)^* = \sum_{n,m\in\mathbb{N}^*} \alpha^{(2)}_{m,n}\varphi_n(x)^*\varphi_m(y),$$

where the second equality is to to the exchange notation of n and m. Then, taking the inner product of $K_1(x,y)$ and $\tilde{K}_2(x,y) := K_2(y,x)^*$, we get

$$\left\langle K_1,\tilde{K}_2\right\rangle = \int_{\mathbb{G}^2} K_1(x,y)K_2(y,x)\mu(\mathrm{d}x)\mu(\mathrm{d}y) = \sum_{n,m\in\mathbb{N}^*} \alpha^{(1)}_{n,m}\alpha^{(2)}_{m,n} = \mathrm{tr}(\mathcal{K}),$$

where the last equality is due to (16.A.23) and the fact that $\sum_{n,m\in\mathbb{N}^*}\left|\alpha^{(1)}_{n,m}\alpha^{(2)}_{m,n}\right| < \infty$ by Cauchy-Schwarz inequality. *Step 2: Proof for* $N \geq 3$. Let K' be the kernel associated to $\mathcal{K}' = \mathcal{K}_2\ldots\mathcal{K}_N$, then

$$\begin{aligned}
\mathrm{tr}(\mathcal{K}) &= \mathrm{tr}(\mathcal{K}_1\mathcal{K}') \\
&= \int_{\mathbb{G}^2} K_1(x,y)K'(y,x)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\
&= \int_{\mathbb{G}^2} K_1(x,y)\left(\int_{\mathbb{G}^{N-2}} K_2(y,z_1)K_3(z_1,z_2)\ldots K_N(z_{N-2},x)\mu(\mathrm{d}z_1)\ldots\mu(\mathrm{d}z_{N-2})\right)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\
&= \int_{\mathbb{G}^N} K_1(x,y)K_2(y,z_1)K_3(z_1,z_2)\ldots K_N(z_{N-2},x)\mu(\mathrm{d}z_1)\ldots\mu(\mathrm{d}z_{N-2})\mu(\mathrm{d}x)\mu(\mathrm{d}y),
\end{aligned}$$

where the second equality is due to Step 1, the third equality is due to (16.A.5), and the last equality is due to Fubini's theorem and (16.A.7). □

16.A.5 Hilbert-Schmidt operators

We adress now the following question: When is a linear operator on $L^2_{\mathbb{C}}(\mu,\mathbb{G})$ an integral operator?

Definition 16.A.20. Hilbert-Schmidt operator *[82, Definition p.210]. A bounded linear operator $\mathcal{K}$ on a separable Hilbert space $\mathcal{H}$ is called* Hilbert-Schmidt *if* $\mathrm{tr}(\mathcal{K}^*\mathcal{K}) < \infty$.

Lemma 16.A.21. *Let $\mathcal{K}$ be a linear operator on a separable Hilbert space $\mathcal{H}$.*

(i) If $\mathcal{K}$ is trace class, then it is Hilbert-Schmidt.

(ii) If $\mathcal{K}$ is Hilbert-Schmidt, then its is compact.

That is,

$$\{\textit{trace class operators on } \mathcal{H}\} \subset \{\textit{Hilbert-Schmidt operators on } \mathcal{H}\} \subset \{\textit{compact operators on } \mathcal{H}\}.$$

Proof. (i) Let $\{\nu_n\}_{n\in\mathbb{N}^*}$ be the singular values of $\mathcal{K}$, then

$$\operatorname{tr}(\mathcal{K}^*\mathcal{K}) = \sum_{n\in\mathbb{Z}} |\nu_n|^2 \leq \left(\sum_{n\in\mathbb{Z}} |\nu_n|\right)^2 = \left[\operatorname{tr}\left(\sqrt{\mathcal{K}^*\mathcal{K}}\right)\right]^2.$$

Assume that $\mathcal{K}$ is trace class. Then $\mathcal{K}$ is Hilbert-Schmidt by the above inequality.
(ii) Cf. [82, Theorem VI.22(e) p.210]. □

Lemma 16.A.22. Hilbert-Schmidt versus integral operator. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$ and assume that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable. Then a linear operator $\mathcal{K}$ on $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is Hilbert-Schmidt iff $\mathcal{K}$ is an integral operator associated to some kernel $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. Moreover, in this case*

$$\operatorname{tr}(\mathcal{K}^*\mathcal{K}) = \int_{\mathbb{G}^2} |K(x,y)|^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

Proof. Cf. [82, Theorem VI.23 p.210]. □

16.A.6 Restriction property

Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$ and let $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. For any $D \in \mathcal{G}$, we define the integral operator $\mathcal{K}_D$ on $L^2_{\mathbb{C}}(\mu, D)$ as in Definition 16.A.8 with D in place of $\mathbb{G}$. With a slight abuse of notation, we will consider $L^2_{\mathbb{C}}(\mu, D)$ as the set of functions in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ whose support is in D (i.e., functions defined on $\mathbb{G}$ which vanish outside D) and therefore, Equation (16.A.1) becomes, for any $f \in L^2_{\mathbb{C}}(\mu, D)$,

$$\mathcal{K}_D f(x) = \mathbf{1}_{\{x\in D\}} \int_D K(x,y) f(y)\mu(\mathrm{d}y), \quad x \in \mathbb{G}. \tag{16.A.24}$$

The following lemma shows how $\mathcal{K}_D$ is related to $\mathcal{K}_{\mathbb{G}}$.

Lemma 16.A.23. Restriction effect on integral operator. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$, let $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$, and let $\mathcal{K}_{\mathbb{G}}$ be the associated integral operator. Let $D \in \mathcal{G}$, let P_D be the* projection *from $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ to $L^2_{\mathbb{C}}(\mu, D)$ defined by, for any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,*

$$P_D f(x) = \mathbf{1}_{\{x\in D\}} f(x), \quad x \in \mathbb{G}, \tag{16.A.25}$$

and let $\mathcal{K}_D$ be the integral operator defined by (16.A.24). Then the following results hold.

(i) Restriction property:

$$\mathcal{K}_D = P_D \mathcal{K}_{\mathbb{G}} P_D. \tag{16.A.26}$$

(ii) In the conditions of Proposition 16.A.13, let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu,\mathbb{G})$ composed of eigenvectors of $\mathcal{K}_{\mathbb{G}}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding respective eigenvalues. Then for any $f\in L^2_{\mathbb{C}}(\mu,D)$,

$$\mathcal{K}_D f = \sum_{n\in\mathbb{N}^*} \langle f, P_D\varphi_n\rangle \lambda_n P_D\varphi_n,$$

where the equality is in $L^2_{\mathbb{C}}(\mu,D)$.

Proof. (i) For any $f\in L^2_{\mathbb{C}}(\mu,\mathbb{G})$, it follows from (16.A.1) that

$$\mathcal{K}_{\mathbb{G}} P_D f(x) = \int_{\mathbb{G}} K(x,y) P_D f(y)\mu(\mathrm{d}y) = \int_D K(x,y) f(y)\mu(\mathrm{d}y).$$

Then

$$P_D\mathcal{K}_{\mathbb{G}} P_D f(x) = \mathbf{1}_{\{x\in D\}} \int_D K(x,y) f(y)\mu(\mathrm{d}y) = \mathcal{K}_D f(x),$$

where the last equality is due to (16.A.24). (ii) Since $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mu,\mathbb{G})$, then for any $f\in L^2_{\mathbb{C}}(\mu,\mathbb{G})$,

$$f = \sum_{n\in\mathbb{N}^*} \langle f,\varphi_n\rangle \varphi_n, \quad \text{and} \quad \mathcal{K}_{\mathbb{G}} f = \sum_{n\in\mathbb{N}^*} \langle f,\varphi_n\rangle \lambda_n\varphi_n,$$

where the second equality is due to the fact that $\mathcal{K}_{\mathbb{G}}$ is a continuous operator (which follows from the fact that it is bounded by Lemma 16.A.9(ii)). In particular, replacing f by $P_D f$ in the above equalities, we get

$$P_D f = \sum_{n\in\mathbb{N}^*} \langle P_D f,\varphi_n\rangle \varphi_n, \quad \text{and } \mathcal{K}_{\mathbb{G}} P_D f = \sum_{n\in\mathbb{N}^*} \langle P_D f,\varphi_n\rangle \lambda_n\varphi_n.$$

Then

$$P_D\mathcal{K}_{\mathbb{G}} P_D f = \sum_{n\in\mathbb{N}^*} \langle P_D f,\varphi_n\rangle \lambda_n P_D\varphi_n.$$

For $f\in L^2_{\mathbb{C}}(\mu,D)$, since $\langle P_D f,\varphi_n\rangle = \langle f, P_D\varphi_n\rangle$, then

$$\mathcal{K}_D f = \sum_{n\in\mathbb{N}^*} \langle f, P_D\varphi_n\rangle \lambda_n P_D\varphi_n,$$

which concludes the proof. □

16.A.7 Canonical and pre-canonical kernels

We shall consider a particular class of kernels K, called *canonical*, which is defined below.

Definition 16.A.24. Canonical kernels. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$, $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be a sequence in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be nonnegative real numbers such that $\sum_{n\in\mathbb{N}^*} \lambda_n \|\varphi_n\|^2 < \infty$. Let $K : \mathbb{G}^2 \to \mathbb{C}$ be a measurable function such that*

$$K(x,y) = \begin{cases} \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x)\varphi_n(y)^* & \text{if } x, y \in \mathbb{G}_1, \\ 0 & \text{otherwise,} \end{cases} \tag{16.A.27}$$

where

$$\mathbb{G}_1 = \left\{ x \in \mathbb{G} : \sum_{n\in\mathbb{N}^*} \lambda_n |\varphi_n(x)|^2 < \infty \right\}. \tag{16.A.28}$$

Then K is called a pre-canonical kernel *associated to $\{\varphi_n\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$. If $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is moreover orthonormal in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, then K is called a* canonical kernel *associated to $\{\varphi_n\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$.*

By the Cauchy-Schwarz inequality, $K(x,y)$ is well defined and finite for any $x, y \in \mathbb{G}$. Note moreover, that for any $k \in \mathbb{N}^*$ and any $x_1, \ldots, x_k \in \mathbb{G}_1$, the matrix $(K(x_i, x_j))_{1\le i,j\le k}$ is Hermitian nonnegative-definite; as a sum of Hermitian nonnegative-definite matrices. (In fact, this holds true for all $x_1, \ldots, x_k \in \mathbb{G}$; cf. Exercise 5.8.1.)

We give below the basic properties of pre-canonical kernels.

Lemma 16.A.25. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$, $\{\varphi_n\}_{n\in\mathbb{N}^*}$ a sequence in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ and $\{\lambda_n\}_{n\in\mathbb{N}^*}$ nonnegative real numbers such that $\sum_{n\in\mathbb{N}^*} \lambda_n \|\varphi_n\|^2 < \infty$. Let $\mathbb{G}_1$ be given by (16.A.28) and K be the pre-canonical kernel defined by (16.A.27). Then the following results hold.*

(i) $\mu(\mathbb{G}\backslash\mathbb{G}_1) = 0$. *More generally,* $\mu^k(\mathbb{G}^k \backslash \mathbb{G}_1^k) = 0$ *for any* $k \in \mathbb{N}^*$.

(ii) The function defined on $\mathbb{G}$ by $x \mapsto K(x,x)$ is in $L^1_{\mathbb{C}}(\mu, \mathbb{G})$ and

$$\int_{\mathbb{G}} K(x,x)\mu(\mathrm{d}x) = \sum_{n\in\mathbb{N}^*} \lambda_n \|\varphi_n\|^2. \tag{16.A.29}$$

(iii) $\sum_{n\in\mathbb{N}^*} |\lambda_n \varphi_n(x)\varphi_n(y)| < \infty$ *for any* $x, y \in \mathbb{G}_1$.

(iv) The function defined on $\mathbb{G}^2$ by $(x,y) \mapsto \sum_{n\in\mathbb{N}^*} |\lambda_n \varphi_n(x)\varphi_n(y)|$ is in $L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. In particular, $K \in L^2_{\mathbb{C}}(\mu^2, \mathbb{G}^2)$. Moreover, for each $x \in \mathbb{G}_1$, the function defined on $\mathbb{G}$ by $y \mapsto \sum_{n\in\mathbb{N}^*} |\lambda_n \varphi_n(x)\varphi_n(y)|$ is in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ (the kernel K is then called a Carleman kernel*).*

(v) For any $N \in \mathbb{N}^*$, define the function $K^{(N)}$ on $\mathbb{G}^2$ by

$$K^{(N)}(x,y) = \sum_{n=1}^{N} \lambda_n \varphi_n(x)\varphi_n(y)^*, \quad x, y \in \mathbb{G}.$$

Then for all $k \in \mathbb{N}^*$ and all $x_1, \ldots, x_k \in \mathbb{G}$, the matrices

$$\left(K^{(N)}(x_i, x_j)\right)_{1 \le i,j \le k}$$

are Hermitian nonnegative-definite and $\det\left(K^{(N)}(x_i, x_j)\right)_{1 \le i,j \le k}$ *is non-decreasing with N. Moreover, for any $k \in \mathbb{N}^*$ and any $x_1, \ldots, x_k \in \mathbb{G}_1$,*

$$\lim_{N \to \infty} \det\left(K^{(N)}(x_i, x_j)\right)_{1 \le i,j \le k} = \det\left(K(x_i, x_j)\right)_{1 \le i,j \le k}. \tag{16.A.30}$$

(vi) For any $k \in \mathbb{N}^*$ and any $x_1, \ldots, x_k \in \mathbb{G}_1$,

$$\det\left(K(x_i, x_j)\right)_{1 \le i,j \le k} = \sum_{1 \le n_1 < \ldots < n_k} \lambda_{n_1} \ldots \lambda_{n_k} \left|\det\left(\varphi_{n_j}(x_i)\right)_{1 \le i,j \le k}\right|^2.$$

Proof. (i) Observe that

$$\int_{\mathbb{G}} \left(\sum_{n \in \mathbb{N}^*} \lambda_n \left|\varphi_n(x)\right|^2\right) \mu(\mathrm{d}x) = \sum_{n \in \mathbb{N}^*} \lambda_n \left\|\varphi_n\right\|^2 < \infty. \tag{16.A.31}$$

Then $\sum_{n \in \mathbb{N}^*} \lambda_n \left|\varphi_n(x)\right|^2 < \infty$ for μ-almost all $x \in \mathbb{G}$. That is, for $\mathbb{G}_1$ be given by (16.A.28) we have $\mu(\mathbb{G} \backslash \mathbb{G}_1) = 0$. Then $\mu^k\left(\mathbb{G}^k \backslash \mathbb{G}_1^k\right) = 0$ for any $k \in \mathbb{N}^*$ by Lemma 14.A.1(i). (ii) Observe that $K(x,x) = \sum_{n \in \mathbb{N}^*} \lambda_n \left|\varphi_n(x)\right|^2$. Then (16.A.31) permits to conclude. (iii) For any $x, y \in \mathbb{G}$, it follows from the Cauchy-Schwarz inequality that

$$\sum_{n \in \mathbb{N}^*} \left|\lambda_n \varphi_n(x) \varphi_n(y)\right| \le \left(\sum_{n \in \mathbb{N}^*} \lambda_n \left|\varphi_n(x)\right|^2\right)^{1/2} \left(\sum_{n \in \mathbb{N}^*} \lambda_n \left|\varphi_n(y)\right|^2\right)^{1/2}, \tag{16.A.32}$$

which is finite for $x, y \in \mathbb{G}_1$. (iv) Taking the square of the inequality (16.A.32) and then integrating over $\mathbb{G}^2$ with respect to $\mu(\mathrm{d}x)\mu(\mathrm{d}y)$, we get

$$\int_{\mathbb{G}^2} \left(\sum_{n \in \mathbb{N}^*} \left|\lambda_n \varphi_n(x) \varphi_n(y)\right|\right)^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y) \le \left(\sum_{n \in \mathbb{N}^*} \lambda_n \left\|\varphi_n\right\|^2\right)^2 < \infty.$$

Moreover, fixing some $x \in \mathbb{G}_1$, it follows again from (16.A.32) that

$$\int_{\mathbb{G}} \left(\sum_{n \in \mathbb{N}^*} \left|\lambda_n \varphi_n(x) \varphi_n(y)\right|\right)^2 \mu(\mathrm{d}y) \le \left(\sum_{n \in \mathbb{N}^*} \lambda_n \left|\varphi_n(x)\right|^2\right) \left(\sum_{n \in \mathbb{N}^*} \lambda_n \left\|\varphi_n\right\|^2\right),$$

which shows that the function $y \mapsto \sum_{n \in \mathbb{N}^*} \left|\lambda_n \varphi_n(x) \varphi_n(y)\right|$ is in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$. (v) Observe that, for any $x, y \in \mathbb{G}_1$, $K(x,y) = \lim_{N \to \infty} K^{(N)}(x,y)$ is well defined and finite by (iii). Let $k \in \mathbb{N}^*$ and $x_1, \ldots, x_k \in \mathbb{G}_1$. Then for any $i, j \in \{1, \ldots, k\}$, $\lim_{N \to \infty} K^{(N)}(x_i, x_j) = \det K(x_i, x_j)$. Since the determinant of a matrix is a multinomial of its components, then (16.A.30) follows. For any $(x_1, \ldots, x_k) \in \mathbb{G}^k$, the matrix $\left(K^{(N)}(x_i, x_j)\right)_{1\leq i,j\leq k}$ is Hermitian nonnegative-definite since it is the sum of the Hermitian nonnegative-definite matrices $\lambda_n \left(\varphi_n(x_i)\varphi_n(x_j)^*\right)_{1\leq i,j\leq k}$ $(n = 1, \ldots, N)$. Observe that, for all $(x_1, \ldots, x_k) \in \mathbb{G}^k$, and all $N \in \mathbb{N}^*$,

$$\left(K^{(N)}(x_i, x_j)\right)_{1\leq i,j\leq k} = \left(K^{(N-1)}(x_i, x_j)\right)_{1\leq i,j\leq k} + \lambda_N \left(\varphi_N(x_i)\varphi_N(x_j)^*\right)_{1\leq i,j\leq k}.$$

It follows from the Minkowski determinant theorem for Hermitian nonnegative-definite matrices [48, §7.8.8 p.482] that

$$\det\left(K^{(N)}(x_i, x_j)\right)_{1\leq i,j\leq k} \geq \det\left(K^{(N-1)}(x_i, x_j)\right)_{1\leq i,j\leq k}, \quad N \in \mathbb{N}^*.$$

Then $\det\left(K^{(N)}(x_i, x_j)\right)_{1\leq i,j\leq k}$ is nondecreasing with N. (vi) For any $N \geq k$, observe that $\left(K^{(N)}(x_i, x_j)\right)_{1\leq i,j\leq k} = AA^*$ where A is the matrix in $\mathbb{C}^{k\times N}$ defined by

$$A_{in} = \sqrt{\lambda_n}\varphi_n(x_i), \quad 1 \leq i \leq k, 1 \leq n \leq N.$$

By the Cauchy-Binet formula, we get

$$\det(AA^*) = \sum_{1\leq n_1,\ldots,n_k\leq N} \left|\det(A[n_1, \ldots, n_k])\right|^2,$$

where $A[n_1, \ldots, n_k]$ is the submatrix of A composed by the columns numbered $n_1, \ldots, n_k$. Moreover, by the very definition of the determinant of a matrix

$$\begin{aligned}\det\left(A[n_1, \ldots, n_k]\right) &= \sum_{\pi\in S_k} \operatorname{sgn}(\pi) \prod_{i=1}^{k} A_{i,\pi(n_i)} \\ &= \sum_{\pi\in S_k} \operatorname{sgn}(\pi) \prod_{i=1}^{k} \sqrt{\lambda_{\pi(n_i)}}\varphi_{\pi(n_i)}(x_i) \\ &= \sqrt{\lambda_{n_1}\ldots\lambda_{n_k}} \sum_{\pi\in S_k} \operatorname{sgn}(\pi) \prod_{i=1}^{k} \varphi_{\pi(n_i)}(x_i),\end{aligned}$$

where S_k is the set of permutations of $\{n_1, \ldots, n_k\}$ and $\operatorname{sgn}(\pi)$ denotes the signature of the permutation π. Letting

$$B_{in} = \varphi_n(x_i), \quad 1 \leq i \leq k, 1 \leq n \leq N,$$

we get

$$\begin{aligned}\det\left(K^{(N)}(x_i, x_j)\right)_{1\leq i,j\leq k} &= \det(AA^*) \\ &= \sum_{1\leq n_1<\ldots<n_k\leq N} \lambda_{n_1}\ldots\lambda_{n_k} \left|\det(B[n_1, \ldots, n_k])\right|^2 \\ &= \sum_{1\leq n_1<\ldots<n_k\leq N} \lambda_{n_1}\ldots\lambda_{n_k} \left|\det\left(\varphi_{n_j}(x_i)\right)_{1\leq i,j\leq k}\right|^2.\end{aligned}$$

Letting $N \to \infty$ and using (16.A.30), we get the announced result. □

The following proposition gives the properties of the integral operator associated to a pre-canonical kernel.

Proposition 16.A.26. Integral operator of a pre-canonical kernel. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$, let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be a sequence in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ and let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be nonnegative real numbers such that $\sum_{n\in\mathbb{N}^*} \lambda_n \|\varphi_n\|^2 < \infty$. Let $\mathcal{K}_{\mathbb{G}}$ be the integral operator associated to the pre-canonical kernel K defined by (16.A.27) where $\mathbb{G}_1$ is given by (16.A.28). Assume that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable. Then the following results hold true.*

(i) For any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$,

$$\mathcal{K}_{\mathbb{G}} f(x) = \sum_{n\in\mathbb{N}^*} \langle f, \varphi_n\rangle \lambda_n \varphi_n(x), \quad \text{for all } x \in \mathbb{G}_1. \tag{16.A.33}$$

(ii) The operator $\mathcal{K}_{\mathbb{G}}$ is Hermitian and nonnegative-definite.

(iii) The trace of the operator $\mathcal{K}_{\mathbb{G}}$ equals

$$\operatorname{tr}(\mathcal{K}_{\mathbb{G}}) = \sum_{n\in\mathbb{N}^*} \lambda_n \|\varphi_n\|^2 = \int_{\mathbb{G}} K(x,x)\mu(\mathrm{d}x). \tag{16.A.34}$$

In particular, the operator $\mathcal{K}_{\mathbb{G}}$ is trace class.

(iv) If $\{\lambda_n\}_{n\in\mathbb{N}^*}$ are positive and $\{\varphi_n\}_{n\in\mathbb{N}^*}$ are orthonormal (in which case, K is a canonical kernel), then $\{\lambda_n\}_{n\in\mathbb{N}^*}$ are the non-null eigenvalues of the integral operator $\mathcal{K}_{\mathbb{G}}$ with respective eigenvectors $\{\varphi_n\}_{n\in\mathbb{N}^*}$. Moreover, the trace of the operator $\mathcal{K}_{\mathbb{G}}$ equals

$$\operatorname{tr}(\mathcal{K}_{\mathbb{G}}) = \sum_{n\in\mathbb{N}^*} \lambda_n = \int_{\mathbb{G}} K(x,x)\mu(\mathrm{d}x). \tag{16.A.35}$$

More generally, for any $k \in \mathbb{N}^*$, the operator $\mathcal{K}^k_{\mathbb{G}}$ is trace class and its trace equals

$$\operatorname{tr}\left(\mathcal{K}^k_{\mathbb{G}}\right) = \sum_{n\in\mathbb{N}^*} (\lambda_n)^k = \int_{\mathbb{G}^k} K(x_1,x_2)K(x_2,x_3)\ldots K(x_k,x_1)\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k). \tag{16.A.36}$$

Proof. (i) For any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$ and $x \in \mathbb{G}_1$,

$$\begin{aligned}
\mathcal{K}_{\mathbb{G}} f(x) &= \int_{\mathbb{G}} \left(\sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x)\varphi_n(y)^*\right) f(y)\,\mu(\mathrm{d}y) \\
&= \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x) \int_{\mathbb{G}} \varphi_n(y)^* f(y)\,\mu(\mathrm{d}y) \\
&= \sum_{n\in\mathbb{N}^*} \langle f, \varphi_n\rangle \lambda_n \varphi_n(x),
\end{aligned}$$

where the interchange of the integral and the sum is due to Fubini's theorem, which is justified since by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_{\mathbb{G}}\left(\sum_{n\in\mathbb{N}^*}\left|\lambda_n\varphi_n(x)\varphi_n(y)\right|\right)|f(y)|\,\mu(\mathrm{d}y)\\
&\le\left[\int_{\mathbb{G}}\left(\sum_{n\in\mathbb{N}^*}\left|\lambda_n\varphi_n(x)\varphi_n(y)\right|\right)^2\mu(\mathrm{d}y)\right]^{1/2}\left[\int_{\mathbb{G}}|f(y)|^2\,\mu(\mathrm{d}y)\right]^{1/2},
\end{aligned}$$

which is finite by Lemma 16.A.25(iv). (ii) Observe that

$$K(x,y)=K(y,x)^*,\quad \text{for all } x,y\in\mathbb{G}_1,$$

then the above equality holds for μ^2-almost all $(x,y)\in\mathbb{G}^2$ by Lemma 16.A.25(i). It follows that $\mathcal{K}_{\mathbb{G}}$ is Hermitian by Lemma 16.A.9(vi). Moreover, $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite, since for any $f\in L^2_{\mathbb{C}}(\mu,\mathbb{G})$,

$$\langle\mathcal{K}_{\mathbb{G}}f,f\rangle=\sum_{n\in\mathbb{N}^*}\lambda_n\left|\langle f,\varphi_n\rangle\right|^2\ge 0, \tag{16.A.37}$$

where the first equality follows from (16.A.33). (iii) Let $\{\psi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_{\mathbb{C}}(\mu,\mathbb{G})$. By Definition 16.A.15,

$$\begin{aligned}
\operatorname{tr}(\mathcal{K}_{\mathbb{G}})&=\sum_{m\in\mathbb{N}^*}\langle\mathcal{K}_{\mathbb{G}}\psi_m,\psi_m\rangle\\
&=\sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*}\lambda_n\left|\langle\psi_m,\varphi_n\rangle\right|^2\\
&=\sum_{n\in\mathbb{N}^*}\lambda_n\sum_{m\in\mathbb{N}^*}\left|\langle\psi_m,\varphi_n\rangle\right|^2=\sum_{n\in\mathbb{N}^*}\lambda_n\left\|\varphi_n\right\|^2,
\end{aligned}$$

where the second equality is due to (16.A.37). The right-hand side of the above equation is finite by assumption. On the other hand, since $\mathcal{K}_{\mathbb{G}}$ is Hermitian and nonnegative-definite, its eigenvalues are nonnegative and $\operatorname{tr}(\mathcal{K}_{\mathbb{G}})$ equals the sum of the eigenvalues. Then the operator $\mathcal{K}_{\mathbb{G}}$ is trace class. The second equality in (16.A.34) follows from Lemma 16.A.25(ii). (iv) Applying (16.A.33) for $f=\varphi_m$ for some $m\in\mathbb{N}^*$, we get

$$\mathcal{K}_{\mathbb{G}}\varphi_m=\sum_{n\in\mathbb{N}^*}\langle\varphi_m,\varphi_n\rangle\lambda_n\varphi_n=\lambda_m\varphi_m.$$

Then φ_m is an eigenvector with associated eigenvalue λ_m. On the other hand, again by (16.A.33), any eigenvector corresponding to a non-null eigenvalue is in the vector subspace generated by $\{\varphi_n : n\in\mathbb{N}^*\}$. Therefore, $\{\lambda_n\}_{n\in\mathbb{N}^*}$ are the non-null eigenvalues of the integral operator $\mathcal{K}_{\mathbb{G}}$ with respective eigenvectors $\{\varphi_n\}_{n\in\mathbb{N}^*}$. Equation (16.A.35) follows from Equation (16.A.34). It remains to prove (16.A.36). Observe form (16.A.33) that, for all $x\in\mathbb{G}_1$,

$$\mathcal{K}^k_{\mathbb{G}}f(x)=\sum_{n\in\mathbb{N}^*}\langle f,\varphi_n\rangle\lambda_n^k\varphi_n(x).$$

Then, $\mathcal{K}_{\mathbb{G}}^{k}$ is the integral operator associated to the kernel

$$K_k(x,y)=\sum_{n\in\mathbb{N}^*}\lambda_n^k\varphi_n(x)\varphi_n(y)^*,$$

which is a canonical kernel since

$$\sum_{n\in\mathbb{N}^*}|\lambda_n|^k\leq\left(\sum_{n\in\mathbb{N}^*}|\lambda_n|\right)^k<\infty.$$

Thus the first equality in (16.A.36) follows form (16.A.35). Using (16.A.27) we get

$$\begin{aligned}
&\int_{\mathbb{G}^k}K(x_1,x_2)K(x_2,x_3)\ldots K(x_k,x_1)\mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right)\\
&=\int_{\mathbb{G}^k}\sum_{n_1\in\mathbb{N}^*}\lambda_{n_1}\varphi_{n_1}(x_1)\varphi_{n_1}(x_2)^*\sum_{n_2\in\mathbb{N}^*}\lambda_{n_2}\varphi_{n_2}(x_2)\varphi_{n_2}(x_3)^*\ldots\\
&\sum_{n_k\in\mathbb{N}^*}\lambda_{n_k}\varphi_{n_k}(x_k)\varphi_{n_k}(x_1)^*\mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right)\\
&=\sum_{n_1,n_2,\ldots,n_k\in\mathbb{N}^*}\lambda_{n_1}\lambda_{n_2}\ldots\lambda_{n_k}\int_{\mathbb{G}}\varphi_{n_1}(x_1)\varphi_{n_k}(x_1)^*\mu\left(\mathrm{d}x_1\right)\int_{\mathbb{G}}\varphi_{n_2}(x_2)\varphi_{n_1}(x_2)^*\mu\left(\mathrm{d}x_2\right)\ldots\\
&\int_{\mathbb{G}}\varphi_{n_k}(x_k)\varphi_{n_{k-1}}(x_k)^*\mu\left(\mathrm{d}x_k\right)\\
&=\sum_{n\in\mathbb{N}^*}\left(\lambda_n\right)^k,
\end{aligned}$$

where the second equality is due to Fubini's theorem which is justified since by (16.A.32)

$$\begin{aligned}
&\int_{\mathbb{G}^k}\sum_{n_1\in\mathbb{N}^*}|\lambda_{n_1}\varphi_{n_1}(x_1)\varphi_{n_1}(x_2)|\sum_{n_2\in\mathbb{N}^*}|\lambda_{n_2}\varphi_{n_2}(x_2)\varphi_{n_2}(x_3)|\ldots\\
&\sum_{n_k\in\mathbb{N}^*}|\lambda_{n_k}\varphi_{n_k}(x_k)\varphi_{n_k}(x_1)|\,\mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right)\\
&\leq\int_{\mathbb{G}^k}\sum_{n_1\in\mathbb{N}^*}\lambda_{n_1}\left|\varphi_{n_1}(x_1)\right|^2\sum_{n_2\in\mathbb{N}^*}\lambda_{n_2}\left|\varphi_{n_2}(x_2)\right|^2\ldots\sum_{n_k\in\mathbb{N}^*}\lambda_{n_k}\left|\varphi_{n_k}(x_k)\right|^2\\
&\qquad\mu\left(\mathrm{d}x_1\right)\ldots\mu\left(\mathrm{d}x_k\right)\\
&=\left(\sum_{n\in\mathbb{N}^*}\lambda_n\right)^k<\infty.
\end{aligned}$$

□

16.A.8 Continuous kernels

The following theorem permits to express a kernel in terms of the eigenvalues and vectors of the associated integral operator under some conditions including, in particular, the kernel being *continuous*.

We will need the following lemma.

Lemma 16.A.27. Dini's lemma. *Let $\mathbb{G}$ be a compact metric space, $f : \mathbb{G} \to \mathbb{R}$ a continuous function, and $f_n : \mathbb{G} \to \mathbb{R}$ ($n \in \mathbb{N}^*$) an increasing sequence of functions (i.e., $f_n(x) \leq f_{n+1}(x)$, for all $n \in \mathbb{N}^*, x \in \mathbb{G}$) converging pointwise to f. Then f_n converges uniformly to f on $\mathbb{G}$. The same result holds if the sequence $\{f_n\}_{n\in\mathbb{N}}$ is decreasing instead of increasing.*

Proof. (i) Assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is increasing. Let $\varepsilon \in \mathbb{R}_+^*$. For each $n \in \mathbb{N}^*$, let $g_n = f - f_n$ and $A_n = \{x \in \mathbb{G} : g_n(x) < \varepsilon\}$ which is open since g_n is continuous. For any $x \in \mathbb{G}$, $\lim_{n\to\infty} g_n(x) = 0$, then $x \in A_n$ for some $n \in \mathbb{N}^*$. Thus $\mathbb{G} = \bigcup_{n\in\mathbb{N}^*} A_n$. Since $\mathbb{G}$ is compact, there is a finite collection of A_n's that covers $\mathbb{G}$. Since the sequence $\{g_n\}_{n\in\mathbb{N}}$ is decreasing, then $A_n \subset A_{n+1}$, thus $\mathbb{G} = A_N$ for some $N \in \mathbb{N}^*$. Thus $\|g_N\|_\infty \leq \varepsilon$ which implies $\|g_n\|_\infty \leq \varepsilon$ for any $n \geq N$. It follows that $\lim_{n\to\infty} \|g_n\|_\infty = 0$. (ii) Assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is decreasing. Applying (i) to $\tilde{f}_n = -f_n$ permits to conclude. □

Theorem 16.A.28. Mercer's theorem. *Let μ be a finite measure on a compact metric space $\mathbb{G}$ (endowed with its Borel σ-algebra) and let $K : \mathbb{G}^2 \to \mathbb{C}$ be a continuous function such that the associated integral operator $\mathcal{K}_\mathbb{G}$ is Hermitian and nonnegative-definite. Let $\{\varphi_n\}_{n\in\mathbb{N}^*}$ be an orthonormal basis of $L^2_\mathbb{C}(\mu, \mathbb{G})$ composed of eigenvectors of $\mathcal{K}_\mathbb{G}$ and let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be the corresponding eigenvalues. Then the following results hold.*

(i) The function φ_n is continuous for all $n \in \mathbb{N}^*$ such that $\lambda_n \neq 0$.

(ii)

$$K(x, y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x) \varphi_n(y)^*, \quad x, y \in \operatorname{supp}(\mu),$$

where $\operatorname{supp}(\mu)$ *is the support of the measure μ defined by (14.B.1).*

(iii) The series in the right-hand side of the above equality is absolutely and uniformly convergent on $\operatorname{supp}(\mu)^2$.

Proof. Since K is continuous on a compact set $\mathbb{G}$, then K is bounded. Let $\|K\|_\infty := \sup_{(x,y)\in\mathbb{G}^2} |K(x, y)|$. Thus,

$$\int_\mathbb{G} |K(x, y)|^2 \mu(\mathrm{d}y) \leq \|K\|_\infty^2 \mu(\mathbb{G}), \quad \text{for all } x \in \mathbb{G}.$$

It follows from Proposition 16.A.13(vii) that for any $f \in L^2_\mathbb{C}(\mu, \mathbb{G})$,

$$\mathcal{K}_\mathbb{G} f(x) = \int_\mathbb{G} K(x, y) f(y) \mu(\mathrm{d}y) = \sum_{n\in\mathbb{N}^*} \langle f, \varphi_n \rangle \lambda_n \varphi_n(x), \quad \text{for any } x \in \mathbb{G}, \tag{16.A.38}$$

where the series in the right-hand side of the above equality converges uniformly in x. (i) It follows from the first equality in (16.A.38) that $\mathcal{K}_{\mathbb{G}}f$ is a continuous function on $\mathbb{G}$ by the dominated convergence theorem. In particular, for all $n \in \mathbb{N}^*$ such that $\lambda_n \neq 0$, the function $\varphi_n = \frac{1}{\lambda_n}\mathcal{K}_{\mathbb{G}}\varphi_n$ is continuous. (ii) Since $\mathbb{G}$ is a compact metric spaces, it is second countable. (Cf. [75, Exercise 30.4 p.194]. Indeed, for each $n \in \mathbb{N}^*$, consider an open covering of $\mathbb{G}$ with balls of radius $1/n$. By compactness of $\mathbb{G}$, extract an open subcovering A_n. Then $B = \bigcup_{n\in\mathbb{N}^*} A_n$ is a countable basis of the topology.) Then by Lemma 14.B.3, any integral with respect to μ over $\mathbb{G}$ equals to the corresponding integral over $\operatorname{supp}(\mu)$. Then it is enough to consider the case

$$\operatorname{supp}(\mu) = \mathbb{G}.$$

For any $N \in \mathbb{N}^*$, let $K^{(N)}$ be the function defined on $\mathbb{G}^2$ by

$$K^{(N)}(x,y) = \sum_{n=1}^{N} \lambda_n \varphi_n(x)\varphi_n(y)^*, \quad x,y \in \mathbb{G},$$

and let $\mathcal{K}_{\mathbb{G}}^{(N)}$ be the associated integral operator. Observe that for any $f \in L^2_{\mathbb{C}}(\mu,\mathbb{G})$,

$$\mathcal{K}_{\mathbb{G}}^{(N)} f = \sum_{n=1}^{N} \lambda_n \langle f,\varphi_n\rangle \varphi_n. \tag{16.A.39}$$

Observe that

$$\begin{aligned}\int_{\mathbb{G}^2}\left[K(x,y) - K^{(N)}(x,y)\right] f(y) f(x)^* \mu(\mathrm{d}x)\mu(\mathrm{d}y) &= \left\langle \left(\mathcal{K}_{\mathbb{G}} - \mathcal{K}_{\mathbb{G}}^{(N)}\right) f, f\right\rangle \\ &= \sum_{n\geq N+1} \lambda_n \left|\langle f,\varphi_n\rangle\right|^2 \geq 0,\end{aligned} \tag{16.A.40}$$

where the first equality is due to (16.A.3) and the second equality follows from (16.A.13)-(16.A.39). Then the operator $\mathcal{K}_{\mathbb{G}} - \mathcal{K}_{\mathbb{G}}^{(N)}$ is Hermitian nonnegative-definite. It follows that

$$K^{(N)}(x,x) \leq K(x,x), \quad \text{for any } x \in \mathbb{G}.$$

(The proof is by contradiction. Indeed, assume that there exist some $x_0 \in \mathbb{G}$ such that $K^{(N)}(x_0,x_0) > K(x_0,x_0)$. Then by continuity, there exist an open neighborhood A of x_0, such that $K^{(N)}(x,y) > K(x,y)$ for any $x,y \in A$. Let $f(x) = \mathbf{1}_A(x)$, then by (16.A.40), $\int_{A^2}\left[K(x,y) - K^{(N)}(x,y)\right]\mu(\mathrm{d}x)\mu(\mathrm{d}y) = 0$. Recall that we assume $\operatorname{supp}(\mu) = \mathbb{G}$, then by Lemmas 14.B.5-14.B.6, $K(x,y) - K^{(N)}(x,y) = 0$ for any $x,y \in A$; in particular $K^{(N)}(x_0,x_0) = K(x_0,x_0)$, a contradiction.) It follows from the above display that, for any $x \in \mathbb{G}$, the series $\sum_{n=1}^{\infty} \lambda_n |\varphi_n(x)|^2$ is convergent and its sum is $\leq K(x,x)$. Observe from the

Cauchy-Schwarz inequality that, for any $x, y \in \mathbb{G}$ and any $m, N \in \mathbb{N}^*$,

$$\left(\sum_{n=m}^{N} |\lambda_n \varphi_n(x)\varphi_n(y)^*|\right)^2 \leq \sum_{n=m}^{N} \lambda_n |\varphi_n(x)|^2 \sum_{n=m}^{N} \lambda_n |\varphi_n(y)|^2$$
$$\leq K(y,y) \sum_{n=m}^{N} \lambda_n |\varphi_n(x)|^2 \leq \|K\|_\infty \sum_{n=m}^{N} \lambda_n |\varphi_n(x)|^2 . \tag{16.A.41}$$

It follows that, for any fixed $x \in \mathbb{G}$, the series $\tilde{K}(x,y) := \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x)\varphi_n(y)^*$ converges absolutely and uniformly in $y \in \mathbb{G}$; then the function $y \mapsto \tilde{K}(x,y)$ is continuous on $\mathbb{G}$. Moreover, for any continuous function $f : \mathbb{G} \to \mathbb{C}$,

$$\int_{\mathbb{G}} \tilde{K}(x,y) f(y) \mu(\mathrm{d}y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x) \int_{\mathbb{G}} f(y) \varphi_n(y)^* \mu(\mathrm{d}y)$$
$$= \sum_{n\in\mathbb{N}^*} \langle f, \varphi_n \rangle \lambda_n \varphi_n(x), \quad \text{for any } x \in \mathbb{G}.$$

Combining the above equality and (16.A.38), we get

$$\int_{\mathbb{G}} \left(K(x,y) - \tilde{K}(x,y)\right) f(y) \mu(\mathrm{d}y) = 0.$$

Taking $f(y) := \left(K(x,y) - \tilde{K}(x,y)\right)^*$ and invoking Lemma 14.B.5, we deduce that

$$K(x,y) = \tilde{K}(x,y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x)\varphi_n(y)^*, \quad \text{for any } x, y \in \mathbb{G}. \tag{16.A.42}$$

(iii) It follows from the above display that

$$K(x,x) = \sum_{n\in\mathbb{N}^*} \lambda_n |\varphi_n(x)|^2 , \quad \text{for any } x \in \mathbb{G}.$$

Since the above series has nonnegative terms and its sum $K(x,x)$ is a continuous function of x, it follows from Dini's lemma 16.A.27 that the series converges uniformly in $x \in \mathbb{G}$. Inequality (16.A.41), then shows that the series in (16.A.42) converges absolutely and uniformly in $(x,y) \in \mathbb{G}^2$. □

Remark 16.A.29. Bibliographic notes. *Theorem 16.A.28 is a generalization of the classical Mercer's theorem [84, §98 p.245] when μ is the Lebesgue measure and $\mathbb{G}$ is a compact interval in $\mathbb{R}$.*

Related statements to Theorem 16.A.28 are given in [51, Theorem 8.11 p.195] and [60, Lemma 1 p.53].

16.A.9 Nonnegative-definite property

We will now address the following question: what are the properties of a kernel $K \in L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$ guaranteeing that the associated integral operator $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite? There is no simple answer to this question in full generality as observed in [51, p.194, last paragraph]. Nevertheless, simple answers may be given for some particular cases. For example, remind the necessary condition in Proposition 5.2.9(iii). We shall give a converse result when K is continuous. In this regard, we introduce the notion of Hermitian, nonnegative-definite function.

Definition 16.A.30. Hermitian, nonnegative-definite functions. *Let X be a set and $K : X \times X \to \mathbb{C}$.*

(i) K is said to be Hermitian *on X if $K(y, x) = K(x, y)^*$ for any $x, y \in X$.*

(ii) K is said to be nonnegative-definite *on X if the matrix $(K(x_i, x_j))_{1\le i,j\le k}$ is* nonnegative-definite *for any $k \in \mathbb{N}^*$ and any $x_1, \ldots, x_k \in X$.*

Proposition 16.A.31. *Let X be a set and $K : X \times X \to \mathbb{C}$.*

(i) K is Hermitian and nonnegative-definite iff there exist a Gaussian process indexed by X with covariance function K.

(ii) Assume that $K : X \times X \to \mathbb{C}$ is Hermitian. Then K is nonnegative-definite iff $\det(K(x_i, x_j))_{1\le i,j\le k} \ge 0$ for any $k \in \mathbb{N}^*$ and any $x_1, \ldots, x_k \in X$.

Proof. (i) Cf. [32, Theorem 3.1 p.72]. (ii) Cf. [96, Theorem 1.17 p.72]. □

Theorem 16.A.32. *Let μ be a finite measure on a compact metric space $\mathbb{G}$ (endowed with its Borel σ-algebra) and let $K : \mathbb{G}^2 \to \mathbb{C}$ be a continuous Hermitian function. Then K is nonnegative-definite on $\operatorname{supp}(\mu)$ iff the associated integral operator $\mathcal{K}_{\mathbb{G}}$ is nonnegative-definite.*

Proof. Observe first that since K is Hermitian, then by Lemma 16.A.9(vi), $\mathcal{K}_{\mathbb{G}}$ is Hermitian. Necessity is obvious from Mercer's theorem 16.A.28. It remains to prove sufficiency. By the same argument as in the proof of Theorem 16.A.28(ii), it is enough to consider the case $\operatorname{supp}(\mu) = \mathbb{G}$. Assume that K is nonnegative-definite on $\mathbb{G}$. For any $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$, it follows from (16.A.3) that

$$\langle \mathcal{K}_{\mathbb{G}} f, f\rangle = \int_{\mathbb{G}^2} K(x, y) f(y) f(x)^* \mu(\mathrm{d}x)\mu(\mathrm{d}y). \tag{16.A.43}$$

(i) Let $f : \mathbb{G} \to \mathbb{C}$ be continuous. Then, each Riemann sum of the above integral has the form

$$\sum_{i,j=1}^{n} K(x_i, x_j) f(x_j) f(x_i)^* \mu(A_i)\mu(A_j),$$

for some $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{G})$ and $x_i \in A_i$. Since the function K is nonnegative-definite, the above sum is nonnegative. Since the integral in the right-hand side of (16.A.43) is the limit of such sums, it follows that $\langle \mathcal{K}_{\mathbb{G}} f, f \rangle \geq 0$. (ii) Let $f \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Since the measure μ is finite and $\mathbb{G}$ is a metric space, then μ is regular by [56, Theorem 17.10 p.107]. Since $\mathbb{G}$ is a metric space it is *Hausdorff* (i.e., distinct points have disjoint neighborhoods). Since $\mathbb{G}$ is moreover compact (and therefore locally compact), then by [26, Proposition 7.4.2, p.227], there exist a sequence of continuous functions $f_n : \mathbb{G} \to \mathbb{C}$ with relatively compact supports, converging to f in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$. By Lemma 16.A.9(ii), the operator $\mathcal{K}_{\mathbb{G}}$ is bounded, and therefore $\mathcal{K}_{\mathbb{G}} f_n$ converges to $\mathcal{K}_{\mathbb{G}} f$ in $L^2_{\mathbb{C}}(\mu, \mathbb{G})$. Then, by the bicontinuity of the inner product in Hilbert spaces [21, Theorem 1.3.3 p.57], $\langle \mathcal{K}_{\mathbb{G}} f, f \rangle$ is the limit of $\langle \mathcal{K}_{\mathbb{G}} f_n, f_n \rangle$ which are nonnegative by Item (i). □

Remark 16.A.33. Bibliographic notes. *The result in Theorem 16.A.32 for the particular case where $\mathbb{G}$ is a compact interval of $\mathbb{R}$ is due to [71, p.462].*

The question treated in the present section is addressed in [4, Theorem 4.1.1 p.45] for the particular case when K is bounded (not necessarily continuous).

16.B Operator determinant

16.B.1 Fredholm determinant

We will describe the Fredholm approach to define the *determinant* $\det(\mathcal{I} + \mathcal{K})$ for a trace class operator $\mathcal{K}$ on a separable Hilbert space $\mathcal{H}$ (where $\mathcal{I}$ is the identity operator).

Let $\mathcal{H}$ be a separable Hilbert space and let $n \in \mathbb{N}^*$. For $\varphi_1, \varphi_2 \in \mathcal{H}$, let $\varphi_1 \otimes \varphi_2$ be their *tensor product*; cf. [82, Definition p.50]. We denote by $\mathcal{H}^{\otimes n}$ the n-fold *tensor product* $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$; cf. [82, Definition p.50]. The space $\mathcal{H}^{\otimes n}$ is equipped with an inner product such that

$$\langle \varphi_1 \otimes \cdots \otimes \varphi_n, \psi_1 \otimes \cdots \otimes \psi_n \rangle = \prod_{k=1}^{n} \langle \varphi_k, \psi_k \rangle ,$$

for any $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in \mathcal{H}$.

Remark 16.B.1. *We have already used the notation $\mathcal{G}^{\otimes n}$ for the n-th power of a σ-algebra $\mathcal{G}$ (in the sense of products of σ-algebra). It should be clear from the context whether we consider the n-th power of a σ-algebra or the n-fold tensor product of a separable Hilbert space.*

Define a linear operator $\wedge^n$ on $\mathcal{H}^{\otimes n}$ by

$$\wedge^n (\varphi_1 \otimes \cdots \otimes \varphi_n) := \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, \varphi_{\pi(1)} \otimes \cdots \otimes \varphi_{\pi(n)}, \quad \varphi_1, \ldots, \varphi_n \in \mathcal{H}, \tag{16.B.1}$$

where S_n is the set of permutations of $\{1, \ldots, n\}$ and $\operatorname{sgn}(\pi)$ denotes the *signature* of the permutation π. $\wedge^n (\varphi_1 \otimes \cdots \otimes \varphi_n)$ is called *antisymmetric tensor product* of $\varphi_1, \cdots, \varphi_n$. (Note that $\wedge^n (\varphi_1 \otimes \ldots \otimes \varphi_n)$ is sometimes denoted as $\varphi_1 \wedge \ldots \wedge \varphi_n$ by some authors.)

The set $\mathcal{H}^{\wedge n} := \wedge^n (\mathcal{H}^{\otimes n})$ is called the n-fold *antisymmetric tensor product* of $\mathcal{H}$.

Example 16.B.2. Tensor product of L^2 spaces *[82, p.51]. let μ be a σ-finite measure on measurable spaces $(\mathbb{G}, \mathcal{G})$ and assume that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable. Let $\mathcal{H} = L^2_{\mathbb{C}}(\mu, \mathbb{G})$, then $\mathcal{H}^{\otimes 2} = L^2_{\mathbb{C}}(\mu, \mathbb{G}) \otimes L^2_{\mathbb{C}}(\mu, \mathbb{G})$ may be identified with $L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$. (Indeed, if $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mu, \mathbb{G})$, then $\{\varphi_n(x)\varphi_k(y)\}_{n,k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$ by Lemma 16.A.3.) With this identification, for all $f, g \in L^2_{\mathbb{C}}(\mu, \mathbb{G})$, we have*

$$(f \otimes g)(x, y) = f(x) g(y), \quad x, y \in \mathbb{G}.$$

Moreover, $\mathcal{H}^{\wedge 2}$ is the set of functions φ in $L^2_{\mathbb{C}}\left(\mu^2, \mathbb{G}^2\right)$ such that $\varphi(x, y) = -\varphi(y, x)$ for any $x, y \in \mathbb{G}$.

Given a linear operator $\mathcal{K}$ on $\mathcal{H}$, we define the n-fold tensor product $\mathcal{K}^{\otimes n}$ as the linear operator on $\mathcal{H}^{\otimes n}$ such that

$$\mathcal{K}^{\otimes n}(\varphi_1 \otimes \cdots \otimes \varphi_n) = \mathcal{K}(\varphi_1) \otimes \cdots \otimes \mathcal{K}(\varphi_n),$$

for any $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$; cf. [82, Proposition p.298]. Combining the above equation and (16.B.1), it follows that

$$\mathcal{K}^{\otimes n}(\wedge^n(\varphi_1 \otimes \ldots \otimes \varphi_n)) = \wedge^n\left(\mathcal{K}^{\otimes n}(\varphi_1 \otimes \ldots \otimes \varphi_n)\right).$$

Then $\mathcal{K}^{\otimes n}$ leaves $\mathcal{H}^{\wedge n}$ invariant; that is $\mathcal{K}^{\otimes n}(\mathcal{H}^{\wedge n}) \subset \mathcal{H}^{\wedge n}$. We denote the restriction of $\mathcal{K}^{\otimes n}$ to $\mathcal{H}^{\wedge n}$ by

$$\wedge^n(\mathcal{K}) := \mathcal{K}^{\otimes n}\big|_{\mathcal{H}^{\wedge n}}.$$

When $n = 0$ we define $\mathcal{H}^{\wedge 0} := \mathbb{C}$ and $\wedge^0(\mathcal{K})$ as the mapping $\mathbb{C} \to \mathbb{C}; z \mapsto 1$. By [83, Eq (187) p.321], we have, for any linear operators $\mathcal{K}$ and $\mathcal{L}$ on $\mathcal{H}$,

$$\wedge^n(\mathcal{KL}) = \wedge^n(\mathcal{K}) \wedge^n(\mathcal{L}). \tag{16.B.2}$$

Example 16.B.3. Finite dimensional case *[83, Lemma 2 p.321]. If $\mathcal{H}$ has finite dimension k, then $\mathcal{H}^{\wedge k}$ is one dimensional and $\wedge^k(\mathcal{K})$ is the linear operator multiplying each element of $\mathcal{H}^{\wedge k}$ by the number* $\det(\mathcal{K})$.

We now give the definition of the so-called Fredholm determinant.

Definition 16.B.4. Fredholm determinant. *Let $\mathcal{K}$ be trace class operator on a separable Hilbert space $\mathcal{H}$. The* Fredholm determinant *of $\mathcal{I}+\mathcal{K}$ is defined by [83, Eq (188) p.323]*

$$\det(\mathcal{I} + \mathcal{K}) := \sum_{n=0}^{\infty} \operatorname{tr}(\wedge^n(\mathcal{K})). \tag{16.B.3}$$

Note that the series in the right-hand side of the above formula is convergent by [83, Lemma 4 p.323]. Here are further properties of the Fredholm determinant.

Lemma 16.B.5. Fredholm determinant properties. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{K}$ and $\mathcal{L}$ be linear operators on $\mathcal{H}$. Assume that $\mathcal{K}$ is trace class and $\mathcal{L}$ is bounded. Then the following results hold true.*

(i) For any $n \in \mathbb{N}^*$, the operator $\wedge^n(\mathcal{K})$ is trace class and satisfies

$$|| \wedge^n (\mathcal{K}) ||_1 \leq \frac{\|\mathcal{K}\|_1^n}{n!},$$

where $\|\mathcal{K}\|_1 := \operatorname{tr}\left(\sqrt{\mathcal{K}^*\mathcal{K}}\right)$ with analogous notation for $|| \wedge^n (\mathcal{K}) ||_1$.

(ii)

$$\det(\mathcal{I} + \mathcal{K}\mathcal{L}) = \det(\mathcal{I} + \mathcal{L}\mathcal{K}). \tag{16.B.4}$$

(iii) If $\mathcal{L}$ is trace class, then

$$\det(\mathcal{I} + \mathcal{K}) \det(\mathcal{I} + \mathcal{L}) = \det(\mathcal{I} + \mathcal{K} + \mathcal{L} + \mathcal{K}\mathcal{L}). \tag{16.B.5}$$

(iv) If $||\mathcal{K}|| < 1$, then the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{tr}(\mathcal{K}^n)$ converges absolutely and

$$\det(\mathcal{I} + \mathcal{K}) = \exp\left[-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{tr}(\mathcal{K}^n)\right].$$

(v) For any $t \in \mathbb{R}$,

$$\det(\mathcal{I} + t\mathcal{K}) = \sum_{n=0}^{\infty} t^n \operatorname{tr}(\wedge^n(\mathcal{K})),$$

where the above series is absolutely convergent.

(vi) Plemelj-Smithies formula*: For any $n \in \mathbb{N}$,*

$$\operatorname{tr}(\wedge^n(\mathcal{K})) = \frac{1}{n!} \det \begin{pmatrix} \operatorname{tr}(\mathcal{K}) & n-1 & 0 & \cdots & 0 \\ \operatorname{tr}\left(\mathcal{K}^2\right) & \operatorname{tr}(\mathcal{K}) & n-2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \operatorname{tr}\left(\mathcal{K}^{n-1}\right) & \operatorname{tr}\left(\mathcal{K}^{n-2}\right) & \operatorname{tr}\left(\mathcal{K}^{n-3}\right) & \cdots & 1 \\ \operatorname{tr}(\mathcal{K}^n) & \operatorname{tr}\left(\mathcal{K}^{n-1}\right) & \operatorname{tr}\left(\mathcal{K}^{n-2}\right) & \cdots & \operatorname{tr}(\mathcal{K}) \end{pmatrix}.$$

Proof. (i) Cf. [83, Lemma 3 p.323]. (ii) Recall that $\mathcal{K}\mathcal{L}$ and $\mathcal{L}\mathcal{K}$ are trace class, then

$$\begin{aligned} \det(\mathcal{I} + \mathcal{K}\mathcal{L}) &= \sum_{n=0}^{\infty} \operatorname{tr}(\wedge^n(\mathcal{K}\mathcal{L})) \\ &= \sum_{n=0}^{\infty} \operatorname{tr}(\wedge^n (\mathcal{K}) \wedge^n (\mathcal{L})) \\ &= \sum_{n=0}^{\infty} \operatorname{tr}(\wedge^n (\mathcal{L}) \wedge^n (\mathcal{K})) \\ &= \sum_{n=0}^{\infty} \operatorname{tr}(\wedge^n (\mathcal{L}\mathcal{K})) = \det(\mathcal{I} + \mathcal{L}\mathcal{K}), \end{aligned}$$

where the first and last equalities are due to (16.B.3), the second and fourth equalities are due to (16.B.2), and the third equality si due to [82, Theorem VI.25 p.212]. (iii) Cf. [83, Theorem XIII.105(a) p.325]. (vi) Cf. [83, Lemma 6 p.331]. (v) It follows from (16.B.3) that

$$\det(\mathcal{I}+t\mathcal{K}) = \sum_{n=0}^{\infty} t^n \operatorname{tr}(\wedge^n(\mathcal{K})), \quad t \in \mathbb{R},$$

where the series in the right-hand side of the above formula is convergent; cf. [83, Lemma 4 p.323]. Since this is a power series in t, it is absolutely convergent for any real t by [59, Proposition 1.1.1 p.1]. (vi) For any $t < 1/||\mathcal{K}|$,

$$\det(\mathcal{I}+t\mathcal{K}) = \exp\left[-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\operatorname{tr}(\mathcal{K}^n)t^n\right] = \sum_{n=0}^{\infty} t^n \operatorname{tr}(\wedge^n(\mathcal{K})).$$

where the first equality is due to Item (v) and the second equality follows from Item (vi). Lemma 13.A.30 then shows the announced equality. □

The following lemma expresses the determinant and the trace of trace class operator in terms of its eigenvalues.

Lemma 16.B.6. *Let $\mathcal{K}$ be trace class operator on a separable Hilbert space $\mathcal{H}$. Then the following results hold.*

(i) $\mathcal{K}$ is compact; in particular the set of nonzero eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$ of $\mathcal{K}$ accounting for their multiplicities is at most countable.

(ii)

$$\det(\mathcal{I}+\mathcal{K}) = \prod_{n=1}^{\infty}(1+\lambda_n), \tag{16.B.6}$$

(iii) Lidskii's theorem*:*

$$\operatorname{tr}(\mathcal{K}) = \sum_{n\in\mathbb{N}^*}\lambda_n, \tag{16.B.7}$$

where the series is absolutely convergent.

Proof. (i) Since $\mathcal{K}$ is trace class operator, then it is compact by Lemma 16.A.21. Since $\mathcal{H}$ is a separable Hilbert space, then the set of eigenvalues of $\mathcal{K}$ is at most countable by Riesz-Schauder theorem [82, Theorem VI.15]. (ii) Cf. [83, Theorem XIII.106 p.326]. (iii) The fact that the series $\sum_{n\in\mathbb{N}^*}\lambda_n$ is absolutely convergent follows from [83, Theorem XIII.103 p.318]; its sum equals $\operatorname{tr}(\mathcal{K})$ by [83, p.328]. □

Note that Lidskii's theorem (Lemma 16.B.6(iii)) extends Corollary 16.A.18 to trace class operators (not necessarily Hermitian).

Remark 16.B.7. Finite dimensional approximating approach. *For a trace class operator $\mathcal{K}$ on a separable Hilbert space $\mathcal{H}$, the* determinant $\det(\mathcal{I}+\mathcal{K})$ *is defined alternatively by Gohberg et al. [41, Equation (5.7) p.61] as limit of determinants of finite dimensional operators approximating $\mathcal{K}$. Since (16.B.6) holds also for Gohberg's determinant; cf. [41, Theorem IV.6.1 p.63], then the two definitions are equivalent.*

Example 16.B.8. Muliplication operator. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G},\mathcal{G})$, let $g:\mathbb{G}\to\mathbb{R}_+$ be a bounded measurable function, and let $\|g\|_\infty=\sup_{x\in\mathbb{G}}|g(x)|$. Define an operator $\mathcal{K}$ on $L^2_{\mathbb{C}}(\mu,\mathbb{G})$ by, for any $f\in L^2_{\mathbb{C}}(\mu,\mathbb{G})$,*

$$\mathcal{K}f(x)=g(x)f(x),\quad x\in\mathbb{G},$$

called the multiplication operator*. This is a well defined operator; i.e., $\mathcal{K}f\in L^2_{\mathbb{C}}(\mu,\mathbb{G})$ since*

$$\int_{\mathbb{G}}|\mathcal{K}f(x)|^2\mu(\mathrm{d}x)\le\|g\|_\infty^2\int_{\mathbb{G}}|f(x)|^2\mu(\mathrm{d}x)<\infty.$$

Moreover, the above equality shows that $\mathcal{K}$ is a bounded linear operator. With a slight abuse of notation, we will denote this operator $\mathcal{K}$ simply by g.

Lemma 16.B.9. Multiplication operator determinant. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G},\mathcal{G})$, $K\in L^2_{\mathbb{C}}\left(\mu^2,\mathbb{G}^2\right)$, and $\mathcal{K}_{\mathbb{G}}$ the associated integral operator assumed trace class. Let $g:\mathbb{G}\to\mathbb{R}_+$ be a bounded measurable function and let $\mathcal{K}$ and $\mathcal{L}$ be the integral operators with respective kernels $K(x,y)g(y)$ and $g(x)K(x,y)$. Then*

$$\mathcal{K}=\mathcal{K}_{\mathbb{G}}g,\quad \mathcal{L}=g\mathcal{K}_{\mathbb{G}}. \tag{16.B.8}$$

Moreover,

$$\det(\mathcal{I}+\mathcal{K})=\det(\mathcal{I}+\mathcal{K}_{\mathbb{G}}g)=\det(\mathcal{I}+g\mathcal{K}_{\mathbb{G}})=\det(\mathcal{I}+\mathcal{L}). \tag{16.B.9}$$

Proof. Recall that g in (16.B.8) denotes the multiplication operator which is a bounded linear operator; cf. Example 16.B.8. In particular, $\mathcal{K}_{\mathbb{G}}g$ denotes the composition of the operator $\mathcal{K}_{\mathbb{G}}$ by the multiplication operator associated to g. Thus, by (16.A.1),

$$(\mathcal{K}_{\mathbb{G}}g)f(x)=\mathcal{K}_{\mathbb{G}}(g\times f)(x)=\int_{\mathbb{G}}K(x,y)g(y)f(y)\mu(\mathrm{d}y)=\mathcal{K}f(x).$$

The second equality in (16.B.8) follows in the same lines. Using (16.B.4) and (16.B.8) permits to prove (16.B.9). □

16.B.2 Expansion of integral operator's determinant

For the particular case of integral operators, $\det(\mathcal{I}+\mathcal{K}_{\mathbb{G}})$ admits the following useful series expansion.

Proposition 16.B.10. Integral operator's determinant expansion. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G}, \mathcal{G})$ such that $L^2_{\mathbb{C}}(\mu, \mathbb{G})$ is separable, $K : \mathbb{G}^2 \to \mathbb{C}$ be square integrable with respect to μ^2 such that*

$$\int_{\mathbb{G}} |K(x,x)|\, \mu(\mathrm{d}x) < \infty, \quad \text{and} \int_{\mathbb{G}} K(x,x)\, \mu(\mathrm{d}x) = \mathrm{tr}(\mathcal{K}_{\mathbb{G}}), \tag{16.B.10}$$

and $\mathcal{K}_{\mathbb{G}}$ the associated integral operator assumed trace class. Then, for any $t \in \mathbb{R}$,

$$\det(\mathcal{I} + t\mathcal{K}_{\mathbb{G}}) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_{\mathbb{G}^n} \det(K(x_i, x_j))_{1 \le i,j \le n}\ \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n). \tag{16.B.11}$$

Moreover, the series in the right-hand side of the above equality is absolutely convergent.

Proof. If $\mathcal{K}_{\mathbb{G}}$ is the null operator, then the announced expansions are obvious. Assume now that $\mathcal{K}_{\mathbb{G}}$ is non-null. **Step 1**. *Determinant's derivatives.* Let f be the function defined on $\mathbb{R}$ by

$$f(t) := \det(\mathcal{I} + t\mathcal{K}_{\mathbb{G}}), \quad t \in \mathbb{R}.$$

By Lemma 16.B.5(v), the function f admits an (absolutely convergent) series expansion at 0 with infinite radius of convergence. By [59, Corollary 1.1.10 p.9], this series has the form

$$f(t) = \det(\mathcal{I} + t\mathcal{K}_{\mathbb{G}}) = 1 + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} t^n, \quad t \in \mathbb{R}. \tag{16.B.12}$$

Let $R = 1/\|\mathcal{K}_{\mathbb{G}}\|$. By Lemma 16.B.5(iv), the function $g(t) = \log(f(t))$ admits the following series expansion

$$g(t) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{tr}(\mathcal{K}_{\mathbb{G}}^n) t^n, \quad t \in (-R, R).$$

In particular, again by [59, Corollary 1.1.10 p.9], g is infinitely differentiable at 0 and

$$g^{(n)}(0) = -\frac{(-1)^n}{n} \mathrm{tr}(\mathcal{K}_{\mathbb{G}}^n) n!, \quad n \in \mathbb{N}^*.$$

Applying (13.A.31), we deduce that, for any $n \in \mathbb{N}^*$,

$$\begin{aligned} f^{(n)}(0) &= \sum_{k=1}^{n} \frac{1}{k!} \sum_{\xi \in (\mathbb{N}^*)^k : |\xi| = n} \frac{n!}{\xi_1! \ldots \xi_k!} \prod_{p=1}^{k} g^{(\xi_p)}(0) \\ &= (-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\xi \in (\mathbb{N}^*)^k : |\xi| = n} \frac{n!}{\xi_1 \ldots \xi_k} \prod_{p=1}^{k} \mathrm{tr}(\mathcal{K}_{\mathbb{G}}^{\xi_p}), \end{aligned}$$

where $|\xi| = \xi_1 + \ldots + \xi_k$ for any $\xi = (\xi_1, \cdots, \xi_k) \in (\mathbb{N}^*)^k$. **Step 2**. *A combinatorial formula.* Let S_n be the set of permutations of $\{1, \ldots, n\}$; S_n is called the symmetric group. Recall that the conjugacy class [33, p.123] of a permutation $\pi \in S_n$, denoted by $[\pi]$, is determined by the lengths $j_1 \geq \ldots \geq j_k \geq 1$ of cycles in π; cf. [33, Proposition 11 p.126]. Thus each conjugacy class in S_n corresponds to exactly one integer partition of n; i.e., a way of writing n as a sum of positive integers $j_1 \geq \ldots \geq j_k \geq 1$. Let $\pi \in S_n$ be such that, for each $i \in \{1, \ldots, s\}$, π has k_i cycles of length m_i (so that $\sum_{i=1}^{s} k_i m_i = n$). Then the number of permutations in the corresponding conjugacy class equals [33, Exercise 33 p.132]

$$\operatorname{card}([\pi]) = \frac{n!}{m_1^{k_1} \ldots m_s^{k_s} k_1! \ldots k_s!}.$$

Let $j_1 \geq \ldots \geq j_k \geq 1$ be the lengths of the cycles in π; that is a rearrangement of the m_i with multiplicity k_i in the decreasing order. Then the above formula may be written as

$$\begin{aligned}
\operatorname{card}([\pi]) &= \frac{n!}{m_1^{k_1} \ldots m_s^{k_s} k_1! \ldots k_s!} \\
&= \frac{n!}{j_1 \ldots j_k} \frac{1}{k_1! \ldots k_s!} \\
&= \frac{n!}{j_1 \ldots j_k} \frac{1}{k!} \operatorname{card}\left(\left\{\xi \in (\mathbb{N}^*)^k : \xi^* = (j_1, \ldots, j_k)\right\}\right),
\end{aligned}$$

where $\xi^* = (\xi_1^*, \ldots, \xi_k^*)$ is a rearrangement of $\xi = (\xi_1, \ldots, \xi_k)$ in such a way that $\xi_1^* \geq \ldots \geq \xi_k^*$. Thus, given $j_1 \geq \ldots \geq j_k \geq 1$ such that $j_1 + \cdots + j_k = n$, the number of permutations in the corresponding conjugacy class equals

$$\sum_{\substack{\pi \in S_n: \\ [\pi] = (j_1, \ldots, j_k)}} 1 = \frac{1}{k!} \sum_{\substack{\xi \in (\mathbb{N}^*)^k: \\ \xi^* = (j_1, \ldots, j_k)}} \frac{n!}{\xi_1 \ldots \xi_k}. \tag{16.B.13}$$

Step 3. Combining the results of the above two steps, we get

$$\begin{aligned}
f^{(n)}(0) &= (-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\substack{j_1 \geq \ldots \geq j_k \geq 1: \\ j_1 + \ldots + j_k = n}} \left[\sum_{\substack{\xi \in (\mathbb{N}^*)^k: \\ \xi^* = (j_1, \ldots, j_k)}} \frac{n!}{\xi_1 \ldots \xi_k} \right] \prod_{p=1}^{k} \operatorname{tr}(\mathcal{K}_{\mathbb{G}}^{j_p}) \\
&= (-1)^n \sum_{k=1}^{n} (-1)^k \sum_{\substack{j_1 \geq \ldots \geq j_k \geq 1: \\ j_1 + \ldots + j_k = n}} \left[\sum_{\substack{\pi \in S_n: \\ [\pi] = (j_1, \ldots, j_k)}} 1 \right] \prod_{p=1}^{k} \operatorname{tr}(\mathcal{K}_{\mathbb{G}}^{j_p}).
\end{aligned}$$

It follows from (16.A.22), and (16.B.10) that

$$\begin{aligned}\prod_{p=1}^{k}\operatorname{tr}(\mathcal{K}_{\mathbb{G}}^{j_p}) &= \prod_{p=1}^{k}\int_{\mathbb{G}^{j_p}} K(x_1,x_2)K(x_2,x_3)\dots K(x_{j_p},x_1)\mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_{j_p}\right)\\ &= \int_{\mathbb{G}^n}\prod_{i=1}^{n}K(x_i,x_{\pi(i)})\mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_n\right), \qquad (16.B.14)\end{aligned}$$

where the second equality is due to Fubini's theorem which is justified by (16.A.7). Combining the last two equations, we get

$$\begin{aligned}f^{(n)}(0) &= (-1)^n\sum_{k=1}^{n}(-1)^k\sum_{\substack{\pi\in S_n:\\ \operatorname{cyc}(\pi)=k}}\int_{\mathbb{G}^n}\prod_{i=1}^{n}K(x_i,x_{\pi(i)})\mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_n\right)\\ &= (-1)^n\sum_{\pi\in S_n}(-1)^{\operatorname{cyc}(\pi)}\int_{\mathbb{G}^n}\prod_{i=1}^{n}K(x_i,x_{\pi(i)})\mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_n\right)\\ &= (-1)^n\int_{\mathbb{G}^n}\sum_{\pi\in S_n}(-1)^{\operatorname{cyc}(\pi)}\prod_{i=1}^{n}K(x_i,x_{\pi(i)})\mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_n\right)\\ &= \int_{\mathbb{G}^n}\det\left(K\left(x_i,x_j\right)\right)_{1\le i,j\le n}\ \mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_n\right),\end{aligned}$$

where $\operatorname{cyc}(\pi)$ denotes the number of cycles in π and the last equality is due to the fact that, for any matrix $A=(A_{ij})_{1\le i,j\le n}\in\mathbb{C}^{n\times n}$, $\det(A)=\sum_{\pi\in S_n}(-1)^{n-\operatorname{cyc}(\pi)}\prod_{i=1}^{n}A_{i\pi(i)}$. Combining (16.B.12) with the above display permits to conclude. □

Remark 16.B.11. *Proposition 16.B.10 in the particular case when K is continuous and $\mathbb{G}$ is an interval of $\mathbb{R}$ is proved in [90, Theorem 3.10 p.36].*

Corollary 16.B.12. *Under the assumption of Proposition 16.B.10, for any $n\in\mathbb{N}^*$,*

$$\operatorname{tr}(\wedge^n(\mathcal{K}_{\mathbb{G}})) = \frac{1}{n!}\int_{\mathbb{G}^n}\det\left(K\left(x_i,x_j\right)\right)_{1\le i,j\le n}\ \mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_n\right).$$

Proof. By Lemma 16.B.5(v), the power series

$$\det(\mathcal{I}+t\mathcal{K}_{\mathbb{G}}) = \sum_{n=0}^{\infty}t^n\operatorname{tr}(\wedge^n(\mathcal{K}_{\mathbb{G}})), \quad t\in\mathbb{R},$$

has an infinite radius of convergence. On the other hand, by Proposition 16.B.10, we have

$$\det\left(\mathcal{I}+t\mathcal{K}_{\mathbb{G}}\right) = 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\int_{\mathbb{G}^n}\det\left(K\left(x_i,x_j\right)\right)_{1\le i,j\le n}\ \mu\left(\mathrm{d}x_1\right)\dots\mu\left(\mathrm{d}x_n\right), \quad t\in\mathbb{R},$$

where the power series in the right-hand side of the above equality is convergent. Identifying the corresponding terms of the above two power series gives the announced result. □

We give now a series expansion for $\det(\mathcal{I}+t\mathcal{K}_{\mathbb{G}})^{-1/\alpha}$ where $\mathcal{K}_{\mathbb{G}}$ is an integral operator and $\alpha \in \mathbb{R}^*$ (which generalizes (16.B.11)) in terms of α-determinant of a finite dimensional matrix defined by (15.A.12).

Proposition 16.B.13. *Let μ be a σ-finite measure on a measurable space $(\mathbb{G},\mathcal{G})$ such that $L^2_{\mathbb{C}}(\mu,\mathbb{G})$ is separable, $K:\mathbb{G}^2\to\mathbb{C}$ be square integrable with respect to μ^2 satisfying (16.B.10), and $\mathcal{K}_{\mathbb{G}}$ the associated integral operator assumed trace class. Then the following expansion*

$$\det(\mathcal{I}+t\mathcal{K}_{\mathbb{G}})^{-1/\alpha} = \\ 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{t^n}{\alpha^n}\int_{\mathbb{G}^n}\det{}_{\alpha}\left(K(x_i,x_j)\right)_{1\le i,j\le n}\ \mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n)\,, \quad (16.B.15)$$

holds and the above series converges absolutely in either of the following cases:

(i) $\alpha\in\mathbb{R}^*$ *and* $t\in(-1/\|\mathcal{K}_{\mathbb{G}}\|,1/\|\mathcal{K}_{\mathbb{G}}\|)$;

(ii) $\alpha\in\{-1/m: m\in\mathbb{N}^*\}$ *and* $t\in\mathbb{R}$.

Proof. If $\mathcal{K}_{\mathbb{G}}$ is the null operator, then the announced expansions are obvious. Assume now that $\mathcal{K}_{\mathbb{G}}$ is non-null. (i) **Step 1**. *Determinant's power derivatives.* Let $R=1/\|\mathcal{K}_{\mathbb{G}}\|$, and f be the function defined on the open interval $(-R,R)$ by

$$f(t):=\det(\mathcal{I}+t\mathcal{K}_{\mathbb{G}})^{-1/\alpha}\,,\quad t\in(-R,R)\,.$$

By Lemma 16.B.5(iv), the function $g(t)=\log(f(t))$ admits a series expansion at 0 with radius of convergence at least R. Thus, by Lemma 15.B.1, $f(t)=e^{g(t)}$ admits a series expansion at 0 with radius of convergence at least R; that is

$$f(t)=1+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}t^n,\quad t\in(-R,R)\,, \tag{16.B.16}$$

where the above series is absolutely convergent by [59, Proposition 1.1.1 p.1]. On the other hand, it follows from Lemma 16.B.5(iv) that

$$g(t)=\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\operatorname{tr}(\mathcal{K}_{\mathbb{G}}^n)t^n,\quad t\in(-R,R)\,.$$

In particular, g is infinitely differentiable at 0 and

$$g^{(n)}(0)=\frac{1}{\alpha}\frac{(-1)^n}{n}\operatorname{tr}(\mathcal{K}_{\mathbb{G}}^n)n!,\quad n\in\mathbb{N}^*.$$

Applying (13.A.31), we deduce that, for any $n\in\mathbb{N}^*$,

$$\begin{aligned} f^{(n)}(0) &= \sum_{k=1}^{n}\frac{1}{k!}\sum_{\xi\in(\mathbb{N}^*)^k:|\xi|=n}\frac{n!}{\xi_1!\dots\xi_k!}\prod_{p=1}^{k}g^{(\xi_p)}(0)\\ &= (-1)^n\sum_{k=1}^{n}\frac{\alpha^{-k}}{k!}\sum_{\xi\in(\mathbb{N}^*)^k:|\xi|=n}\frac{n!}{\xi_1\dots\xi_k}\prod_{p=1}^{k}\operatorname{tr}(\mathcal{K}_{\mathbb{G}}^{\xi_p}), \end{aligned}$$

where $|\xi| = \xi_1 + \ldots + \xi_k$ for any $\xi = (\xi_1, \cdots, \xi_k) \in (\mathbb{N}^*)^k$. **Step 2**. *Determinant's power expansion.* Recall the combinatorial formula (16.B.13) where S_n is the set of permutations of $\{1, \ldots, n\}$, $[\pi]$ is the conjugacy class of a permutation $\pi \in S_n$, and $\xi^* = (\xi_1^*, \ldots, \xi_k^*)$ is a rearrangement of $\xi = (\xi_1, \ldots, \xi_k)$ in such a way that $\xi_1^* \geq \ldots \geq \xi_k^*$. Combining the above display with (16.B.13), we get

$$\begin{aligned} f^{(n)}(0) &= (-1)^n \sum_{k=1}^{n} \frac{\alpha^{-k}}{k!} \sum_{\substack{j_1 \geq \ldots \geq j_k \geq 1: \\ j_1 + \ldots + j_k = n}} \left[\sum_{\substack{\xi \in (\mathbb{N}^*)^k: \\ \xi^* = (j_1, \ldots, j_k)}} \frac{n!}{\xi_1 \ldots \xi_k} \right] \prod_{p=1}^{k} \mathrm{tr}(\mathcal{K}_{\mathbb{G}}^{j_p}) \\ &= (-1)^n \sum_{k=1}^{n} \alpha^{-k} \sum_{\substack{j_1 \geq \ldots \geq j_k \geq 1: \\ j_1 + \ldots + j_k = n}} \left[\sum_{\substack{\pi \in S_n: \\ [\pi] = (j_1, \ldots, j_k)}} 1 \right] \prod_{p=1}^{k} \mathrm{tr}(\mathcal{K}_{\mathbb{G}}^{j_p}). \end{aligned}$$

Using (16.B.14), we get

$$\begin{aligned} f^{(n)}(0) &= (-1)^n \sum_{k=1}^{n} \alpha^{-k} \sum_{\substack{\pi \in S_n: \\ \mathrm{cyc}(\pi) = k}} \int_{\mathbb{G}^n} \prod_{i=1}^{n} K(x_i, x_{\pi(i)}) \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n) \\ &= (-1)^n \sum_{\pi \in S_n} \alpha^{-\mathrm{cyc}(\pi)} \int_{\mathbb{G}^n} \prod_{i=1}^{n} K(x_i, x_{\pi(i)}) \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n) \\ &= (-1)^n \int_{\mathbb{G}^n} \sum_{\pi \in S_n} \alpha^{-\mathrm{cyc}(\pi)} \prod_{i=1}^{n} K(x_i, x_{\pi(i)}) \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n) \\ &= (-1)^n \alpha^{-n} \int_{\mathbb{G}^n} \det{}_\alpha \left(K(x_i, x_j) \right)_{1 \leq i,j \leq n} \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_n), \end{aligned}$$

where the last equality is due to (15.A.12). Combining the above display with (16.B.16) permits to conclude. (ii) Let $\alpha = -1/m$ for some $m \in \mathbb{N}^*$ and

$$h(t) := \det(\mathcal{I} + t\mathcal{K}_{\mathbb{G}}), \quad t \in \mathbb{R}.$$

By Lemma 16.B.5(v), the function h admits a series expansion at 0 with infinite radius of convergence, then so is the function h^m by [59, Proposition 1.1.4 p.4]. On the other hand, by Item (i), $h(t)^m = \det(\mathcal{I} + t\mathcal{K}_{\mathbb{G}})^{-1/\alpha}$ admits the power expansion (16.B.15) for any $t \in (-1/\|\mathcal{K}_{\mathbb{G}}\|, 1/\|\mathcal{K}_{\mathbb{G}}\|)$. The uniqueness of the power series representation [59, Corollary 1.1.10 p.9] of h^m permits to conclude. □

Remark 16.B.14. Bibliographic notes. *For a related statement to Proposition 16.B.10 cf. [41, Theorem VI.3.1 p.121]. Proposition 16.B.13 is due to [88, Theorem 2.4] (which generalizes [97, Equation (3)] for finite dimensional matrices).*

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